Complex and quaternionic structures on symmetric spaces – correspondence with Freudenthal-Kantor triple systems

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Abstract. It is proved that, under the general correspondence between graded Lie algebras and generalized Jordan pairs and -triple systems, the five-graded Lie algebras correspond to Freudenthal-Kantor pairs and -triple systems. We interprete this in terms of geometric structures on symmetric spaces and show that the balanced Freudenthal-Kantor pairs correspond to those five-graded Lie algebras that arise from complexifications of quaternionic symmetric spaces.

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Key words: Jordan pair, Freudenthal-Kantor triple system, symmetric space, quaternionic structure.

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0. Introduction

A $2\nu+1$-*graded Lie algebra* is a Lie algebra of the form $\mathfrak{g} = \oplus_{k=-\nu}^{\nu} \mathfrak{g}_\nu$ such that $[\mathfrak{g}_k, \mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$. As is well-known, 3-*graded Lie algebras* are essentially in bijection with certain Jordan-theoretic objects called *Jordan pairs* and which in some sense are generalizations of *Jordan algebras* (cf. [Lo75]). I.L. Kantor has remarked that more general graded Lie algebras correspond to *generalized Jordan triple systems* which are non-commutative in the sense that they are in general non-symmetric in the outer variables. Classification results have been given by S. Kaneyuki and H. Asano in [KanAs88].

In this note we wish to contribute to the understanding of the *geometry* corresponding to graded Lie algebras. In the book [Be00] we have investigated the geometric structures corresponding to commutative Jordan algebraic structures, i.e. to 3-graded Lie algebras, with or without involution. The geometric structures that are important here are *symmetric spaces* with some additional structure such as *invariant twisted complex or para-complex structures.* For a general ν-graded Lie algebra the situation is much more complicated: we can associate a symmetric pair $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ to these data just by taking for $\mathfrak{h}$ the even and for $\mathfrak{q}$ the odd part of the graded Lie algebra $\mathfrak{g}$, but the grading will in general not define any *invariant* structure on the corresponding symmetric space G/H. On the other hand, there is an associated homogeneous space G/G_0 (where G_0 is the centralizer of the characteristic element of the grading) which has very interesting invariant structures – see, for instance, [Kan92] and [DHKN99]. Only in the 3-graded case these two geometries coincide; in the general case their interplay raises interesting problems, which to our knowledge are not yet fully investigated.

Fur the purposes of this note, we restrict our attention to the 5-graded case (which includes the 3-graded case). In the first two chapters we show that, geometrically, this case corresponds to symmetric spaces with a so-called *twisted polarisation* which, however, is in general not invariant. As mentioned above, this implies that the associated generalized Jordan pair is no longer commutative, but since we are in the 5-graded case we still have some additional identity which is known to define the so-called *Freudenthal-Kantor pairs* or *-triple systems* (cf. [Kam89], [Kam94]).

In general, non-invariant structures on symmetric spaces are not very interesting, but there is one particular exception to this rule, namely the case of *quaternionic symmetric spaces*: we have three "twisted" complex structures which are not invariant individually but generate a quaternion algebra which, as a whole, is invariant. As is easily seen, the complexification of this situation gives rise to complex 5-graded Lie algebras which have one-dimensional extremal parts $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ generating a $\mathfrak{sl}_2$-subalgebra which contains the grading element. Under the correspondence with Jordan structures, these correspond to Freudenthal-Kantor triple systems which have been called *balanced* by N. Kamiya (cf. [Kam89]). On the other hand, J. Wolf has shown in [Wo65] that complexifications of (semisimple) quaternionic symmetric spaces are in one-to-one correspondence with complex simple Lie algebras, and thus we end up with a bijection between complex simple Lie algebras and complex simple balanced Freudenthal-Kantor triple systems. In particular, all complex exceptional Lie algebras can be constructed from balanced Freudenthal-Kantor triple systems. This has already been proved in a purely algebraic way by N. Kamiya ([Kam98]); the geometric interpretation of this correspondence given here is not used in that work.

The results presented in this note are all very easy to prove once one admits the correspondence between symmetric spaces and Lie triple systems (we recall the basic facts from [Lo69] in Chapter 1), and much of it might be already known to specialists although it seems to have been never completely published. But there are some deep and open problems for which we hope that this note will at least provide a sort of language in which they can be reasonably discussed: first of all, Freudenthal triple systems have been invented as a tool in understanding *exceptional geometries*, and here the most outstanding problem (which to the author seems to be still open) is a complete interpretation of "Freudenthal's magic square" (see various contributions in [Fr62]). Jordan algebras have also been frequently used in the description of exceptional geometries. Thus there is a strong evidence that there should be some profound link between Jordan theory and exceptional geometries, but its precise form is not clear at present. It seems likely that quaternionic symmetric spaces play an important role here. Secondly, a version of the homogeneous space G/G_0 mentioned above can, in the context of quaternionic symmetric spaces, be interpreted as the so-called *twistor space* which is a certain complex manifold associated to a quaternion-kähler manifold (cf. Section 4.3). What is its significance in our Jordan-theoretic approach ? We have the impression that this could be understood in the framework of *generalized projective geometries over skew-fields* – in fact, in case of a *commutative* field or ring the concept of a *generalized projective geometry* proposed by the author in [Be01] turned out to be a good framework for understanding (commutative) Jordan structures; but non-commutativity of the base field is a very serious obstacle when trying to generalize this approach. It seems indeed necessary to introduce a sort of twistor space in this context; we intend to return to this problem in subsequent work.

It is a great pleasure for me to dedicate this note to Professor Soji Kaneyuki at the occasion of his retirement from Sophia University, and I thank him and Professor Reiko Miyaoka very much for their great hospitality and for the kind invitation to the workshop "Lie Groups and the Theory of Manifolds" held in Tokyo, march 2001; during this conference I learned about the work [Kan01] and the work of N. Kamiya to whom I am grateful for making to me accessible his papers [Kam89, 92, 94, 98] which motivated me to write the present note.

1. Complex structures and polarizations on symmetric spaces

1.1. Lie triple systems and symmetric spaces. A *symmetric space* is a manifold with a complete torsionfree affine connection having covariantly constant curvature. Every connected symmetric space is obtained as a homogeneous space $M = G/H$, where G is a Lie group and H an open subgroup of the fixed point group G^σ of some non-trivial involutive automorphism σ of G. The point $o := eH$ is called the *base point* of M. If we denote by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the ± 1-eigenspace decomposition of the Lie algebra $\mathfrak{g}$ of G under the differential $\dot\sigma$ of σ, then the evaluation map

$$\mathfrak{q} \to T_oM, \quad X \mapsto X_o \tag{1.1}$$

is a bijection with kernel $\mathfrak{h}$ (where we assume G to act faithfully on M and consider $\mathfrak{g}$ as a Lie algebra of vector fields on M). We will consider (1.1) as an identification of $\mathfrak{q}$ with the tangent space T_oM. Since $\dot\sigma$ as an automorphism of order 2, the fixed space $\mathfrak{h}$ of $\dot\sigma$ is a subalgebra, and we have $[\mathfrak{h},\mathfrak{q}] \subset \mathfrak{q}$, $[\mathfrak{q},\mathfrak{q}] \subset \mathfrak{h}$, whence $[[\mathfrak{q},\mathfrak{q}],\mathfrak{q}] \subset \mathfrak{q}$. Under the identification (1.1), the trilinear map

$$\mathfrak{q} \times \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q}, \quad (X,Y,Z) \mapsto [X,Y,Z] := [[X,Y],Z] \tag{1.2}$$

corresponds (up to a sign) to the *curvature tensor (at o) of the canonical connection of M*, and therefore we also use the notation

$$R(X,Y)Z := -[X,Y,Z]. \tag{1.3}$$

Then the following identities are verified:

(Lt1) $[X,Y,Z] = -[Y,X,Z]$

(Lt2) $[X,Y,Z] + [Z,X,Y] + [Y,Z,X] = 0$

(Lt3) $R(X,Y)$ is a derivation of R, i.e.

$$R(X,Y)[U,V,W] = [R(X,Y)U,V,W] + [U,R(X,Y)V,W] + [U,V,R(X,Y)W]. \tag{1.4}$$

A (real or complex) vector space $\mathfrak{q}$ with a trilinear map $[\cdot,\cdot,\cdot]$ satisfying (Lt1) – (Lt3) is called a (real or complex) *Lie triple system* (Lts). Every Lts is obtained as a -1-eigenspace of an involution of a Lie algebra: the direct sum $\mathfrak{q} \oplus \mathrm{Der}(\mathfrak{q})$ of $\mathfrak{q}$ with its derivation algebra carries a natural Lie bracket, and $\mathfrak{q} \oplus R(\mathfrak{q} \otimes \mathfrak{q})$ is a subalgebra; for $\mathfrak{g}$ we may take any subalgebra between these two. The minimal choice $\mathfrak{g} := \mathfrak{g}(\mathfrak{q}) := \mathfrak{q} \oplus R(\mathfrak{q} \otimes \mathfrak{q})$ is called the *standard imbedding* of the Lts $\mathfrak{q}$.

The functor associating to a symmetric space with base point o its Lts $\mathfrak{q} = T_o M$ is called the *Lie functor*; it is an equivalence of categories between finite-dimensional real Lts and connected simply connected symmetric spaces with base point (see [KoNo69] or [Lo69]).

1.2. Twisted complex structures on Lie triple systems. Let $\mathfrak{q}$ be a real Lts and $J : \mathfrak{q} \to \mathfrak{q}$ a complex structure, i.e. an endomorphism such that $J^2 = -\mathrm{id}_{\mathfrak{q}}$. We say that J is *invariant* if the identity

$$[X,Y,JZ] = J[X,Y,Z] \tag{1.5}$$

holds. If this is the case, J gives rise to an invariant tensor field (an almost complex structure) on M, and we are in the situation considered in [Be00, Ch.3]. In particular it is shown there (loc. cit. Cor. V.1.12) that, if M is a *simple* symmetric space, then an invariant complex structure J is either *straight* (i.e. R is J-trilinear) or *twisted* which means that the identity $R(JX,Y) = -R(X,JY)$ holds; in view of (1.5), the latter condition is equivalent to the identity

$$J[X,Y,Z] = [JX,Y,Z] + [X,JY,Z] + [X,Y,JZ] \tag{1.6}$$

which means that J is a *derivation* of R. Let us call also in the general (not necessarily invariant) case a complex structure *twisted* if the condition (1.6) holds. Then e^{tJ} is an automorphism of R for all $t \in \mathbb{R}$; that is, J generates a circle group of automorphisms of $\mathbb{R}$. In particular, $J = e^{\frac{\pi}{2}J}$ itself is an automorphism of R. Thus we may think of J both as an element of the stabilizer group H and as an element of its Lie algebra $\mathfrak{h}$.

1.3. Twisted polarizations. We apply the complexification functor $\otimes_{\mathbb{R}}\mathbb{C}$ to the set-up from the preceding section: $(\mathfrak{q}_{\mathbb{C}}, R_{\mathbb{C}})$ is now a complex trilinear Lie triple system with imaginary unit denoted by i and complex conjugation with respect to $\mathfrak{q}$ denoted by τ. The complex linear prolongation of J is denoted by $J_{\mathbb{C}}$. Then $(iJ_{\mathbb{C}})^2 = \mathrm{id}_{\mathfrak{q}_{\mathbb{C}}}$, i.e. $I := iJ_{\mathbb{C}}$ is a *polarization* of $\mathfrak{q}_{\mathbb{C}}$. The eigenspace decomposition is denoted by

$$\mathfrak{q}_{\mathbb{C}} = \mathfrak{q}_+ \oplus \mathfrak{q}_-$$

with $\mathfrak{q}_\pm = \{X \pm IX \mid X \in \mathfrak{q}\}$. If J is twisted then I will also be *twisted* in the sense that

$$IR_{\mathbb{C}}(X,Y)Z = R_{\mathbb{C}}(IX,Y)Z + R_{\mathbb{C}}(X,IY)Z + R_{\mathbb{C}}(X,Y)IZ \tag{1.7}$$

since the derivations of $R_{\mathbb{C}}$ form a complex vector space. Note, however, that I will no longer be an automorphism since automorphisms do not form a complex vector space.

Now let us forget the construction of I via a complex structure. A *polarization* of a real Lts $(\mathfrak{q}, R)$ is a linear map $I : \mathfrak{q} \to \mathfrak{q}$ such that $I^2 = \mathrm{id}_{\mathfrak{q}}$; it will be called *twisted* if it is a derivation of R, i.e. if (1.7) holds for $R_{\mathbb{C}}$ replaced by R. We may think of I as an element of $\mathfrak{h}$ but in general not as an element of H.

1.4. Correspondence with five-graded Lie algebras. If I is a twisted polarization of a Lts $(\mathfrak{q}, R)$, we denote by $\mathfrak{q} = \mathfrak{q}_+ \oplus \mathfrak{q}_-$ its ± 1-eigenspace decomposition; then, since I is a derivation, we have the rule

$$[\mathfrak{q}_\alpha, \mathfrak{q}_\beta, \mathfrak{q}_\gamma] \subset \mathfrak{q}_{\alpha+\beta+\gamma} \tag{1.8}$$

for eigenspaces $\mathfrak{q}_\alpha$, $\mathfrak{q}_\beta$, $\mathfrak{q}_\gamma$. It implies that the spaces $\mathfrak{q}_\pm$ are abelian subtriples of $\mathfrak{q}$.

We let $\mathfrak{h} = R(\mathfrak{q} \otimes \mathfrak{q}) \subset \mathrm{Der}(\mathfrak{q})$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Then since $[I, R(X,Y)] = R(IX,Y) + R(X,IY)$, I normalizes $\mathfrak{h}$ and we have a Lie algebra-derivation $\mathrm{ad}(I) : \mathfrak{g} \to \mathfrak{g}$ which preserves $\mathfrak{q}$ and $\mathfrak{h}$. By the rule $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for eigenspaces of a Lie algebra-derivation, we have

$$[\mathfrak{q}_+, \mathfrak{q}_+] \subset \mathfrak{h}_2, \quad [\mathfrak{q}_-, \mathfrak{q}_-] \subset \mathfrak{h}_{-2}, \quad [\mathfrak{q}_+, \mathfrak{q}_-] \subset \mathfrak{h}_0, \tag{1.9}$$

and since $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{h}$, we deduce that $\mathfrak{h} = \mathfrak{h}_{-2} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_2$ is the direct sum of three eigenspaces of $\mathrm{ad}(I)$, and thus

$$\mathfrak{g} = (\mathfrak{q}_+ \oplus \mathfrak{q}_-) \oplus (\mathfrak{h}_0 \oplus \mathfrak{h}_{-2} \oplus \mathfrak{h}_2) \tag{1.10}$$

is a five-graded Lie algebra with $\mathfrak{g}_{\pm 1} = \mathfrak{q}_\pm$, $\mathfrak{g}_i = \mathfrak{h}_i$, $i = -2, 0, 2$.

Conversely, if $\mathfrak{g}$ is five-graded, then the characteristic map D of this grading, defined by $D|_{\mathfrak{g}_i} = i\,\mathrm{id}_{\mathfrak{g}_i}$, $i = -2, -1, \ldots, 2$, is a derivation, and it is a polarization on of the subspace $\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$ which is a sub-LTS of $\mathfrak{g}$; hence we are back in the set-up from the beginning. Summing up, we have a bijection between twisted polarized Lts and five-graded Lie algebras $\mathfrak{g}$ with "minimal" even part $\mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ (i.e. $\mathfrak{g}$ is generated by the odd part $\mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$).

As in the case of complex structures, we say that a polarization I on a Lts $\mathfrak{q}$ is *invariant* if it commutes with $\mathfrak{h} = R(\mathfrak{q} \otimes \mathfrak{q})$; in other words, we have $\mathfrak{h} = \mathfrak{h}_0$, and thus the structure I is invariant iff $\mathfrak{g}$ is actually 3-graded. This case is geometrically interesting because then I gives rise to an invariant tensor field which is actually integrable (see [Be00] and also the work of Kozai and Kaneyuki [KanKo85] on para-Hermitian symmetric spaces which are special cases of this situation), and it is algebraically interesting because 3-graded Lie algebras correspond to *Jordan pairs* in the sense of O. Loos [Lo75]; this correspondence will be explained and generalized in the following chapter.

2. Freudenthal-Kantor pairs and -triple systems

2.1. Freudenthal-Kantor pairs. Assume I is a twisted polarization on a real Lts $(\mathfrak{q}, R)$. We let $V := \mathfrak{q}$, $V_\pm := \mathfrak{q}_\pm$ and define trilinear maps

$$T_+ : V_+ \times V_- \times V_+ \to V_+, \quad (X,Y,Z) \mapsto T_+(X,Y,Z) := [X,Y,Z] = [[X,Y],Z],$$

$$T_- : V_- \times V_+ \times V_- \to V_-, \quad (X,Y,Z) \mapsto T_-(X,Y,Z) := [X,Y,Z] = [[X,Y],Z].$$

According to the commutator relations (1.8) this is well-defined. We let $T_\pm(X,Y)Z := T_\pm(X,Y,Z)$. Then we have, for $X, A, C \in V_+$, $Y, B \in V_-$,

$$\begin{aligned}
&T_+(X,Y)T_+(A,B,C) \\
&= \mathrm{ad}[X,Y] \cdot [A,B,C] \\
&= [[X,Y,A],B,C] + [A,[X,Y,B],C] + [A,B,[X,Y,C]] \\
&= [[X,Y,A],B,C] - [A,[Y,X,B],C] + [A,B,[X,Y,C]] \\
&= T_+(T_+(X,Y)A,B,C) - T_+(A,T_-(Y,X)B,C) + T_+(A,B,T_+(X,Y)C),
\end{aligned}$$

and similarly for T_-. Thus $(V_\pm, T_\pm)$ satisfies the defining identity

$$\begin{aligned} &T_+(X,Y)T_+(A,B,C) \\ &= T_+(T_+(X,Y)A,B,C) - T_+(A,T_-(Y,X)B,C) + T_+(A,B,T_+(X,Y)C) \end{aligned} \tag{J}$$

of a *generalized Jordan pair* (cf. [KanAs88]; there not pairs but triple systems are considered). Usual (linear) Jordan pairs are defined by adding the property of symmetry in the outer variables (cf. [Lo75]). However, in our setting the compositions $T_\pm$ will in general be non-symmetric in the outer variables; in fact, we have

$$\begin{aligned} T_+(X,Y,Z) - T_+(Z,Y,X) &= [X,Y,Z] - [Z,Y,X] \\ &= -([Y,X,Z] + [Z,Y,X]) = [X,Z,Y] = \mathrm{ad}[X,Z]Y, \end{aligned}$$

and $\mathrm{ad}[X,Z] \in \mathfrak{h}_2$ acts in general non-trivially from $\mathfrak{q}_-$ to $\mathfrak{q}_+$. Following notation introduced by N. Kamiya [Kam89], we let for $X, Z \in V_+$,

$$K_+(X,Z) := T_+(X,\cdot,Z) - T_+(Z,\cdot,X) : V_- \to V_+. \tag{2.1}$$

Then $K_+(X,Z) = \mathrm{ad}[X,Z]$, and similarly we define $K_-(X,Z) : V_+ \to V_-$ for $X, Z \in V_-$. We may see $K_\pm$ also as trilinear maps

$$K_\pm : \mathfrak{q}_\pm \times \mathfrak{q}_\pm \times \mathfrak{q}_\mp \to \mathfrak{q}_\pm, \quad (X,Y,Z) \mapsto K_\pm(X,Y)Z = [X,Y,Z].$$

Since $[\mathfrak{q}_+, \mathfrak{q}_+, \mathfrak{q}_+] \subset \mathfrak{h}_3 = 0$, the fact that $\mathrm{ad}[X,Z]$ is a derivation of $\mathfrak{g}$ gives us the following identity for $A, C \in V_+$, $X, Z, B \in V_-$,

$$\begin{aligned} K_-(X,Z)T_+(A,B,C) &= \mathrm{ad}[X,Z] \cdot [A,B,C] \\ &= [\mathrm{ad}[X,Z]A,B,C] + 0 + [A,B,\mathrm{ad}[X,Z]C] \\ &= [\mathrm{ad}[X,Z]A,B,C] - [B,A,\mathrm{ad}[X,Z]C] \\ &= K_-(K_-(X,Z)A,B)C - T_-(B,A)K_-(X,Z)C, \end{aligned}$$

and similarly for K_+ instead of K_- Thus we have

$$K_\mp(X,Z)T_\pm(A,B) = K_\mp(K_\mp(X,Z)A,B) - T_\mp(B,A)K_\mp(X,Z). \tag{F}$$

By definition, a *Freudenthal-Kantor pair* (over a field $\mathbb{K}$) is a pair (V_+, V_-) of $\mathbb{K}$-vector spaces with $\mathbb{K}$-trilinear maps $T_\pm : V_\pm \times V_\mp \times V_\pm \to V_\pm$ such that, putting $K_\pm(X,Y) = T_\pm(X,\cdot,Y) - T_\pm(Y,\cdot,X)$, the identities (J) and (F) hold. Thus we have seen that Lts with twisted polarization give rise to Freudenthal-Kantor pairs.

Conversely, given a Freudenthal-Kantor pair (V_+, V_-), we let $\mathfrak{q} := V_+ \oplus V_-$ and define a trilinear bracket R by $T_\pm$ on the parts $V_\pm \times V_\mp \times V_\pm$ and by $K_\pm$ on the parts $V_\pm \times V_\pm \times V_\mp$ and by extending by antisymmetry on $V_\mp \times V_\pm \times V_\pm$. Then (Lt1) holds by definintion of the bracket and (Lt2) holds since $K_\pm$ is defined via antisymmetrization from $T_\pm$; finally, (Lt3) holds in all possible cases by (J) and (F). Therefore $(\mathfrak{q}, R)$ is a Lie triple system, and the linear map I which is one on V_+ and minus one on V_- is a derivation since the commutator relations (1.8) hold by our definitions. Summing up, we have shown that there is a bijection between the following objects:

(1) Lie triple systems with twisted polarizations,
(2) 5-graded Lie algebras with minimal even part,
(3) Freudenthal-Kantor pairs.

These bijections are essentially known from the work of N. Kamiya (cf. [Kam94]); they contain as a special case bijections between the following objects which are studied in [Be00]:

(1a) Lie triple systems with invariant twisted polarization,

(2a) 3-graded Lie algebras with minimal even part,

(3a) (linear) Jordan pairs.

In all cases, there are natural notions of *homomorphisms*, and thus we get categories. It is then easily seen that the bijection between (1) and (3) is an equivalence of categories, whereas the category (2) will not be equivalent to (1) and (3) – the arguments are the same as those given in [Be00, Ch. III.3] for the equivalence of (1a) and (3a) which are not equivalent to (2a). Homomorphisms of (2) give always rise to homomorphisms of (1) and (3); the converse is false, but for instance, if $\mathfrak{q}$ is semisimple, then any homomorphism $\varphi : \mathfrak{q} \to \mathfrak{q}'$ can be extended to a homomorphism $\widetilde{\varphi} : \mathfrak{g}(\mathfrak{q}) \to \mathfrak{g}(\mathfrak{q}')$ of the standard imbeddings (this result is due to N. Jacobson, cf. [Be00, Th. V.1.9]). Since φ is compatible with polarizations I, I', it follows that then φ and $\widetilde{\varphi}$ are graded homomorphisms. The homomorphisms $\widetilde{\varphi}$ generalize, in the split setting, the "(H_1)-homomorphisms" from [Sa80, p. 84].

2.2. Freudenthal-Kantor triple systems. As shown in [Lo75], *Jordan triple systems* are the same as Jordan pairs with involution. In our more general situation this is explained as follows: assume that $(\mathfrak{q}, R, I)$ is a Lie triple system with twisted polarization I. An *antiautomorphism* of these data is an $\mathbb{R}$-linear automorphism τ of $(\mathfrak{q}, R)$ which anticommutes with I: $\tau I = -I\tau$. Then $\tau(\mathfrak{q}_\pm) = \mathfrak{q}_\mp$, that is, τ is determined by two bijections $\tau_\pm : V_\pm \to V_\mp$, and we write in matrix form

$$\tau = \begin{pmatrix} 0 & \tau_- \\ \tau_+ & 0 \end{pmatrix}.$$

On V_+ we define a trilinear map by

$$T(X,Y,Z) := T_+(X, \tau_+ Y, Z), \tag{2.2}$$

and on V_- we define a trilinear map by $\widetilde{T}(X,Y,Z) := T_-(X, \tau_- Y, Z)$. Then, since τ is an automorphism of R, τ_+ is an isomorphism from (V_+, T) onto $(V_-, \widetilde{T})$ and τ_- is an isomorphism from $(V_-, \widetilde{T})$ onto (V_+, T). Note that τ^2 is an automorphism commuting with I. We will only be interested in the following two cases:

(a) $\tau^2 = \mathrm{id}_V$ – then τ may be seen as a "para-real form for the paracomplex structure I", and (V_+, T) and $(V_-, \widetilde{T})$ are isomorphic to the fixed point space of τ;

(b) $\tau^2 = -\mathrm{id}_V$ – then τ is a complex structure anticommuting with I, and there is no direct geometric interpretation of (V_+, T) and $(V_-, \widetilde{T})$, but see the next chapter for its significance in the quaternionic case.

We let

$$K(X,Y)Z = T(X,Z,Y) - T(Y,Z,X) = K_+(X,Y)\tau_+ Z. \tag{2.3}$$

If $\tau^2 = \varepsilon\, \mathrm{id}_{\mathfrak{q}}$, $\varepsilon = \pm 1$, the identities (J) and (F) imply the following identities for T (we just have to replace T_+, K_+ by T, K and T_-, K_- by εT, εK):

$$\begin{aligned} &T(X,Y)T(A,B,C) = \\ &T(T(X,Y)A,B,C) - \varepsilon T(A, T(Y,X)B, C) + T(A,B,T(X,Y)C) \end{aligned} \tag{J}$$

$$\varepsilon K(X,Z)T(A,B) = K(K(X,Z)A,B) - T(B,A)K(X,Z). \tag{F}$$

By definition, $(\mathfrak{q}_+, T)$ is then called a *Freudenthal-Kantor triple system.* Comparing with the conventions in Kamiya's papers [Kam98] we see that our case $\varepsilon = 1$ corresponds to the negative sign there and vice versa.

3. Quaternionic symmetric spaces and balanced FK-triple systems

3.1. Quaternionic symmetric spaces. A *quaternionic symmetric space* is a symmetric space $M = G/H$ having an invariant distribution $(\mathbb{H}_p)_{p\in M}$ of *twisted quaternion algebras*; that is

(q1) the subalgebra $\mathbb{H}_p \subset \mathrm{End}(T_pM)$ is generated by three pairwise anticommuting twisted complex structures I_p, J_p, K_p,

(q2) invariance: for all $g \in G$, $g.\mathbb{H}_p = \mathbb{H}_{g.p}$.

By (q1), I_p, J_p and K_p generate a Lie algebra isomorphic to $\mathfrak{su}(2)$ which can be seen as a subalgebra of the Lie algebra $\mathfrak{g}_p$ of the stabilizer group G_p; therefore G_p contains a subgroup isomorphic to $\mathrm{Sp}(1) = \mathrm{SU}(2)$. By (q2), the algebra $\mathfrak{su}(2)$ and the group $\mathrm{SU}(2)$ just mentioned are normalized by G_p. Condition (q2) can in general *not* be replaced by requiring that the individual complex structures I_p, J_p, K_p be invariant – in fact, this would mean that the isotropy representation $H \to \mathrm{GL}(T_oM)$ were quaternionic; but this is never the case if we assume the symmetric space M to be semisimple (cf. [Be00, Cor. V.1.13]).

Semisimple quaternionic symmetric spaces M are classified as follows: first of all, one can show that, if M is semisimple, then M has actually to be simple (cf. [Bes87, 14.49]), and J. Wolf has shown that the compact quaternionic symmetric spaces (or, equivalently, their complexifications) are in one-to-one correspondence with simple complex Lie groups (cf. [W65], [Bes87, Section 14.54]). From these observations one gets the following classification of the compact spaces (cf. [Bes87]); we list also the rank of these spaces which turns out never to be bigger than four and their dimension which turns out to be closely related to the dimensions of certain (commutative) Jordan algebras or -triple systems – a fact that can only be understood after a much finer structure theory (cf. [Cl01]) (Sym denotes algebras of symmetric and Herm algebras of Hermitian matrices; $\mathbb{O}$ is the octonion algebra).

space $M = G/H$	dimension, rank r
$\mathbb{HP}^n = \mathrm{Sp}(n+1)/(\mathrm{Sp}(n)\times \mathrm{Sp}(1))$	$4n = \dim_{\mathbb{R}}(\mathbb{H}^n)$, $r = 1$
$\mathrm{Gr}_2(\mathbb{C}^{n+2}) = \mathrm{U}(n+2)/(\mathrm{U}(n)\times \mathrm{U}(2))$	$4n = \dim_{\mathbb{R}}(M(2,n;\mathbb{C}))$, $r = 2$
$\mathrm{Gr}_4(\mathbb{R}^{n+4}) = \mathrm{O}(n+4)/(\mathrm{O}(n)\times \mathrm{O}(4))$	$4n = \dim(M(4,n;\mathbb{R}))$, $r = 3$ $(n=3)$, $r = 4$ $(n \geq 4)$
$G_2/\mathrm{SO}(4)$	$8 = \dim(\mathbb{O}) = 2\dim(\mathbb{H})$, $r = 2$
$F_4/(\mathrm{Sp}(3)\times \mathrm{SU}(2))$	$28 = 4\dim(\mathrm{Sym}(3,\mathbb{R})) + 4$, $r = 4$
$E_6/(\mathrm{SU}(6)\times \mathrm{SU}(2))$	$40 = 4\dim(\mathrm{Herm}(3,\mathbb{C})) + 4$, $r = 4$
$E_7/(\mathrm{SO}(12)\times \mathrm{SU}(2))$	$64 = 4\dim(\mathrm{Herm}(3,\mathbb{H})) + 4$, $r = 4$
$E_8/(E_7\times \mathrm{SU}(2))$	$112 = 4\dim(\mathrm{Herm}(3,\mathbb{O})) + 4$, $r = 4$

The following low-dimensional isomorphisms occur: the space $\mathbb{RP}^4$ (third line, $n = 1$) is locally isomorphic to $\mathbb{HP}^1$ (first line, $n = 1$), and the space $\mathrm{Gr}_4(\mathbb{R}^6)$ (third line, $n = 2$) is locally isomorphic to $\mathrm{Gr}_2(\mathbb{C}^4)$ (second line, $n = 2$).

3.2. Invariant $\mathfrak{sl}_2$-algebras and split quaternionic symmectric spaces. We assume that $M = G/H$ is a quaternionic symmetric space. Then, by (q1), the associated Lts $(\mathfrak{q}, R)$ has three twisted complex structures, and therefore the complexified Lts $(\mathfrak{q}_{\mathbb{C}}, R_{\mathbb{C}})$ has three (pairwise anticommuting) twisted polarizations which generate a subalgebra isomorphic to $(\mathfrak{su}(2))_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sp}(1,\mathbb{C})$. We fix the usual basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.1}$$

of $\mathfrak{sl}(2,\mathbb{C})$, with commutator relations $[H,P]=2P$, $[H,Q]=-2Q$, $[P,Q]=-H$. The decomposition $\mathfrak{q}_{\mathbb{C}}=\mathfrak{q}_+\oplus\mathfrak{q}_-$ and the five-grading of $\mathfrak{g}=\mathfrak{q}\oplus\mathfrak{h}$ are now taken with respect to the twisted polarization given by the element $H\in\mathfrak{h}_{\mathbb{C}}$. By Weyl's theorem, all $\mathfrak{sl}(2,\mathbb{C})$-modules appearing in this context are completely reducible, and by standard $\mathfrak{sl}_2$-theory, the irreducible submodules are characterized by the weights of the action of H. Therefore $\mathfrak{q}_{\mathbb{C}}$ is a direct sum of two-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules, and because of the grading $\mathfrak{h}=\mathfrak{h}_{-2}\oplus\mathfrak{h}_0\oplus\mathfrak{h}_2$, only one-dimensional or three-dimensional irreducible submodules can appear in the decomposition of $\mathfrak{h}$, where H acts on the threedimensional submodules by the matrix

$$\begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}.$$

One such module is our fixed copy of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{h}$ itself. There can't be any other such module for it has to contain an eigenvector X of H with $[H,X]=2X$, but since (by (q2)) $\mathfrak{sl}(2,\mathbb{C})$ is normalized by $\mathfrak{h}$, it follows that then $X\in\mathfrak{sl}(2,\mathbb{C})$. Summing up, we have shown that in the decomposition (1.10) the spaces $\mathfrak{h}_2$ and $\mathfrak{h}_{-2}$ are one-dimensional, $\mathfrak{h}_2=\mathbb{C}P$, $\mathfrak{h}_{-2}=\mathbb{C}Q$ and

$$\mathfrak{h}_2\oplus\mathfrak{h}_{-2}\oplus\mathbb{C}H\cong\mathfrak{sl}(2,\mathbb{C}). \tag{3.2}$$

Now we forget about the origin of this situation by complexification of a quaternionic symmetric space and define: a *split quaternionic symmetric space* is a symmetric space $M=G/H$ having an invariant distribution $(\mathbb{A})_{p\in M}$ of *twisted split quaternion algebras*, that is

(sq1) the subalgebra $\mathbb{A}_p\subset\operatorname{End}(T_pM)$ is generated by two twisted polarisations H,F together with a twisted complex structure J satisfying the usual relations of the matrices $H,F=P+Q$ and $J=Q-P$ (defined by (3.1)) in the algebra $M(2,2;\mathbb{R})$,

(sq2) invariance: for all $g\in G$, $g.\mathbb{A}_p=\mathbb{A}_{g.p}$.

If we fix a base point $o\in M$ and a twisted polarization $H\in\mathbb{A}_o$, then the preceding arguments show that the components $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ of the associated five-graded Lie algebra $\mathfrak{g}$ are one-dimensional and satisfy (3.2), and in this way we get a bijection between such Lie algebras and split quaternionic symmetric spaces with base point and a fixed twisted polarization. Next we are going to describe the triple system-version of this concept.

3.3. Balanced Freudenthal-Kantor triple systems. We assume that $\mathfrak{q}=\mathfrak{q}_+\oplus\mathfrak{q}_-$ is a Lts with a split quaternionic structure; the fixed polarization is denoted by H and the one-dimensional extremal parts of the 5-grading of $\mathfrak{g}$ by $\mathfrak{g}_2=\mathbb{K}P$ and $\mathfrak{g}_{-2}=\mathbb{K}Q$, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. As explained in Section 2.1, $\mathfrak{q}$ carries the structure of a Freudenthal-Kantor pair with respect to the twisted polarization H. It has the special feature that, for all $X,Y\in\mathfrak{q}_+$, there is a scalar $\omega_+(X,Y)$ such that

$$[X,Y]=\omega_+(X,Y)P, \quad \text{i.e.} \quad K_+(X,Y)=\omega_+(X,Y)\operatorname{ad}(P), \tag{3.3}$$

and similarly for $\mathfrak{q}_-$. The (skew-symmetric) bilinear form $\omega_+:\mathfrak{q}_+\times\mathfrak{q}_+\to\mathbb{K}$ depends on the way we complete H to an $\mathfrak{sl}_2$-triple H,P,Q and therefore is only defined up to a scalar factor.

This ambiguity can be overcome by observing that our Freudenthal-Kantor pair in fact carries naturally the structure of a FK-triple system: by the commutator relations of $\mathfrak{sl}_2$, the maps $P:\mathfrak{q}_-\to\mathfrak{q}_+$ and $Q:\mathfrak{q}_+\to\mathfrak{q}_-$ are bijections such that $P\circ Q=-\operatorname{id}_{\mathfrak{q}_+}$, $Q\circ P=-\operatorname{id}_{\mathfrak{q}_-}$. Let us consider $Q:\mathfrak{q}_+\to\mathfrak{q}_-$ as an identification. Then H,P,Q are really represented by the matrices (3.1) on $\mathfrak{q}\cong\mathfrak{q}_+\oplus\mathfrak{q}_+$, and the matrix

$$\begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}=J \tag{3.4}$$

is an exponential of an element of this fixed $\mathfrak{sl}_2$-copy; it therefore is an Lts-automorphism which anticommutes with the twisted polarization given by the matrix H. Then $J|_{\mathfrak{q}_+} : \mathfrak{q}_+ \to \mathfrak{q}_-$ is nothing but $Q : \mathfrak{q}_+ \to \mathfrak{q}_-$, and therefore J turns the FK-pair $(\mathfrak{q}_+, \mathfrak{q}_-)$ into a FK-triple system $\mathfrak{q}_+$ with $\varepsilon = -1$ (Section 2.2). Comparing with (3.3) we see that there is a skew-symmetric bilinear form $\omega : \mathfrak{q}_+ \times \mathfrak{q}_+ \to \mathbb{C}$ such that for all $X, Y \in \mathfrak{q}_+$,

$$K(X,Y) = \omega(X,Y)\,\mathrm{id}, \tag{B}$$

and this condition is indeed independent of the way we complete H to an $\mathfrak{sl}_2$-triple. We call a Freudenthal-Kantor triple system with $\varepsilon = -1$ *balanced* if there is a non-zero skew-symmetric form ω such that condition (B) is satisfied.

Conversely, if (V, T) is a balanced Freudenthal-Kantor triple system, then it is immediately clear that the spaces $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ of the associated five-graded Lie algebra have to be one-dimensional, $\mathfrak{g}_2 = \mathbb{K}P$, $\mathfrak{g}_{-2} = \mathbb{K}Q$, with P, Q the matrices defined by (3.1) on $V \oplus V$, and they generate together with the polarization H an $\mathfrak{sl}_2$-algebra. Summing up, we have bijections between the following objects:

(1b) Lie triple systems with an invariant split quaternionic structure and a fixed H-element,
(2b) 5-graded Lie algebras with one-dimensional extremal parts $\mathfrak{g}_2, \mathfrak{g}_{-2}$ generating an $\mathfrak{sl}_2$-algebra which contains the grading element,
(3b) balanced Freudenthal-Kantor triple systems.

By the remarks from the preceding chapter, the bijection between (1b) and (3b) is in fact an equivalence of categories. Note that in this category we do not have direct products since the direct sum of the $\mathfrak{g}_{\pm 2}$-components would lead to higher dimensional eigenspaces; this explains at least partially that semisimple objects in this category will actually be simple (cf. the corresponding fact for quaternionic symmetric spaces mentioned above).

Recall (Section 2.1) that in the semisimple case, homomorphisms $\varphi : \mathfrak{q} \to \mathfrak{q}'$ of triple systems always extend to homomorphisms $\widetilde{\varphi} : \mathfrak{g} \to \mathfrak{g}'$ of the standard imbedding; therefore homomorphisms are in this case homomorphisms of graded Lie algebras. In particular, $\widetilde{\varphi}(\mathfrak{g}_{\pm 2}) \subset \mathfrak{g}'_{\pm 2}$. If $\widetilde{\varphi}$ is not zero, it has to be injective since the algebras in question are simple, and since the spaces $\mathfrak{g}_{\pm 2}$ are one-dimensional, restriction of a homomorphism gives rise to a bijection of the corresponding $\mathfrak{sl}_2$-algebras. In particular, we deduce that $\widetilde{\varphi}(H) = H'$, i.e. the grading elements are mapped onto each other. Thus in this case homomorphisms are "H_2-homomorphisms" in the sense of [Sa80, p. 84].

For a structure theory of balanced FK-triple systems see the work of N. Kamiya [Kam89] and references given there; in particular, N. Kamiya proves equivalence with the earlier concept of Freudenthal triple systems used by Meyberg [Mey68] in which the four-linear form

$$\mu : \mathfrak{q}_- \times \mathfrak{q}_+ \times \mathfrak{q}_+ \times \mathfrak{q}_+ \to \mathbb{C}, \quad (X, U, V, W) \mapsto [[[X,U],V],W] = \mu(X,U,V,W)P,$$

i.e.

$$\mu(X,U,V,W) = \omega(T(X,U)V, W),$$

plays an important rôle. It would be interesting to relate this structure theory directly to the geometry of quaternionic symmetric spaces. In the following section we propose a concept of *one-parameter subspaces* which might be useful in such an attempt.

3.4. One-parameter subspaces. A *one-parameter subspace* of a symmetric space M is just a (non-trivial) homomomorphism of the one-dimensional symmetric space $\mathbb{R}$ into M. One can define similarly one-parameter subspaces of symmetric spaces corresponding to commutative Jordan structures (cf. [Be00, Ch. X.5]). Now assume M is a quaternionic symmetric space. The smallest possible dimension of M is four, and according to the table given in Section 3.1, there are two *different* equivalence classes of compact quaternionic symmetric spaces of dimension four:

(a) $\mathbb{HP}^1 = \mathrm{Sp}(2)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$; it is isomorphic to the space $\mathbb{RP}^4 = \mathrm{O}(5)/(\mathrm{O}(4) \times \mathrm{O}(1))$ (third line, $n = 1$). Its split version is

$$\begin{aligned} \mathrm{Sp}(2,\mathbb{R})/(\mathrm{Sp}(1,\mathbb{R}) \times \mathrm{Sp}(1,\mathbb{R})) &= \mathrm{Sp}(2,\mathbb{R})/(\mathrm{Sl}(2,\mathbb{R}) \times \mathrm{Sl}(2,\mathbb{R})) \\ &\cong \mathrm{SO}(3,2)/(\mathrm{Sl}(2,\mathbb{R}) \times \mathrm{Sl}(2,\mathbb{R}). \end{aligned} \tag{A}$$

(b) $M = \mathbb{CP}^2 = \mathrm{U}(3)/(\mathrm{U}(2) \times \mathrm{U}(1))$. Its split version is

$$M = \mathrm{GL}(3,\mathbb{R})/(\mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(1,\mathbb{R})). \tag{B}$$

A *one-parameter subspace* of a quaternionic symmetric space M is a non-trivial homomorphism of a four-dimensional quaternionic symmetric space L into M (where "homomorphism" refers to the category defined in Section 3.3). The one-parameter subspace is called *compact* if L is compact and *non-compact* if L is of non-compact type (recall that the distinction of these two cases must already be made in the case of commutative Jordan structures, cf. [Be00, X.5.1]), and it is called *of type (a)*, resp. *of type (b)*, if L is isomorphic to $\mathbb{HP}^1$, resp. to $\mathbb{CP}^2$ or to its non-compact dual. Similarly, one-parameter subspaces of type (a) and (b) of split quaternionic symmetric spaces are defined.

In [Wa01, Section 5], using classification results, a fairly detailed structure theory with respect to one-parameter subspaces (in the above sense) is given. The case C_m plays a rather special rôle: in this case there exist one-parameter subspaces of type (a), but not of type (b); in all other cases, there are one-parameter subspaces of type (a). Then the Lie algebra $\mathfrak{g}$ of G is turned into an $\mathfrak{sl}(3,\mathbb{K})$-module; the decomposition into irreducible submodules is given in loc. cit., Proposition 2, and the H_0-orbits (where H_0 is the centralizer of the grading element) in $\mathfrak{q}_\pm$ are studied. In [Cl01], a system of *strongly orthogonal roots* is constructed in a similar way as in Harish Chandra's theory of Hermitian symmetric spaces; it is shown that the rank of such a system is at most four. It would be interesting to interprete this construction in terms of one-parameter subspaces.

4. Integrability and consequences

4.1. Integrability of invariant twisted polarizations. It is well-known that *invariant* almost complex structures or polarizations on symmetric spaces are integrable. Let us describe briefly the the consequences of integrability in case I is an invariant twisted polarization on the symmetric space M (cf. [Be00]; for the special case of a para-Hermitian symmetric space see also the work of S. Kaneyuki): through a fixed base point $o \in M$ there are maximal integral submanifolds M_+ and M_- integrating the eigenspaces with eigenvalues $+1$, respectively -1, of I. The action $G \times M \to M$ preserves the foliation and hence gives rise to a "local" action of G on M_+ and on M_-. Since the subspaces $\mathfrak{q}_\pm$ are abelian subalgebras (cf. Section 1.4), M_+ and M_- carry a natural vector space structure, isomorphic to the one of $\mathfrak{q}_+$ and $\mathfrak{q}_-$, respectively. Thus M has a natural chart, called *Jordan coordinates* in [Be00]. Again using integrability, it is seen that the local action of G on $M_\pm$ is actually algebraic, thus allowing to globalize everything and to define the *conformal completion* of $M_\pm$ and of M by

$$X_\pm := G/P_\pm, \qquad M \subset X_+ \times X_-, \tag{4.1}$$

where $P_\pm$ are subgroups with Lie algebras $\mathfrak{p}_\pm = \mathfrak{g}_0 \oplus \mathfrak{q}_\mp$, respectively. It is remarkable that in this setting, everything can be expressed by algebraic formulas only involving the associated Jordan pair defined in Section 2.1. It then becomes clear that the pair of spaces (X, X') behaves very much like the pair given by a projective space and its dual space (see [Be00, Ch. VI]). This observation finally led to the approach [Be01] which formalizes the essential features for the case of a general base field or -ring, where the spaces are possibly infinite dimensional.

4.2. Integrability in the general case. Now assume that I is a twisted polarization on the Lts $\mathfrak{q}$, not necessarily invariant. Then there is no canonical extension to a polarization $(I_p)_{p\in M}$ on the associated symmetric space $M = G/H$, and therefore it makes no sense to ask wether I is integrable or not. However, this question does make sense if we pass to a bigger space: let

$$\mathfrak{g}_0 \oplus (\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2)$$

be the graduation (1.10) of $\mathfrak{g}$ associated to I and denote by G_0 the centralizer in G of the grading element $I \in \mathfrak{g}_0$. Let $\widetilde{M} := G/G_0$; then the tangent space at the base point of the homogeneous space $\widetilde{M}$ can be written

$$\mathfrak{m} = \mathfrak{m}_+ \oplus \mathfrak{m}_-, \quad \mathfrak{m}_\pm = \mathfrak{g}_{\pm 1} \oplus \mathfrak{g}_{\pm 2}, \tag{4.2}$$

and the spaces $\mathfrak{m}_\pm$ are subalgebras of $\mathfrak{g}$ which are normalized by $\mathfrak{g}_0$. Thus they define a G-invariant polarization of $\widetilde{M}$ (however, the map which is ± 1 on the subspaces comes not from the characteristic element of the grading !). It follows from results in [Kan92] and [DHKN99] that this polarization is indeed integrable. As has been observed by S. Kaneyuki [Kan01], one can then define, as in the 3-graded case described above, a "conformal completion" $\widetilde{M} \subset (X_+ \times X_-)$.

One may ask now which parts of the theory from [Be00] carry over to this more general situation, in particular, to what extent the corresponding geometry can be described by the associated Freudenthal-Kantor triple system. Possibly the algebra structure on $\mathfrak{m}$ given by

$$\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}, \quad (X, Y) \mapsto \mathrm{proj}_{\mathfrak{m}}[X, Y]$$

plays also an important rôle here; in [Kam94] it is shown that it defines a *generalized structurable algebra*, cf. Th. 8 loc. cit. In a second step, one could try to generalize the algebraic approach from [Be01] to this situation. Here the balanced case will be particularly interesting.

4.3. The balanced case and the the twistor space of a quaternionic symmetric space. In the balanced case, the spaces $\mathfrak{q}_{\pm 2}$ are one-dimensional, and hence the projection

$$\widetilde{M} \to M, \quad gG_0 \mapsto gH$$

has two-dimensional fibers. It is the split analog of the well-known *twistor space of a quaternionic symmetric space* $M = G/(H_1 \times \mathrm{SU}(2))$ which can be realized as the map

$$Z := G/(H_1 \times S^1) \to M, \quad g(H_1 S^1) \mapsto g(H_1\, \mathrm{SU}(2)).$$

(This is a special case of a general construction associating a twistor space to a quaternion-kähler manifold, cf. [Bes87, ch. 14].) The fibers of this map are isomorphic to a 2-sphere S^2, and the space Z admits a G-invariant almost complex structure since the group S^1 belongs to the center of the stabilizer group. This almost complex structure is actually integrable (cf. [Bes87] for the general case of quaternion-kähler manifold).

We conjecture that the analog of the approach [Be01] in this situation should be a theory of *generalized projective geometries over skew-fields.* However, for the moment it is not clear at all how a correct formalization of such a concept should look like. The commutativity of the base field or -ring is essential in the approach [Be01]; if we drop this assumption, then we certainly need an additional structure comparable to a "twistor space" or to its split analog.

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