GRADED SETS, GRADED GROUPS, AND CLIFFORD ALGEBRAS

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Abstract. We define a general notion of *centrally* Γ-*graded sets and groups* and of their *graded products*, and prove some basic results about the corresponding categories: most importantly, they form *braided monoidal categories*. Here, Γ is an arbitrary (generalized) ring. The case $\Gamma = \mathbb{Z}/2\mathbb{Z}$ is studied in detail: it is related to Clifford algebras and their discrete Clifford groups (also called Salingaros Vee groups).

Dedicated to the memory of my collegue Lionel Bérard-Bergery (1945 – 2019)

Introduction

Let $(\Gamma, +)$ be an abelian group (our main emphasis will be on the case $\Gamma = \mathbb{Z}/2\mathbb{Z}$). Generally, Γ-*graded* structures play an important rôle in mathematics. In this work we study a class of Γ-gradings, which, when combined with what we call a *central* Γ-*action*, has very interesting and pleasant properties.

0.1. **Additively and multiplicatively graded sets.** A first point in the present work is to distinguish between *additively*, and *multiplicatively* graded structures:

Definition 0.1. *Let X and M be sets.*

(1) An additive Γ-grading on M *is a* disjoint union structure, *written*

$$M = \sqcup_{\gamma \in \Gamma} M_\gamma, \ or \ M = \coprod_{\gamma \in \Gamma} M_\gamma,$$

i.e., M is the coproduct *of a family of sets* $(M_\gamma)_{\gamma \in \Gamma}$. *We allow some of the M_γ to be empty. Thus, an additive grading is equivalent to a* degree-map,

$$d : M \to \Gamma, \quad such \ that \quad M_\gamma = d^{-1}(\gamma).$$

(2) A multiplicative Γ-grading on X *is a* direct product structure *of the form*

$$X = \times_{\gamma \in \Gamma} X_\gamma, \ or \ X = \prod_{\gamma \in \Gamma} X_\gamma.$$

If M is additively graded, then the space $X = K^M$ of functions from M to a set K is multiplicatively graded (a function $f : M \to K$ corresponds to the family $(f|_{M_\gamma})_{\gamma \in \Gamma}$); in particular, taking $K = \{0, 1\}$, we see that the power set $\mathcal{P}(M)$ of M is multiplicatively graded: multiplicative gradings often arise as "exponentials" of additive ones. This furnishes important links between both types of gradings.

0.2. **Graded groups, and Clifford groups.** In this work, we focus on *additively* graded sets, and in particular on *(additively) graded groups or monoids*:

Definition 0.2. *A Γ-graded monoid is a monoid G which is additively Γ-graded, such that the grading map $d : G \to (\Gamma, +)$ is a morphism of monoids. It is called* centrally graded *given a morphism $Z : (\Gamma, +) \to G$, $x \mapsto Z(x) =: Z^x$ such that:*

(1) *Z takes values in the center of G: for all $x \in \Gamma$ and $g \in G$: $Z^x g = g Z^x$,*
(2) *$d \circ Z = 0$, i.e., $\forall x \in \Gamma : d(Z^x) = 0$.*

When $\Gamma = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$, then the element $Z = Z(\bar{1})$ is called the *grading element*, and the power Z^x is the same as $Z(x)$. A prototype of graded group G then is what in certain texts (e.g., [L]) is called the *(discrete) Clifford group*, and elsewhere *Salingaros Vee-groups*, (cf. [S84, S82, A]): the canonical basis elements $e_1, \ldots, e_n$ of the Clifford algebra $\mathrm{Cl}_{p,q}(\mathbb{K})$ (see Appendix A for notation) generate a group $G = Q_{p,q}$ of invertible elements in the Clifford algebra. This group is Γ-graded in the sense defined above: the grading morphism d is the unique morphism $d : Q_{p,q} \to \mathbb{Z}/2\mathbb{Z}$ such that $d(e_i) = \bar{1}$ (odd) for all i, and the grading element is $Z = -1$ (since $\langle e_i, e_j \rangle = 0$ for $i \neq j$, we have $e_i e_j = -e_j e_i$, so the group G contains the element -1, which clearly is central in G). The underlying philosophy is to replace the central element -1 from the Clifford case by an "abstract central element". Thus the approach presented here will give an abstract, group-theoretic presentation of the discrete Clifford group. Such a group-theoretic approach has already been advocated by the above quoted authors, but here we will go much further by putting it into a general categorical approach, as follows.

0.3. **The braided monoidal category of graded sets.** A key feature in the theory of Clifford algebras is the relation (see, e.g., [BtD], Prop. I.6.6)

$$(0.1) \qquad\qquad \mathrm{Cl}(U \oplus W) = \mathrm{Cl}(U) \widehat{\otimes} \mathrm{Cl}(W)$$

expressing the Clifford algebra of a quadratic space $U \oplus W$ by the *graded tensor product* of the corresponding Clifford algebras. When U, W carry the zero quadratic form, we get the corresponding rules for *Grassmann algebras*. Our approach is designed to furnish an analog of this for general (additvely) *centrally Γ-graded sets*:

Definition 0.3. *A Γ-graded set, with grading map $d : M \to \Gamma$, is called* centrally Γ-graded *if it is equipped with a (left) Γ-action*

$$z : \Gamma \times M \to M, \quad (x, m) \mapsto z(x, m) = x.m$$

preserving the grading, i.e., such that $\forall x \in \Gamma, \forall m \in M : d(x.m) = d(m)$.

A centrally graded monoid G is a centrally graded set: just let $z(x, m) = Z^x m$. First of all, we define the graded product of sets:

Theorem 0.4. *Assume $d_i : M_i \to \Gamma$, $i = 1, 2$, are two centrally Γ-graded sets. Let M be the quotient of $M_1 \times M_2$ for the equivalence relation defined by $\forall x \in \Gamma$: $(x.m_1, m_2) \sim (m_1, x.m_2)$. Then M, together with grading map and Γ-action*

$$d([m_1, m_2]) = d(m_1) + d(m_2), \quad x.[m_1, m_2] = [x.m_1, m_2] = [m_1, x.m_2],$$

is again a centrally Γ-graded set, which we denote by $M = M_1 \times_z M_2$. The usual set-theoretic identification $(M_1 \times M_2) \times M_3 = M_1 \times (M_2 \times M_3)$ induces a natural isomorphism of centrally Γ-graded sets

$$(M_1 \times_z M_2) \times_z M_3 = M_1 \times_z (M_2 \times_z M_3).$$

The set $E = \Gamma$ with $d(m) = 0$, $x.m = x + m$, is a "unit" : $E \widehat{\times}_z M = M = M \times_z E$. Summing up, centrally Γ-graded sets together with $\times_z$ form a monoidal category.

Now, the crucial point about graded tensor products is that they are associative, but *not commutative*: this is encoded by the *braiding map* of graded tensor products, defined for homogeneous elements v, w of degree $|v|, |w|$ by

$$(0.2) \qquad \beta : V \widehat{\otimes} W \to W \widehat{\otimes} V, \quad v \otimes w \mapsto (-1)^{|v| \cdot |w|} w \otimes v.$$

Likewise, in our setting, braidings can be chosen as additional structure: first of all, there is the "usual", or "standard" braiding, $(x_1, x_2) \mapsto (x_2, x_1)$, which leads us back to usual set-theory. But now assume that Γ is also a *generalized ring*, meaning that it carries a bi-additive "product" map $\Gamma^2 \to \Gamma$, $(a, b) \mapsto ab$.

Definition 0.5. *Let M_i, $i = 1, 2$, be two centrally Γ-graded sets. Then the map*

$$\beta = \beta_{M_1, M_2} : M_1 \times_z M_2 \to M_2 \times_z M_1, \quad [x_1, x_2] \mapsto z\big(d(x_2) \cdot d(x_1), [x_2, x_1]\big)$$

is well-defined, called the braiding *given by the generalized ring Γ.*

Theorem 0.6. *For any generalized ring, the braiding maps are isomorphisms of centrally graded sets and define a structure of (strict) braided monoidal category of centrally Γ-graded sets. If the generalized ring product is* skew, *meaning that $ab + ba = 0$ for all $a, b \in \Gamma$, then this structure is* symmetric, *i.e., we always have*

$$\beta_{M_1, M_2}^{-1} = \beta_{M_2, M_1}.$$

In the symmetric case, there is a simple formula, in terms of *permutation inversions*, for the action of the symmetric group on iterated products (Theorem 2.1). Next, braiding maps are used to define *graded products of graded groups*:

Theorem 0.7. *We fix a generalized ring structure on Γ.*

(1) For every centrally Γ-graded group G there is a braided dual group (b-dual) *given by $G^\vee = G$, with same grading and central action as G, and product*

$$x \bullet y = Z^{d(y)d(x)} \, xy.$$

(2) There is a group structure, denoted by $G_1 \widehat{\times}_Z G_2$, on the product $G_1 \times_z G_2$ of two centrally Γ-graded groups G_i, $i = 1, 2$, given by the group law

$$[g_1, g_2] \cdot [h_1, h_2] = Z^{d(h_1)d(g_2)} [g_1 h_1, g_2 h_2].$$

(3) Iteration of the preceding construction is associative: the natural map $(G_1 \widehat{\times}_Z G_2) \widehat{\times}_Z G_3 \cong G_1 \widehat{\times}_Z (G_2 \widehat{\times}_Z G_3)$ is a group isomorphism.

(4) When Γ is skew, then the braiding map β_{G_1, G_2} is an isomorphism from $G_1 \widehat{\times}_Z G_2$ onto $G_2 \widehat{\times}_Z G_1$.

Dually to the result mentioned above, there is also a simple formula for the product of n elements in $G_1 \widehat{\times}_Z G_2$ (Remark 2.4) and in $G^\vee$ (Remark 2.2).

0.4. **The discrete Clifford category.** Turning to concrete examples, we have already mentioned that (discrete) Clifford groups provide non-trivial examples illustrating the preceding facts. More abstractly, given a generalized ring Γ (which for simplicity we assume to be skew), we define a sequence of groups: let $Q_0 := (\Gamma, +)$,

$$Q_{1,0} := \Gamma^2 = \Gamma \times \Gamma, \quad Q_{0,1} := (Q_{1,0})^\vee, \ldots \quad Q_{p,q} := \widehat{\times}_z^p Q_{1,0} \widehat{\times}_z (\widehat{\times}_z^q Q_{0,1}),$$

where Γ^2 is an abelian group, but with non-trivial grading and action (see Example 2.1). When $\Gamma = \mathbb{Z}/2\mathbb{Z}$, then this sequence is exactly the sequence of discrete Clifford groups. In this case, the first few groups are

$Q_0 = (\Gamma, +) = C_2 = \mathbb{Z}/2\mathbb{Z}$,
$Q_{1,0} = V = C_2 \times C_2$ (the Klein four-group),
$Q_{0,1} = C = C_4$ (the cyclic group of order 4),
$Q_{2,0} = Q_{1,0} \widehat{\times}_Z Q_{1,0} = D = D_4$ (the dihedral group of symmetries of the square),
$Q_{0,2} = Q_{0,1} \widehat{\times}_Z Q_{0,1} = Q$ (the quaternion group of order 8).,
$Q_{1,1} \cong D_4$ (but with a different grading than above).

We investigate this *discrete Clifford category* in some detail in Section 3. Most facts are, essentially, known from the theory of Clifford algebras: Salingaros denotes these groups by $G_{p,q}$ and calls them *Vee-groups* (see [S84, S82], see also references given in [A, AM]). In the present work, we push the group theoretic approch given by these authors much further, by putting it in the large framework of graded monoidal categories, allowing much more general rings than $\mathbb{Z}/2\mathbb{Z}$. For instance, when $\Gamma = \mathbb{R}^2$ with its canonical skew-symmetric product $x \cdot y = (x_1 y_2 - x_2 y_1, 0)$, we get a sequence of groups of Heisenberg-type, called the *Heisenberg category*.

0.5. **Group algebra, and Clifford algebra.** The group algebra $\mathbb{K}[G]$ of the groups $G = Q_{p,q}$ is a super-algebra which decomposes as a direct sum of two ideals $\mathbb{K}[G]^+$ (which in turn is a group algebra of an abelian group), and $\mathbb{K}[G]^-$, which is precisely the Clifford algebra (Theorem 4.2 and 4.1). More generally, when $G = G_1 \widehat{\times}_Z G_2$, then $\mathbb{K}[G]^+$ is the ordinary (ungraded) tensor product $\mathbb{K}[G_1] \otimes \mathbb{K}[G_2]$ of group algebras, and $\mathbb{K}[G]^-$ is the *graded* tensor product (Theorem 4.1).

0.6. **Concluding remarks.** It seems as if the category of centrally $\mathbb{Z}/2\mathbb{Z}$-graded sets supports a good deal of what can be done on ordinary, ungraded sets. One may ask if this carries as far as differential calculus and analysis – indeed, the joint paper with J. Haut, [BeH] proposes a setting of "categorical differential calculus" which seems to be suitable for an adaptation to a "graded", or "super", framework. The important challenge is to understand if, and how, such a calculus were related to other known categorical approaches ([Mol, Sa, Sch]). We hope to investigate this question in future work.

Another aspect, already present in, e.g., [A, AM, L, S84], becomes clearified in the present setting, namely, the relation with the theory of *central extensions of groups*: indeed, all our groups can be viewed as central extensions, and as such, can be described by certain *cocycles* (Section 3.3). It becomes apparent that these cocycles are all related to algebra and combinatorics of *totally ordered (finite) index sets*, in particular, via *inversions* (*Fehlstand*, in German).

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Je souhaite dédier ce travail à la mémoire de mon collègue Lionel Bérard-Bergery (1945 – 2019). Les discussions que nous avions, autour de sujets de géométrie, où les graduations étaient toujours présentes, m'ont été une source inépuisable d'inspiration dans ce domaine. Bien que Lionel n'ait pas eu le temps de formaliser et de finaliser ces idées, j'espère qu'elles verront le jour, tôt ou tard, et que ce travail y contribuera.

1. Γ-GRADED SETS AND MONOIDS

In this section, $(\Gamma, +)$ is an abelian group.

1.1. **Additively and multiplicatively graded sets, and grading automorphism.** The definition of *(additively)* Γ*-graded sets and monoids* has been given above (Def. 0.1 and 0.2). As explained in the introduction, we'll focus on *additively* graded structures, and simply call them "graded". Let's turn them into a category. We have the choice between a "strong" or "weak" definition of morphisms (preserving, or permuting, the grading):

Definition 1.1. *Let X, X' be sets and $(\Gamma, +)$ an abelian group. A* morphism of additively Γ-graded sets $(X, d), (X', d')$*, of degree $d(f) \in \Gamma$, is a map $f : X \to X'$ such that*

$$\forall x \in X : \qquad d(f(x)) = d(f) + d(x).$$

When $d(f) = 0$, we say that f is a grading-preserving morphism.

A morphism of centrally Γ-graded sets *is a grading-preserving morphism commuting with the Γ-action ($f(\gamma.x) = \gamma.f(x)$, for all $x \in M$ and $\gamma \in \Gamma$).*

A morphism of centrally Γ-graded monoids or groups *is a morphism of groups or monoids satisfying the preceding condition.*

Definition 1.2. *On every centrally Γ-graded set (M, d, z), we define the* grading automorphism *by*

$$\alpha : M \to M, \quad x \mapsto \alpha(x) = d(x).x = z(d(x), x).$$

Clearly, α is a grading-preserving morphism. Its inverse is $\alpha^{-1}(x) = z(-d(x), x)$.

Definition 1.3. *We say that $(M, -d)$ is the* negative grading *of (M, d).*

Lemma 1.4. *If (G, d, Z) is a centrally Γ-graded group, then $\alpha : G \to G$, $g \mapsto Z^{d(g)}g$ is an automorphism of graded groups (again called* grading automorphism*).*

Proof. $\alpha(gh) = Z^{d(gh)}gh = Z^{d(g)}Z^{dh}gh = Z^{d(g)}gZ^{d(h)}h = \alpha(g)\alpha(h)$, $\alpha(e) = e$. $\square$

When $Z : \Gamma \to G$ is injective, then the grading can be recovered from α via $G_\gamma = \{x \in G \mid \forall u \in \Gamma : ux = \gamma x\}$ (a kind of "pre-eigenspace"). As already noticed, examples of centrally Γ-graded sets arise from centrally graded groups:

Example 1.1. Let $M := \mathcal{P}(\mathsf{n})$ the power set of $\mathsf{n} = \{1, \dots, n\}$. It is an abelian group for the symmetric difference $A\Delta B$ of sets, and from elementary set theory we have a group morphism

$$d : \mathcal{P}(\mathsf{n}) \to \Gamma := \mathbb{Z}/2\mathbb{Z}, \quad A \mapsto |A| \mod 2.$$

So $M_0 = \mathcal{P}(\mathsf{n})_0$ is the subset of sets having even cardinality, and $M_1 = \mathcal{P}(\mathsf{n})_1$ the one of subsets having odd cardinality. A central action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{P}(\mathsf{n})$ is obtained by choosing some set $B \subset \mathsf{n}$ of even cardinality and letting $z(\overline{0}, A) = A$, $z(\overline{1}, A,) = A\Delta B,$. We have $\alpha(A) = A$ if A is even, and $\alpha(A) = A\Delta B$ if A is odd.

Example 1.2. All dihedral groups D_n are $\mathbb{Z}/2\mathbb{Z}$-graded, via the quotient morphism $D_n \to D_n/C_n = C_2$ (where C_n is the cyclic group of order n). But only for n even they are *centrally* $\mathbb{Z}/2\mathbb{Z}$-graded: only in this case the center is non-trivial, $Z(D_{2n}) = \{\pm 1\} \cong C_2$, so that $Z : \mathbb{Z}/2\mathbb{Z} \to D_{2n}$ can be taken to be $Z(\overline{0}) = 1, Z(\overline{1}) = -1$. Note that the groups D_{4n} admit several normal subgroups of index 2, containing the element -1, so there are several different central gradings on D_{4n}.

Example 1.3. The quaternion group Q is centrally $\mathbb{Z}/2\mathbb{Z}$-graded: similarly to the case of D_4, its center ± 1 is isomorphic to C_2, and it has several normal subgroups of index 2 containing the element -1. In contrast to the D_4-case, the different normal subgroups in question are conjugate to each other under outer automorphisms.

Remark 1.1 (Central extensions). The quotient (orbit set) of the Γ-action, $M^0 := M/\Gamma$, is again Γ-graded: the map d passes to the quotient $\overline{d} : M^0 \to \Gamma$. If the action of Γ is free, it will sometimes be useful to choose a set-theoretic section of the projection $M \to M^0$, and thus to identify M with $M^0 \times \Gamma$. In particular, the setting of centrally graded groups self-dual in the sense that, reversing arrows, d and Z change their roles, and we have exact sequences of groups

$$\begin{array}{ccccc} G_0 & \to & G & \to & \Gamma, \\ \Gamma & \to & G & \to & G^0 := G/Z^\Gamma, \end{array}$$

the second of them realizing G as *central extension* of G^0 by Γ. Choosing a set-map $G^0 \to G$, section to the projection $G \to G^0$, the set G can be identified with $G^0 \times \Gamma$, with group law expressed via a map $\tau : G^0 \times G^0 \to \Gamma$ (with G^0 written additively),

$$(h, u) \cdot (h', u') = (hh', u + u' + \tau(h, h')).$$

Associativity is equivalent to saying that τ satisfies the *cocycle relation*

$$\tau(h, h') + \tau(hh', h'') = \tau(h, h'h'') + \tau(h', h'').$$

(cf. Section 3.3). Note that d induces a morphism $d : G^0 \to \Gamma$, but the induced $[Z] : \Gamma \to G^0$ would be trivial.

1.2. **Monoidal structure on the category of centrally Γ-graded sets.** We define the product $M = M_1 \times_z M_2$ of two centrally Γ-graded sets as explained in Theorem 0.4. By routine check, M is again centrally Γ-graded. (Note that the equivalence relation is the orbit relation of the action of the "antidiagonal subgroup" $\{(\gamma, -\gamma) \mid \gamma \in \Gamma\}$ on $M_1 \times M_2$, and the grading passes to the quotient.) To complete the proof of Theorem 0.4, we check associativity

$$(M_1 \times_z M_2) \times_z M_3 = M_1 \times_z (M_2 \times_z M_3).$$

Indeed, this follows from associativiy (which we consider to be "strict") of the usual Cartesian product $\times$, which passes to the quotient. Namely, on both sides, the grading is induced by $d_1 \oplus d_2 \oplus d_3$ (by associativity and commutativity of the group

$(\Gamma, +))$, and on both sides, the quotient is taken with respect to the "generalized antidiagonal"

$$N := \{(\gamma_1, \gamma_2, \gamma_3) \in \Gamma^3 \mid \gamma_1 + \gamma_2 + \gamma_3 = 0\}.$$

Thus both sides are identified with the quotient for the equivalence relation

$$z^\gamma[x_1, x_2, x_3] = [z_1^\gamma x_1, x_2, x_3] = [x_1, z_2^\gamma x_2, x_3] = [x_1, x_2, z_3^\gamma x_3].$$

Finally, the set $E = \Gamma$ with trivial grading $d = 0$ and action on itself by $z^\gamma(x) = \gamma + x$ is neutral for the graded product (direct check). And, given morphisms $f_i : M_i \to M_i'$, $i = 1, 2$, of degree $u_i = d(f_i)$, the following is a well-defined morphism from $M_1 \times_z M_2$ to $M_1' \times_z M_2'$ of degree $u_1 + u_2$:

$$(f_1 \times_z f_2)[x_1, x_2] = [f_1(x_1), f_2(x_2)].$$

Summing up, we get a bifunctor, turning centrally Γ-graded sets with their graded product into a *(strict) monoidal category* (see [CWM], Chapter VII, p. 161).

2. BRAIDINGS, AND GRADED PRODUCT OF GROUPS

Next, we discuss "commutativity": what is the relation between $M_1 \times_z M_2$ and $M_2 \times_z M_1$? A *braiding* on a monoidal category is the choice of a "choherent" family of isomorphisms switching the order of product (see [CWM], Chapter XI). In general, a braiding is an *additional* structure. The standard braiding from set theory, $A \times B \cong B \times A$, induces of course a braiding on centrally Γ-graded sets: we call this again the *standard braiding*.

2.1. **Braiding of centrally graded sets.** In order to define other braidings, from now on, we assume that Γ is a generalized ring, with bi-additive product map $\Gamma^2 \to \Gamma$, $(a, b) \mapsto ab$, and define the *braiding map* $\beta_{M_1, M_2} : M_1 \times_Z M_2 \to M_2 \times_Z M_1$ as in Definition 0.5. Let us prove Theorem 0.6. Clearly, β_{M_1, M_2} preserves gradings and commutes with the Γ-action (since $(\Gamma, +)$ is commutative), hence is a morphism of centrally Γ-graded sets. It is an isomorphism, since it admits an inverse

$$\beta^{-1} : [y_1, y_2] \mapsto Z^{-d(y_1)d(y_2)}[y_2, y_1].$$

Note that this inverse corresponds to the original braiding map when replacing d by $-d$ and the "product" ab of Γ by its "opposite product" ba. Thus, if $ab = -ba$, we have the symmetry condition $\beta_{M_1, M_2}^{-1} = \beta_{M_2, M_1}$, as claimed.

Saying that our strict monoidal category is *braided* amounts to the following two "hexagonal diagram identities" (see [CWM], Chapter XI, p. 253): first, the following two morphisms from $M_1 \times_Z M_2 \times_Z M_3$ to $M_2 \times_Z M_3 \times_Z M_1$ coincide (for better readability, we shall write, $\beta_{1,2}$ instead of $\beta_{(M_1, M_2)}$, and $\beta_{1,23}$ instead of $\beta_{(M_1, M_2 \times_z M_3)}$, etc.):

$$(\mathrm{id}_2 \times_Z \beta_{(1,3)}) \circ (\beta_{(1,2)} \times_Z \mathrm{id}_3) = \beta_{(1,23)}.$$

We prove this by direct computation (writing d instead of d_i)

$$(\mathrm{id}_2 \times_Z \beta_{(1,3)}) \circ (\beta_{(1,2)} \times_Z \mathrm{id}_3)[x_1, x_2, x_3] =$$
$$(\mathrm{id}_2 \times_Z \beta_{(1,3)})(Z^{d(x_2)d(x_1)}[x_2, x_1, x_3]) = z^{d(x_2)d(x_1)+d(x_3)d(x_1)}[x_2, x_3, x_1],$$
$$\beta_{(1,23)}[x_1, x_2, x_3] = z^{(d(x_2)+d(x_3))d(x_1)}[x_2, x_3, x_1].$$

By distributivity of the product in Γ, both sides agree. The second "hexagonal" condition amounts to the first condition, when replacing the braiding maps by their inverses. But, as noticed above, the inverses of the braiding are of the same form (replacing d by $-d$ and the product in Γ by its opposite), so the computation is exactly the same.

Remark 2.1. As explained in [CWM], Section XI.5 (Braided Coherence), the preceding arguments amount to proving that the morphisms $s_i := \beta_{(M_i, M_{i+1})}$ satisfy the defining *relations of the generators of the braid group*

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \text{ and } s_i s_j = s_j s_i \text{ whenever } |i - j| \neq 1.$$

Explicitly, let's show that $s_2 s_1 s_2 = s_1 s_2 s_1$, by a computation essentially equivalent to the one given above. We write (23) for s_2 and (12) for s_1:

$$
\begin{aligned}
(23)(12)(23)[x_1, x_2, x_3] &= (23)(12)Z^{d_3 d_2}[x_1, x_3, x_2] \\
&= (23)Z^{d_3 d_2 + d_3 d_1}[x_3, x_1, x_2] \\
&= Z^{d_3 d_2 + d_3 d_1 + d_2 d_1}[x_3, x_2, x_1] \\
(12)(23)(12)[x_1, x_2, x_3] &= (12)(23)Z^{d_2 d_1}[x_2, x_1, x_3] \\
&= (12)Z^{d_2 d_1 + d_3 d_1}[x_2, x_3, x_1] \\
&= Z^{d_2 d_1 + d_3 d_1 + d_3 d_2}[x_3, x_2, x_1]
\end{aligned}
$$

When the braiding is symmetric, the action on the braid group passes to an action of the symmetric group $\mathfrak{S}_n$ (coherence theorem in symmetric monoidal categories, see [CWM], p. 253, Thm.1). We give an explicit formula describing this action in terms of permutation inversions:

Theorem 2.1. *Assume the product in Γ is skew. Then there are well-defined isomorphisms, for every $\sigma \in \mathfrak{S}_n$,*

$$\beta_\sigma^{(M_1, \ldots, M_n)} : M_1 \times_Z \ldots \times_Z M_n \to M_{\sigma(1)} \times_Z \ldots \times_Z M_{\sigma(n)},$$

coinciding for adjacent transpositions $s_i = (i, i+1)$ with the braiding isomorphisms, and in general given by

$$\beta_\sigma^{(M_1, \ldots, M_n)}[x_1, \ldots, x_n] = Z^u[x_{\sigma(1)}, \ldots, x_{\sigma(n)}],$$

with u given in terms of permutation inversions *for σ:*

$$u = \sum_{\substack{(i,j): \\ i < j, \sigma(i) > \sigma(j)}} d(x_j) d(x_i).$$

Proof. Existence of the action follows from the coherence theorem in symmetric monoidal categories (loc cit.) All that remains to be proved is the explicit formula for u in terms of inversions. Indeed, for $n = 2$ and $n = 3$, the explicit formulae given above prove the claim. For general n, the claim is given by induction. To this end, one may decompose any permutation into a product of adjacent transpositions, each of which adding an inversion term (which may cancel out with another term obtained precedingly, and so really counts the inversions: this is where skewness of the product in Γ becomes crucial), and performing a computation which is essentially the same as the one given above for the permutation $(13) = (12)(23)(12)$. $\square$

2.2. **Braided dual of a centrally graded group.** We continue to assume that Γ is a generalized ring. The braiding maps show up in constructions on graded groups (or monoids) from Theorem 0.7. We prove Item (1) of this theorem: the product of $G^\vee$ is associative:

$$
\begin{aligned}
g_1 \cdot^\vee (g_2 \cdot^\vee g_3) &= Z^{d(g_1)\,d(g_2 g_3)}(g_1 \cdot (g_2 \cdot^\vee g_3)) \\
&= Z^{d(g_1)d(g_2)+d(g_1)d(g_3)+d(g_2)d(g_3)} g_1 g_2 g_3 \\
&= Z^{d(g_1 g_2)\,d(g_3)}(g_1 \cdot^\vee g_2) \cdot g_3 = (g_1 \cdot^\vee g_2) \cdot^\vee g_3.
\end{aligned}
$$

The neutral element is e, and the inverse of g is $Z^{(d(g)^2)}g^{-1}$, since $d(g^{-1}) = -d(g)$,

$$
g \cdot^\vee Z^{(d(g)^2)}g^{-1} = Z^{d(g)\cdot d(g^{-1})}Z^{(d(g))^2}gg^{-1} = Z^{-(d(g))^2+(d(g))^2}e = e = Z^{(d(g)^2)}g^{-1} \cdot^\vee g.
$$

Since Γ is commutative and $Z(\Gamma)$ central, the maps d and Z define on this group the structure of a Γ-graded group. Since the product in $(G^\vee)^\vee$ is $(g_1, g_2) \mapsto Z^{d(g_2)d(g_1)+d(g_1)d(g_2)}g_1 g_2$, it follows that $(G^\vee)^\vee = G$ if Γ is skew.

Remark 2.2. By induction, the product of n elements in $G^\vee$ is given by

$$
g_1 \cdot^\vee \ldots \cdot^\vee g_n = Z^u g_1 \cdots g_n, \quad \text{where } u = \sum_{(i,j):i<j} d(g_i)d(g_j).
$$

Note that there are also other "duals": the *opposite group* G^{opp} of a group G, and the braided duals belonging to the products $-ab$ and to ba, and any combination of these operations. We are mainly interested in the case $\Gamma = \mathbb{Z}/2\mathbb{Z}$, where most of them coincide, and will not investigate systematically the family of "duals" thus obtained.

Example 2.1 (Central extensions of Γ). Let $G = \Gamma^2$ with group law here written

$$
(x_0, x_1) \cdot (y_0, y_1) = (x_0 + y_0, x_1 + y_1),
$$

and $d(x_0, x_1) = x_1$ and $Z(x) = (x, 0)$. This clearly defines a (commutative) centrally Γ-graded group, denoted by $\Gamma_{1,0}$. Its b-dual, which we denote by $\Gamma_{0,1} = \Gamma_{1,0}^\vee$, is given by the same d and Z, and product

$$
(x_0, x_1) \cdot^\vee (y_0, y_1) = Z(x_1 y_1) \cdot (x_0 + y_0, x_1 + y_1) = (x_0 + y_0 + x_1 y_1, x_1 + y_1).
$$

Indeed, $\Gamma_{1,0}$ is the central extension of Γ by Γ via the trivial cocycle 0, whereas $\Gamma_{0,1}$ is the extension via the (in general) non-trivial cocycle $(x, y) \mapsto xy$.

For later use, let us have a closer look at the case $\Gamma = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ with its usual (field) product. Let

$$
1 := e_0 := (\bar{0}, \bar{0}), \quad Z := (\bar{1}, \bar{0}), \quad e_1 := (\bar{0}, \bar{1}), \quad Ze_1 := (\bar{1}, \bar{1}).
$$

In the following, we'll write the group law multiplicatively (although at this stage it will still be commutative). So let $t \in \{1, Z\}$. When $t = 1$, the following gives the group law of $\Gamma_{1,0}$, and when $t = Z$, it gives the group law of $\Gamma_{0,1}$:

$\cdot$	1	Z	e_1	Ze_1
1	1	Z	e_1	Ze_1
Z	Z	1	Ze_1	e_1
e_1	e_1	Ze_1	t	tZ
Ze_1	Ze_1	e_1	tZ	t

The elements 1 and Z are even, and e_1 and Ze_1 are odd. For $t = 1$, we see that $\Gamma_{1,0} \cong C_2 \times C_2$ as group (usual direct product). For $t = Z$, we have $\Gamma_{0,1} \cong C_4$, because e_1 becomes here an element of order four.

2.3. **Graded product of centrally graded groups.** We define the *graded product* of two centrally Γ-graded groups G_i, $i = 1, 2$, as in Item 5)2) of Theorem 0.7. Let us show that the new product is associative:

$$
\begin{aligned}
(f_1, f_2) \cdot \big((g_1, g_2) \cdot (h_1, h_2)\big) &= (f_1, f_2) \cdot (Z_1^{d(g_2)\,d(h_1)} g_1 h_1, g_1 h_2) \\
&= (Z_1^{d(f_2)d(g_1 h_1)} Z_1^{d(g_2)d(h_1)} f_1 g_1 h_1, f_2 g_2 h_2) \\
&= (Z_1^{d(f_2)d(g_1)+d(f_2)d(h_1)+d(g_2)d(h_1)} f_1 g_1 h_1, f_2 g_2 h_2) \\
&= (Z_1^{d(f_2)d(g_1)+d(f_2 g_2)d(h_1)} f_1 g_1 h_1, f_2 g_2 h_2) \\
&= (Z_1^{d(f_2)d(g_1)} f_1 g_1, f_2 g_2) \cdot (h_1, h_2) \\
&= \big((f_1, f_2) \cdot (g_1, g_2)\big) \cdot (h_1, h_2).
\end{aligned}
$$

The inverse element of (g_1, g_2) is, as is directly checked, $(Z_1^{d(g_2)d(g_1)} g_1^{-1}, g_2^{-1})$, and so $(G_1 \times G_2, \cdot)$ is a group. Moreover, $d(g_1, g_2) := d(g_1) + d(g_2)$ clearly defines a morphism from this group to $(\Gamma, +)$. The elements (Z_1^k, Z_2^{-k}) belong to the center of this group, and their degree is zero, hence the quotient G/N is again a graded group. The equivalence relation generated by N comes from $(g_1, g_2) \sim (Z_1^k g_1, Z_2^{-k} g_2)$, that is, $(g_1, Z_2^k g_2) \sim (Z_1^k g_1, g_2)$ for all k, and hence the quotient group $G_1 \times G_2/N$ fulfills the conditions from the theorem. Moreover, if the product in Γ is skew, the braiding map is a group isomorphism from $G_1 \widehat{\times}_Z G_2$ onto $G_2 \widehat{\times}_Z G_1$:

$$
\begin{aligned}
\beta([g_1, g_2] \cdot [h_1, h_2]) &= \beta([Z_1^{d(g_2)d(h_1)} g_1 h_1, g_2 h_2]) \\
&= Z^{d(g_2)d(h_1)-d(g_2 h_2)d(g_1 h_1)} [g_2 h_2, g_1 h_1] \\
&= Z^{d(g_2)d(h_1)-(d(g_2)+d(h_2))(d(g_1)+d(h_1))} [g_2 h_2, g_1 h_1] \\
&= Z^{-d(h_2)d(g_1)-d(h_2)d(h_1)-d(g_2)d(g_1)} [g_2 h_2, g_1 h_1] \\
&= Z^{d(g_1)d(h_2)} [Z_2^{d(g_1)d(g_2)+d(h_1)d(h_2)} g_2 h_2, g_1 h_1] \\
&= [Z_2^{d(g_1)d(g_2)} g_2, g_1] \cdot [Z_2^{d(h_1)d(h_2)} h_2, h_1] \\
&= \beta[g_1, g_2] \cdot \beta[h_1, h_2]
\end{aligned}
$$

Remark 2.3 (Product of n elements). With different notation, the formula from the proof for the product of three elements reads, with $d_{ij} := d(g_{ij})$,

$$
(g_{11}, g_{12}) \cdot (g_{21}, g_{22}) \cdot (g_{31}, g_{32}) = Z_1^{d_{12}d_{21}+d_{12}d_{31}+d_{22}d_{31}} (g_{11}g_{21}g_{31}, g_{12}g_{22}g_{32}).
$$

By induction, the product of n elements in G is given by

$$
[g_{11}, g_{12}] \cdots [g_{n1}, g_{n2}] = Z^{\underset{\substack{(i,j)\in n^2: \\ i<j}}{\sum} d_{i2}d_{j1}} \big[g_{11}\cdots g_{n1},\, g_{12}\cdots g_{n2}\big].
$$

Note that the proof of associativity and the proof of the braiding property are essentially equivalent.

To finish the proof of Theorem 0.7, we prove Item (3): the set-theoretic equality $(G_1 \widehat{\times} G_2) \widehat{\times} G_3 = G_1 \widehat{\times} (G_2 \widehat{\times} G_3)$ is an isomorphism of groups: on the one hand,

$$\big((g_1, g_2), g_3\big) \cdot \big((h_1, h_2), h_3\big) = \big(Z^{d(g_3)\, d(h_1,h_2)}(g_1, g_2) \cdot (h_1, h_2), g_3 h_3\big)$$
$$= \big(Z_1^{d(g_3)(d(h_1)+d(h_2))} Z_1^{d(g_2)d(h_1)} g_1 h_1, g_2 h_2, g_3 h_3\big)$$
$$= \big(Z_1^{d(g_3)d(h_1)+d(g_3)d(h_2)+d(g_2)d(h_1)} g_1 h_1, g_2 h_2, g_3 h_3\big).$$

On the other hand, using distributivity in the ring Γ, we get the same result:

$$\big(g_1, (g_2, g_3)\big) \cdot \big(h_1, (h_2, h_3)\big) = \big(Z_1^{d(g_2,g_3)\, d(h_1)} g_1 h_1, (g_2, g_3) \cdot (h_2, h_3)\big)$$
$$= \big(Z_1^{(d(g_2)+d(g_3))\, d(h_1)} Z_1^{d(g_3)d(h_2)} g_1 h_1, g_2 h_2, g_3 h_3\big)$$
$$= \big(Z_1^{d(g_2)d(h_1)+d(g_3)d(h_1)+d(g_3)d(h_2)} g_1 h_1, g_2 h_2, g_3 h_2\big)$$

Remark 2.4. The preceding computation shows that the group product in $G_1 \widehat{\times}_Z G_2 \widehat{\times}_Z G_3$ is given by, with $d_{ij} := d_{(i,j)} := d(g_{ij})$,

$$[g_{11}, g_{12}, g_{13}] \cdot [g_{21}, g_{22}, g_{23}] = Z^{d_{12}d_{21}+d_{13}d_{21}+d_{13}d_{22}}[g_{11}g_{21}, g_{12}g_{22}, g_{13}g_{23}].$$

By induction, we get the product of two elements in $G = G_1 \widehat{\times}_Z G_2 \widehat{\times}_Z \ldots \widehat{\times}_Z G_p$,

$$[g_{11}, \ldots, g_{1p}] \cdot [g_{21}, \ldots, g_{2p}] = Z^{\sum_{i>j} d_{1i}d_{2j}}[g_{11}g_{21}, \ldots, g_{1p}g_{2p}].$$

Note the "duality" with the formula from Remark 2.3! Applying it twice, we get

$$[g_{11}, g_{12}, g_{13}] \cdot [g_{21}, g_{22}, g_{23}] \cdot [g_{31}, g_{32}, g_{33}] = Z^u[g_{11}g_{21}g_{31}, g_{12}g_{22}g_{32}, g_{13}g_{23}g_{33}],$$

with the degree term given by

$$u = d_{12}d_{21} + d_{13}d_{21} + d_{13}d_{22} + (d_{12}+d_{22})d_{31} + (d_{13}+d_{23})d_{31} + (d_{13}+d_{23})d_{32}$$
$$= (d_{12}d_{21} + d_{13}d_{21} + d_{13}d_{22}) + (d_{12}d_{31} + d_{13}d_{31} + d_{13}d_{32}) + (d_{22}d_{31} + d_{23}d_{31} + d_{23}d_{32}).$$

The 9 terms are structured by the graph of the usual total order on $\{1, 2, 3\}$ (the three pairs $(1 < 2), (1 < 3), (2 < 3)$). By induction (the details of which we omit), we get for the product of n elements $g_i = (g_{i1}, \ldots, g_{ip})$, for $i = 1, \ldots, n$:

$$g_1 \cdots g_n = Z^u(g_{11} \cdots g_{n1}, \ldots, g_{1p} \cdots g_{np})$$

with degree term given by a kind of "matrix product"

$$u = \sum_{\substack{(i,j,k,\ell) \in \mathsf{p}^2 \times \mathsf{n}^2: \\ i>j, k<\ell}} d_{ki}d_{\ell j}.$$

Remark 2.5. The group inverse in $G_1 \widehat{\times}_Z G_2 \widehat{\times}_Z G_3$ is given by

$$[g_1, g_2, g_3]^{-1} = Z^{d_2 d_1 + d_3 d_1 + d_3 d_2}[g_1^{-1}, g_2^{-1}, g_3^{-1}].$$

By induction, $[g_1, \ldots, g_n]^{-1} = Z^{\sum_{i<j} d_j d_i}[g_1^{-1}, \ldots, g_n^{-1}]$.

Remark 2.6 (Choices and conventions). The graded product of graded groups could have been defined by other choices and conventions. To explain this, note that we could have defined the group law on $G_1 \times_z G_2$ in a different way: every matrix $X \in M(2, 2; \mathbb{Z})$ gives rise to a bi-additive product on $\Gamma \oplus \Gamma$, via

$$\langle a, b \rangle = (a_1, a_2) X (b_1, b_2)^t = a_1 X_{11} b_1 + a_1 X_{12} b_2 + a_2 X_{21} b_1 + a_2 X_{22} b_2$$

(where products in Γ are given by the fixed bi-additive product, and coefficients from $\mathbb{Z}$ associate with everything). Thus we can define a braided dual group structure on $G_1 \times G_2$, as above, with respect to the bi-additive product on $\Gamma \oplus \Gamma$, and then mod out the normal subgroup N as before. The result is a new group structure on $G_1 \times_z G_2$, depending on the matrix X. When $\Gamma = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, then all 16 matrices from $M(2, 2; \mathbb{F}_2)$ give rise to a group structure. For instance, the matrix $X = E_{11}$ gives rise to the group $G_1^\vee \times_z G_2$, and $X = I = E_{11} + E_{22}$ gives rise to the group $G_1^\vee \times_Z G_2^\vee$. However, for these choices, Item (3) of Theorem 0.7 would fail to hold: for this we need that X is strict (upper or lower) triangular. For $X = 0$, this gives the ungraded product; our choice is $X = E_{12}$, but $X = E_{21}$ would give an isomorphic theory.

Remark 2.7 (Internal graded product). With notation as in Theorem 0.7, G has two normal subgroups

$$\{[g_1, 1] \mid g_1 \in G_1\} \cong G_1,$$
$$\{[1, g_2] \mid g_2 \in G_2\} \cong G_2,$$

such that $G_1 \cap G_2 = Z^\Gamma = \operatorname{im}(Z)$. Commutators of elements of G_1 with elements of G_2 belong to Z^Γ. Indeed, both groups normalize each other:

$$[1, g_2] \cdot [g_1, 1] \cdot [1, g_2]^{-1} = Z^{d(g_2)d(g_1)}[g_1, g_2] \cdot [1, g_2^{-1}]$$
$$= [Z_1^{d(g_2)d(g_1)} g_1, 1] = Z^{d(g_2)d(g_1)}[g_1, 1].$$

These facts imply that G is a certain amalgamated semi-direct product of G_1 with G_2. Conversely, such data can be used to recover the graded product (so we may speak of an "internal graded product" of subgroups of a given group, corresponding to the "external graded product", given by the theorem). Thus we may write $g_1 g_2$ instead of $[g_1, g_2]$, and compute products by using the "commutation relation"

$$(2.1) \qquad\qquad g_2 g_1 = Z^{d(g_2)d(g_1)} g_1 g_2.$$

Example 2.2 (D_4 and Q revisited). To prepare the grounds for the following section, let us compute explicitly the graded product of two groups of type $\Gamma_{1,0}$ or $\Gamma_{0,1}$, as defined in Example 2.1. Let $G_i = \{1, Z_i, e_i, Z_i e_i\}$, given by the table from Example 2.1, with $t_i \in \{1, Z_i\}$, where Z_i corresponds to Z. The cardinality of $G = G_1 \widehat{\times}_Z G_2$ is $\frac{4 \times 4}{2} = 8$, and its elements are

$$1 = [(1, 1)], \quad Z = [(1, Z_2)] = [(Z_1, 1)], \quad e_1 = [(e_1, 1)], \quad e_2 = [(1, e_2)],$$

$$Ze_1, \quad Ze_2, \quad e_{12} := e_1 e_2 = [(e_1, e_2)], \quad Ze_{12}.$$

These elements are multiplied by using the rules: $e_1 e_2 = Z e_2 e_1$, $e_i^2 = t_i$, $Z_i^2 = 1$, Z is central. The elements $1, Z, e_{12}, Ze_{12}$ are even, and e_1, e_2, Ze_1, Ze_2 are odd. For instance $e_{12}^2 = e_1 e_2 e_1 e_2 = Z e_1^2 e_2^2 = Z t_1 t_2$. The group table of G is as follows:

$\cdot$	1	Z	e_{12}	Ze_{12}	e_1	Ze_1	e_2	Ze_2
1	1	Z	e_{12}	Ze_{12}	e_1	Ze_1	e_2	Ze_2
Z	Z	1	Ze_{12}	e_{12}	Ze_1	e_1	Ze_2	e_2
e_{12}	e_{12}	Ze_{12}	Zt_1t_2	t_1t_2	Zt_1e_2	t_1e_2	t_2e_1	Zt_2e_1
Ze_{12}	Ze_{12}	e_{12}	t_1t_2	Zt_1t_2	t_1e_2	Zt_1e_2	Zt_2e_1	t_2e_1
e_1	e_1	Ze_1	t_1e_1	Zt_1e_1	t_1	Zt_1	e_{12}	Ze_{12}
Ze_1	Ze_1	e_1	Zt_1e_1	t_1e_1	Zt_1	t_1	Ze_{12}	e_{12}
e_2	e_2	Ze_2	Zt_2e_1	t_2e_1	Ze_{12}	e_{12}	t_2	Zt_2
Ze_2	Ze_2	e_2	t_2e_1	Zt_2e_1	e_{12}	Ze_{12}	Zt_2	t_2

Since G is non-commutative and of cardinality 8, it is either isomorphic to the dihedral group D_4 or to the quaternion group Q. When $t_1 = Z = t_2$, then G contains 6 elements of order 4, and hence is isomorphic to Q. Its even subgroup is $G_0 \cong C_4$. In the other cases, it contains exactly 2 elements of order 4, and hence is isomorphic to D_4. The even subgroup then is isomorphic to C_4 if $t_1 = 1 = t_2$, and to $C_2 \times C_2$ if $t_1 = Z, t_2 = 1$, or if $t_1 = 1, t_2 = Z$.

Example 2.3. Let $m \geq 2$, and $\Gamma = \mathbb{Z}/m\mathbb{Z}$. We have three groups, of cardinal m^3,

$$C_m^2 \widehat{\times}_Z C_m^2, \qquad C_{m^2} \widehat{\times}_Z C_m^2, \qquad C_{m^2} \widehat{\times}_Z C_{m^2}.$$

It follows from Remark 2.7 that these groups satsify $[G, G] \subset \langle Z \rangle$, hence are 2-step nilpotent. E.g., for $m = 3$, this gives the two non-abelian groups of order 27 (see [D] p. 565 and p. 566 for detailed information on these groups).

3. The discrete Clifford category

3.1. **The Γ-category.** Assume the product in Γ is skew, and recall from Example 2.1 the graded groups

$$Q(1) := \Gamma_{1,0} = \Gamma^2, \qquad Q(Z) := \Gamma_{0,1} = \Gamma_{1,0}^\vee.$$

The symmetric monoidal category of Γ-graded groups generated by $Q(1)$ and $Q(Z)$ will be called the Γ-*category*. Explicitly, these groups are of the form:

Definition 3.1. *For* $\mathbf{t} = (t_1, \ldots, t_n) \in \{1, Z\}^n$,
   if $t_i = 1$, *let* $Q(t_i)$ *be a copy of* $Q(1) = \Gamma^2$,
   if $t_i = Z$, *let* $Q(t_i)$ *be a copy of* $Q(Z) = Q(1)^\vee$, *and define*

$$Q(\mathbf{t}) := Q(t_1) \widehat{\times}_Z \ldots \widehat{\times}_Z Q(t_n).$$

We let also, $Q_0 := \Gamma$ *with trivial grading (as we have seen, this is the neutral element for the graded product), and for* $p + q = n$,

$$Q_{p,q} := Q(\underbrace{1, \ldots, 1}_{p\times}, \underbrace{Z, \ldots, Z}_{q\times}), \quad Q_n := Q_{n,0} = Q(1, \ldots, 1), \quad Q_n^\vee = Q_{0,n}.$$

By Item (3) of Theorem 0.7, we can recast this definition:

Definition 3.2. *Let* K *be a set. For* $(\mathbf{t}, \mathbf{s}) \in K^n \times K^m$, *we denote by*

$$\mathbf{t} \oplus \mathbf{s} := (t_1, \ldots, t_n, s_1, \ldots, s_m) \in K^{n+m}$$

their "juxtaposition". Note that this operation is associative, but not commutative (in fact, it is the composition law of the free monoid over K).

Lemma 3.3. *For* $\mathbf{t} \in \{1, Z\}^n$, $\mathbf{s} \in \{1, Z\}^m$, *we have*

$$Q(\mathbf{t} \oplus \mathbf{s}) = Q(\mathbf{t})\widehat{\times}_Z Q(\mathbf{s}).$$

In particular, $Q_{p,q} = Q_p \widehat{\times}_Z Q_{0,q}$.

Remark 3.1. Fom Item (4) of Theorem 0.7, we get an isomorphism $\phi_{(\mathbf{t},\mathbf{s})} : Q(\mathbf{t}\oplus\mathbf{s}) \to Q(\mathbf{s} \oplus \mathbf{t})$. For instance, re-arranging the order of factors, we get an isomorphism

$$Q_{p,p} \cong Q_{1,1}\widehat{\times}_Z \ldots \widehat{\times}_Z Q_{1,1}.$$

Example 3.1. Let $\Gamma = \mathbb{R}^k$ with $k \geq 2$ and skew-symmetric product $(x_1, x_2) = (0, \ldots, 0, x_1 x_2 - x_2 x_1)$. Then $Q_{1,0}$ is the abelian group $\mathbb{R}^{2k}$, whereas $Q_{0,1}$, and all higher $Q_{p,q}$ are are 2-step nilpotent Lie groups, isomorphic to a *Heisenberg group*. We propose to call the Γ-category, in this case, the *Heisenberg category*. A systematic study of this category is certainly an interesting topic for subsequent work.

3.2. **The discrete Clifford category.** From now on, we assume that $\Gamma = \mathbb{Z}/2\mathbb{Z}$ with its usual ring (field) structure. In this case, the Γ-category will be called the *(discrete) Clifford category*, since the groups $Q_{p,q}$ then are (discrete) Clifford groups, as we shall see. The specific feature of this case is that Γ is both skew and admits a unit element $\overline{1}$ (which is not the case in the Heisenberg category). To give an explicit description of all groups by generators and relations, we follow the notation from Example 2.2: for $m \in \mathbb{N}$, let Z_m be a copy of the element Z (copy called "of m-th generation"). For each $t_m \in \{1, Z_m\}$, we define a graded group of four elements,

$$Q(t_m) = \{e_0, e_m, Z_m, Z_m e_m\},$$

just as in Example 2.2: $1 = e_0$ is neutral, $Z_m^2 = e_0$, and $e_m^2 = t_m e_0$. The elements $e_i, Z_i e_i$ are odd, and e_0, Z_i are even. Thus, as we have seen in Exemple 2.1,

(3.1)
$$\begin{aligned} Q_1 = Q(1) = C_2^2, \\ Q_{0,1} = Q(Z) = C_4. \end{aligned}$$

From Example 2.2, we now get

(3.2)
$$\begin{array}{ccccc} Q_2 & = & Q(1,1) & = & D_4, \\ Q_{0,2} & = & Q(Z,Z) & = & Q, \\ Q_{1,1} & = & Q(1,Z) & = & D_4. \end{array}$$

As already said, $Q(\mathbf{t})$ will be seen to be isomorphic to the discrete Clifford group of the Clifford algebra $\mathrm{Cl}(\mathbf{t})$ (Appendix A). In the following, let us prove some basic structure results on these groups, without using the theory of Clifford algebras. Recall from Example 1.1 the abelian group $(\mathcal{P}(\mathsf{n}), \Delta)$, isomorphic to $(Z/2\mathbb{Z}))^n$.

Theorem 3.4. *Fix* $\mathbf{t} \in \{1, Z\}^n$. *Then:*
  (1) *The group* $G := Q(\mathbf{t})$ *is of cardinality* 2^{n+1}.
  (2) *The quotient group* $G^0 = G/\{1, Z\}$ *is abelian, isomorphic to* $\mathcal{P}(\mathsf{n}) = (\mathbb{Z}/2\mathbb{Z})^n$, *i.e., the following is an exact sequence of groups (central extension)*

$$\{1, Z\} \quad \to \quad Q(\mathbf{t}) \quad \to \quad (\mathcal{P}(\mathsf{n}), \Delta).$$

  Elements of $Q(\mathbf{t})$ *are of order either* 1, 2 *or* 4.

(3) *The group G is generated by the elements $e_i \in Q(t_i)$ (which are identified with the corresonding element in G). For $A \subset \mathsf{n}$ with $k = |A|$, whose elements are ordered, $a_1 < \ldots < a_k$, we let*

$$e_A := e_{a_1} \cdots e_{a_k}.$$

Then G is a disjoint union, as follows:

$$G = \{e_A \mid A \in \mathcal{P}(\mathsf{n})\} \sqcup \{Ze_A \mid A \in \mathcal{P}(\mathsf{n})\}.$$

We often identify the first set with $\mathcal{P}(\mathsf{n})$ (i.e., we fix this set theoretic splitting of the exact sequence from Item (2)). There is a function $\tau : \mathcal{P}(\mathsf{n}) \times \mathcal{P}(\mathsf{n}) \to \mathbb{Z}/2\mathbb{Z}$ such that, for all $A, B \in \mathcal{P}(\mathsf{n})$,

$$e_A e_B = Z^{\tau(A,B)} e_{A\Delta B}.$$

(4) *The group G is 2-step nilpotent: the commutator subgroup $[G, G]$ belongs to the center of G, whence $[G, [G, G]] = \{1\}$.*

(5) *Defining relations between the generators $e_1, \ldots, e_n, Z$ are:*
   (a) *e_0 is neutral,*
   (b) *(for all $i = 1, \ldots, n$) $e_i^2 = t_i e_0$,*
   (c) *(for $k < \ell$) $e_k e_\ell = Z e_\ell e_k$, and*
   (d) *Z is central, even, and of order 2.*

Proof. (1), (2), (3) are essentially contained in Theorem 0.7 and Remark 2.4. More formally, these items are proved by induction: for $n = 1$ and $n = 2$, this has been noticed above. Assume the claim already proved at level $n \in \mathbb{N}$, and write

(3.3) $$Q(t_1, \ldots, t_{n+1}) = Q(\mathbf{t}) \widehat{\times} Q(t_{n+1}),$$

and use "internal notation" (2.1) together with induction hypothesis to write elements of this group as claimed. Concerning the product of two elements, $e_A e_B = e_{a_1} \cdots e_{a_k} e_{b_1} \cdots e_{b_\ell}$, re-order elements in ascending order of indices; when two indices coincide, the square yields either 1 or Z; only terms with indices from the symmetric difference survive, together with a term Z^u with $u = 0$ or $u = 1$. It follows that $(e_A)^4 = 1$ and $(Ze_A)^4 = 1$, for all $A \in \mathcal{P}(\mathsf{n})$.

(4) The morphism from item (2) sends the commutator group $[G, G]$ to $[\mathcal{P}(\mathsf{n}), \mathcal{P}(\mathsf{n})] = 0$ since $\mathcal{P}(\mathsf{n})$ is abelian. Therefore $[G, G] \subset \{1, Z\}$, and hence $[G, G]$ belongs to the center of G. (See Lemma 3.5 for an explicit formula describing commutators.)

(5) Relations (a) – (d) are contained in the preceding items. Conversely, given a group having generators satisfying these relations, it is seen by induction that it is isomorphic to $Q(\mathbf{t})$: for $n = 1$ and $t_1 = 1$, we have the defining relations of $C_2 \times C_2$, and for $n = 1$ and $t_1 = Z$, we have the defining relations of C_4; likewise, for $n = 2$, we have the defining relations of D_4, resp., of Q. For the induction step, assume the claim holds at order n, and let G_{n+1} be a group having generators and relations of the given form at order $n + 1$. Then $H := \{1, e_{n+1}, Z, Ze_{n+1}\}$ is a subgroup, isomorphic to C_4 if $t_{n+1} = Z$ and to C_2^2 if $t_{n+1} = 1$, and the given relations show that G is a homomorphic image of $G_n \widehat{\times}_Z H$. For reasons of cardinality, G is actually equal to this group, i.e., given by (3.3). $\qquad\square$

Remark 3.2. Following the pattern of "classification of real Clifford algebras", we can now "classify" the groups $Q_{p,q}$. In our opinion, this gives a transparent and

conceptual version of that part of Clifford theory. Since, in the main text, we are rather interested in classification-free theory, we relegate the presentation of this issue to Appendix B.

Remark 3.3. The function τ is the cocycle of a central extension. See Section 3.3 for more on this; cf. also [L] p.12–14.

Next, we compute an explicit formula for the inner automorphism given by e_A:

Lemma 3.5. *For all* $A, B \in \mathcal{P}(\mathsf{n})$,

$$e_A \cdot e_B \cdot (e_A)^{-1} = Z^{|A| \cdot |B| - |A \cap B|} e_B.$$

Proof. When $|A| = 1 = |B|$, then the claim is in keeping with the defining relations of $Q(\mathbf{t})$, namely $e_i e_j (e_i)^{-1} = u e_j$, with $u = 1$ if $i = j$, and $u = Z$ else. Now let $|A| = 1$, say, $A = \{i\}$, and B arbitrary. When $i \notin B$, then we have to exchange the position of e_i and each e_j, $j \in B$, which gives a factor $Z^{|B|}$, and when $i \in B$, we get a factor $Z^{|B|-1}$ (since we exchange for all $i \in B$ with $i \neq j$), which again is in keeping with the claim. For general A and B, conjugation by $e_A = e_{a_1} \cdots e_{a_k}$ is composition of conjugation by the e_{a_i}. We count the total number of exchanges of elements e_i and e_j arising in the whole procedure: whenever $(i, j) \in A \times B$ with $i \neq j$, we get an additional factor Z, so in the end we get a factor Z^p with

$$p = |\{(i, j) \in A \times B \mid i \neq j\}| = |A \times B \setminus \mathrm{diag}| = |A| \cdot |B| - |A \cap B|,$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 3.2. For the *pseudoscalar* (case $A = \mathsf{n}$, $A \cap B = B$), $e_\mathsf{n} = e_1 \cdots e_n$, we get:

$$e_\mathsf{n} e_B (e_\mathsf{n})^{-1} = Z^{(n-1)|B|} e_B,$$

which is equal to e_B if n is odd, and equal to $Z^{|B|} e_B$ if n is even.

Theorem 3.6 (Center, inner automorphisms). *Let* $\mathbf{t} \in \{1, Z\}^n$ *and* $G := Q(\mathbf{t})$. *Assume* n *is even. Then:*

(1) *the center of* G *is the subgroup* $\{1, Z\}$,
(2) *there are* $2^n + 1$ *conjugacy classes: two of them are singletons,* $\{1\}$ *and* $\{Z\}$, *and the other are of cardinal two,* $\{e_A, Z e_A\}$, *for* $A \neq \emptyset$,
(3) *the inner automorphism group of* $Q(\mathbf{t})$ *is isomorphic to* $\mathcal{P}(\mathsf{n}) \cong (\mathbb{Z}/2\mathbb{Z})^n$,
(4) *the grading automorphism* $\alpha(e_A) = Z^{|A|} e_A$ *is an inner automorphism.*

Assume n *is odd. Then:*

(1) *the center of* G *is the subgroup* $\{1, Z, e_\mathsf{n}, Z e_\mathsf{n}\}$ *(when* $\mathbf{t} = (1, \ldots, 1)$, *it is isomorphic to* C_4 *if* $n \equiv 1 \mod 4$, *and to* C_2^2 *if* $n \equiv 3 \mod 4$),
(2) *there are* $2^n + 2$ *conjugacy classes, four of them singletons (elements of the center), the other* $2^n - 2$ *classes of the form* $\{e_A, Z e_A\}$, $A \neq \emptyset, A \neq \mathsf{n}$,
(3) *the inner automorphism group is isomorphic to* $\mathcal{P}(\mathsf{n} - 1) \cong (\mathbb{Z}/2\mathbb{Z})^{n-1}$,
(4) *the grading automorphism* α *is an outer automorphism.*

In both cases, α, *together with the inner automorphisms, generates a subgroup of automorphisms isomorphic to* $\mathcal{P}(\mathsf{n}) \cong (C_2)^n$.

Proof. From Lemma 3.5, e_A commutes with all e_i iff $|A| - |A \cap \{i\}|$ is even, for all i, that is, for all i, the set $A \setminus \{i\}$ has an even number of elements. The only possibility to realize this case is $A = \mathsf{n} = \{1, \ldots, n\}$, where n is odd. Since Z is always central, and $e_{\mathsf{n}}^2 = Z^{\frac{n(n-1)}{2}}$, this proves the statements about the center.

Concerning conjugation classes, Lemma 3.5 shows that that each class has at most 2 elements, say e_B and Ze_B. It has 1 element iff the element is central, and 2 elements else. Thus the claims follow from those about the center. Likewise, those on the inner automorphism group $G/Z(G)$ follow from those on the center.

The grading automorphism from Lemma 1.4 satisfies the condition $\alpha(e_k) = Z^{d(e_k)}e_k = Ze_k$, whence $\alpha(e_A) = Z^{|A|}e_A$. As seen in Example 3.2, when n is even, it is inner. When n is odd, then there is no set $A \in \mathcal{P}(\mathsf{n})$ such that $e_A e_k (e_A)^{-1} = Ze_k$ for all $k = 1, \ldots, n$ (since $A = \mathsf{n}$ does not satisfy the condition, and when $A \neq \mathsf{n}$, the value of $e_A e_k (e_A)^{-1}$ depends whether $k \in A$ or $k \notin A$). Thus the grading automorphism then is outer. $\qquad\square$

In general, the group of *all* automorphisms of $Q(\mathbf{t})$ is not a suitable object, for our purposes. We are interested in the case that all t_i are equal, because then permutations of the e_i induce automorphisms. But in this case, the group of all automorphisms may be too big, as is illustrated by the following examples:

Example 3.3. Let $n = 1$, $G = Q(t_1) = \{e_0, e_1, Z, Ze_1\}$, $e_1^2 = t_1$.

(1) When $t_1 = Z$, so $G \cong C_4$, then $\mathrm{Aut}(G) \cong C_2$ (the unique non-trivial automorphism exchanges the two elements of order 4, namely e_1 and Ze_1).

(2) When $t_1 = 1$, so $G \cong C_2^2$, then $\mathrm{Aut}(G) \cong \mathfrak{S}_3$ is the permutation group of the three elements of order 2, namely of e_1, Z, Ze_1.

Example 3.4. Let $n = 2$, $G = Q(t_1, t_2) = \{e_0, e_1, e_2, e_{12}, Z, Ze_1, Ze_2, Ze_{12}\}$, $e_i^2 = t_i$.

(1) When $t_1 = 1 = t_2$, then $\mathrm{Aut}(G) \cong D_4 \cong G$. Conjugation by elements e_1, e_2, e_{12} defines three non-trivial inner automorphisms of order 2, and exchange of e_1 and e_2 yields an outer automorphism of order 2. Conjugation by e_{12} commutes with all automorphisms.

(2) When $t_1 = Z = t_2$, then $G = Q$ and $\mathrm{Aut}(G) \cong \mathfrak{S}_4$. Namely, the inner automorphisms form a Klein 4-group as above, and permutations of i, j, k (that is, of e_1, e_2, e_{12}), form a complementary $\mathfrak{S}_3$-subgroup. However, exchange of e_1 and e_{12} does not preserve the grading (since e_1 is odd and e_{12} even). The subgroup of grading-preserving automorphisms is again a D_4-subgroup of $\mathfrak{S}_4$, having same types of elements as in (1).

Theorem 3.7 (Hyperoctahedral automorphism group)**.** *The symmetric group $\mathfrak{S}_n$ acts by automorphisms on Q_n: for every $\sigma \in \mathfrak{S}_n$, there is a unique automorphism*

$$\Phi_\sigma : Q_n \to Q_n \text{ such that } \forall k = 1, \ldots, n : \Phi_\sigma(e_k) = e_{\sigma(k)}.$$

This group of automorphisms normalizes the group $(C_2)^n$ described in Theorem 3.6 , and together they form a group B_n of grading-preserving automorphisms of cardinality $n!2^n$. The grading automorphism α belongs to the center of this group. The same statements hold for the group $Q_{0,n}$.

Proof. By induction: for $n = 1$ the statement is uninteresting; for $n = 2$, the braiding automorphism the unique automorphism

$$\tau : Q_2 = Q_1 \widehat{\times}_Z Q_1 \to Q_2, \quad [g_1, g_2] \mapsto Z^{d(g_1)d(g_2)}[g_2, g_1]$$

such that $\tau(e_1) = e_2, \quad \tau(e_2) = e_1, \quad \tau(Z) = Z, \quad \tau(e_{12}) = Ze_{12}$ (cf. also Example 3.4). Assuming that $\mathfrak{S}_n$ acts by automorphisms on Q_n, we write

$$Q_{n+1} = Q_n \widehat{\times} Q_1 = Q_{n-1} \widehat{\times} Q_2.$$

Then $\mathfrak{S}_n = \mathfrak{S}_n \times \mathrm{id}_{Q_1}$ acts on Q_{n+1}, and another copy of $\mathfrak{S}_2 = \mathrm{id}_{Q_{n-1}}$ acts on Q_{n+1} (transposition $(n\, n + 1)$). Together, these two actions generate the action of $\mathfrak{S}_{n+1}$. Similarly for $Q_{0,n} = Q_{0,1} \widehat{\times} \ldots \widehat{\times} Q_{0,1}$.

It is clear that the action of $\mathfrak{S}_n$ normalizes the inner automorphisms as well as the grading automorphism α, hence the group $(\mathbb{Z}/2\mathbb{Z})^n$ is a normal subgroup, intersecting trivially the $\mathfrak{S}_n$-subgroup. Thus B_n is a semidirect product, of cardinality $n!2^n$. Clearly, α commutes with the $\mathfrak{S}_n$-action, hence belongs to the center of B_n. □

Remark 3.4. Recall that, by definition, the hyperoctahedral group is the group of *signed permutation matrices* (of size $n \times n$). It is Coxeter group of type B_n, generated by the usual permutation matrices (subgroup $\mathfrak{S}_n$) and the diagonal ± 1-matrices (subgroup C_2^n). It is isomorphic to the automorphism group defined in the preceding theorem. Its center is equal to $\{\mathrm{id}, \alpha\}$.

Definition 3.8. *We call the group B_n of automorphisms defined in the preceding theorem the* hyperoctahedral automorphism group *of Q_n, resp. of $Q_{0,n}$.*

Remark 3.5. The whole group of grading preserving automorphisms is in general bigger than the hyperoctahedral automorphism group.

Remark 3.6. For $Q_{p,q}$, the symmetry group $\mathfrak{S}_n$ must be replaced by $\mathfrak{S}_p \times \mathfrak{S}_q$, permuting the e_i's with $t_i = 1$, resp. those with $t_i = Z$.

Remark 3.7 (Inversions). In the Clifford category, all algebraic operations can be described by explicit formulae. From an algebraic point of view, this amounts to describe structures (central extensions) by *cocycles*, and from a combinatorial point of view, the main tool is given by *inversions*. For instance, to compute a formula for the automorphism Φ_σ from Theorem 3.7, we start by writing $A = \{a_1, \ldots, a_k\}$ with $a_1 < \ldots < a_k$, and

$$\Phi_\sigma(e_A) = \Phi_\sigma(e_{a_1} \cdots e_{a_k}) = e_{\sigma(a_1)} \cdots e_{\sigma(a_k)}.$$

We have to re-order these terms so that the indices are in increasing order. The number of terms Z appearing is $u := |(i, j) \in A \times A \mid i < j, \sigma(i) > \sigma(j)|$, so we end up with the formula of "inversion type"

$$\Phi_\sigma(e_A) = Z^{(u \mod 2)} e_{\sigma(A)}.$$

3.3. Cocycles.
Recall that the central extension of a group H by an abelian group A can be described by a cocycle $C : H \times H \to A$. In our case, by Theorem 3.4 we have the central extension $A = \{1, Z\} \to Q(\mathbf{t}) \to H = \mathcal{P}(\mathbf{n})$, with cocycle denoted by τ, defined by the rule

$$e_A e_B = \tau(A, B) e_{A \triangle B}.$$

Expanding both sides of $(e_A e_B)e_C = e_A(e_B e_C)$, we get the *cocycle relations*:

$$(3.4) \qquad \tau(A,B)\tau(A\Delta B, C) = \tau(A, B\Delta C)\tau(B,C).$$

Since our cocyle τ depends on $\mathbf{t}$ we call it a "relative cocycle". We shall separate it into two parts, the first of which contains the explicit dependence on $\mathbf{t}$, and the second being independent of $\mathbf{t}$ ("absolute").

Definition 3.9. *Let K be a commutative monoid, $\mathbf{t} \in K^n$ and $A \in \mathcal{P}(\mathsf{n})$. We let*

$$\mathbf{t}_\emptyset := 1, \qquad \mathbf{t}_A := \prod_{a \in A} t_a \in K.$$

The map $A \mapsto \mathbf{t}_A$ behaves like a "multiplicative measure": when $A \cap B = \emptyset$, then $\mathbf{t}_{A \cup B} = \mathbf{t}_A \cdot \mathbf{t}_B$, and, for general A, B,

$$\mathbf{t}_A \mathbf{t}_B = \mathbf{t}_{A \cap B}\mathbf{t}_{A \cup B} = \mathbf{t}_{A \cap B}^2 \mathbf{t}_{A \setminus B}\mathbf{t}_{B \setminus A} = \mathbf{t}_{A \cap B}^2 \mathbf{t}_{A \Delta B}.$$

If $k^2 = 1$ for all $k \in K$ (which is the case for $K = \{1, Z\}$), then

$$\mathbf{t}_{A \Delta B} = \mathbf{t}_A \cdot \mathbf{t}_B.$$

Lemma 3.10. *Let $\mathbf{t} \in \{1, Z\}^n$. The map $(A, B) \mapsto \mathbf{t}_{A \cap B}$ defines a cocyle on $\mathcal{P}(\mathsf{n})$.*

Proof. $\mathbf{t}_{A \cap B}\mathbf{t}_{(A \Delta B) \cap C} = \mathbf{t}_{A \cap B}\mathbf{t}_{A \cap C}\mathbf{t}_{B \cap C} = \mathbf{t}_{A \cap (B \Delta C)}\mathbf{t}_{B \cap C}.$ $\qquad \square$

We call the cocycle from the preceding lemma "relative". It will be combined with an "absolute" cocyle γ:

Definition 3.11. *Let $L := \{(a, b) \in \mathsf{n}^2 \mid a < b\}$ be the usual total order relation on n, and $L^{-1} = \{(a, b) \in \mathsf{n}^2 \mid a > b\}$ its opposite order. For a subset $R \subset (\mathsf{n} \times \mathsf{n})$, call*

$$\mathrm{Inv}(R) := R \cap L^{-1} = \{(a, b) \in R \mid a > b\}$$

the set of inversions *in R, and for $(A, B) \in \mathcal{P}(\mathsf{n})^2$, let*

$$\gamma(A, B) := Z^{|\mathrm{Inv}(A \times B)|} = Z^{|\{(a,b) \in A \times B \mid a > b\}|}.$$

Theorem 3.12 (The absolute cocycle)**.** *Let $\mathbf{t} \in \{1, Z\}^n$ and $A, B \in \mathcal{P}(\mathsf{n})$. Then the product $e_A e_B$ in $Q(\mathbf{t})$ is given by*

$$e_A e_B = \mathbf{t}_{A \cap B} \cdot \gamma(A, B) \cdot e_{A \Delta B}.$$

The map $\gamma : \mathcal{P}(\mathsf{n})^2 \to \{1, Z\}$ is the cocylce defining Q_n as central extension of $\mathcal{P}(\mathsf{n})$.

Proof. As in the proof of Theorem 3.4, we re-order the product $e_A e_B = e_{a_1} \cdots e_{a_k} e_{b_1} \cdots e_{b_l}$. After re-ordering, indices $i \in A \cap B$ give rise to a square $e_i^2 = t_i$, whence the factor $\mathbf{t}_{A \cap B}$. While re-ordering, a relation of type $e_i e_j = Z e_j e_i$ is applied when $i > j$, first for $j = b_1$, then for $j = b_2$, and so on, so the number of times we apply it is

$$\sum_{b \in B} |\{a \in A \mid a > b\}| = |\{(a, b) \in A \times B \mid a > b\}| = |\mathrm{Inv}(A, B)|.$$

When $\mathbf{t} = (1, \dots, 1)$ (case of Q_n), the factor $\mathbf{t}_{A \cap B}$ is always 1, so $\gamma(A, B)$ is the cocyle of Q_n (and in particular satisfies again the cocycle relation). $\qquad \square$

Besides satisfying the cocycle relation (3.4), the cocyle γ has several other properties, which in turn can be used to develop Clifford theory based on the discrete Clifford group (see [L] for a systematic exposition of this viewpoint).

4. Group algebra, and Clifford algebras

In the following, we start to develop some "harmonic analysis" on graded groups – far from complete, with the main aim to clarify the rôle of the Clifford algebra in case of the discrete Clifford category. Remarkably, this shows some analogy with the harmonic analysis of the Heisenberg group. This deserves to be developed elsewhere, and in the following we assume that all groups are discrete, or even finite.

4.1. **Group (super) algebra.** From now on we fix a commutative unital base ring $\mathbb{K}$. We assume that the scalar 2 is invertible in $\mathbb{K}$. For a set M, we denote by $\mathbb{K}[M]$ the free $\mathbb{K}$-module with basis denoted $(\delta_m)_{m \in M}$, so elements of $\mathbb{K}[M]$ are finite linear combinations $\sum_{m \in M} \lambda_m \delta_m$; if $M = G$ is a semigroup, then $\mathbb{K}[G]$ carries an associative product $*$ defined by $\delta_x * \delta_y = \delta_{xy}$ (semigroup algebra; cf. [FH]); if G is a monoid with unit e, the basis element δ_e becomes a neutral element of $\mathbb{K}[G]$, and if G is a group, then inversion $G \to G$ induces an anti-automorphism $t : \mathbb{K}[G] \to \mathbb{K}[G]$, sometimes called the *canonical anti-automorphism* or *antipode*.

In the following, let G be a centrally $\mathbb{Z}/2\mathbb{Z}$-graded group, and assume the element $Z(\overline{1})$ is non-trivial (we denote it again by Z, so that the image of $Z : \mathbb{Z}/2\mathbb{Z} \to G$ is the group $\{e, Z\}$). Let $\pi : G \to G^0 = G/\{e, Z\}$ be the canonical projection, and $\mathbb{A} := \mathbb{K}[G]$ be the group algebra. There are two maps of order two on G, which induce two linear maps of order two on $\mathbb{A}$:

(4.1) $Z : G \to G, \, g \mapsto Zg = gZ, \quad \alpha : G \to G, \, g \mapsto Z^{d(g)}g,$

(4.2) $Z_* : \mathbb{A} \to \mathbb{A}, \, f \mapsto Z_*f, \quad \alpha_* : \mathbb{A} \to \mathbb{A}, \, f \mapsto \alpha_*f.$

Since 2 is invertible in $\mathbb{K}$, both linear maps can be diagonalized; since $\alpha \circ Z = Z \circ \alpha$, they can be jointly diagonalized. We denote the eigenspace decompositions by

(4.3)   for $Z_* : \quad \mathbb{A} = \mathbb{A}^+ \oplus \mathbb{A}^-, \quad \mathbb{A}^\pm = \{f \in \mathbb{A} \mid Z_*f = \pm f\},$

(4.4)   for $\alpha_* : \quad \mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}_1, \quad \mathbb{A}_k = \{f \in \mathbb{A} \mid \alpha_*f = (-1)^k f\}.$

Since Z is central, the map Z_* commutes with elements of the group algebra, and hence the eigenspaces $\mathbb{A}^\pm$ are *ideals* of the group algebra. Since α is a group automorphism, by functoriality of the group algebra construction, it follows that the induced linear map is an *algebra automorphism of order two*, and hence the eigenspaces would behave multiplicatively if we indexed them by the eigenvalues ± 1. However, following the usual convention for *super-algebras*, we take 0 and 1 as indices, so the eigenspaces obey the "super algebra rules"

$$\mathbb{A}_i * \mathbb{A}_j \subset \mathbb{A}_{i+j \mod 2}.$$

Each of the two ideals $\mathbb{A}^\pm$ is, in turn, again a superalgebra.

Theorem 4.1 (Eigenspaces of Z_*; superalgebras)**.** *With notation as above:*

(1) *The algebra $\mathbb{K}[G]$ is a direct sum of ideals, $\mathbb{K}[G] = \mathbb{K}[G]^+ \oplus \mathbb{K}[G]^-$, and both eigenspaces are isomorphic as $\mathbb{K}$-modules. The projection $\pi : G \to G^0 = G/\{e, Z\}$ induces an algbra morphism $\pi_* : \mathbb{K}[G] \to \mathbb{K}[G^0]$ having kernel $\mathbb{K}[G]^-$ and image isomorphic to $\mathbb{K}[G]^+$, so $\mathbb{K}[G]^+ \cong \mathbb{K}[G^0]$ as algebra.*

(2) *With respect to α_*, the group algebra and the ideals from the preceding item are super-algebras. The disjoint union $G = G_0 \sqcup G_1$ induces a decomposition $\mathbb{K}[G] = \mathbb{K}[G_0] \oplus \mathbb{K}[G_1]$ such that*

$$\mathbb{K}[G_0] \subset \mathbb{A}_0, \qquad \mathbb{A}_1 \subset \mathbb{K}[G_1].$$

Proof. (1) Let $P^{\pm}$ be the two projectors onto the eigenspaces. To see that both eigenspaces are isomorphic, choose a set H of representatives for the $\{e, Z\}$-cosets in G (i.e., a section of π), so that $G = H \sqcup ZH$ as set, whence $\mathbb{K}[G] = \mathbb{K}[H] \oplus Z_* \mathbb{K}[H]$ as $\mathbb{K}$-module. Then $\mathbb{K}[G]^+$ represents the diagonal, and $\mathbb{K}[G]^-$ the antidiagonal in this decomposition, and the restrictions and corestrictions

$$P^+ : \mathbb{K}[H] \to \mathbb{K}[G]^+, \qquad P^- : \mathbb{K}[H] \to \mathbb{K}[G]^-$$

are linear isomorphisms, so both $\mathbb{K}[G]^{\pm}$ are isomorphic to $\mathbb{K}[H]$ as $\mathbb{K}$-modules. (When G is finite, both spaces have dimension $|H| = |G|/2$).

Since $\pi(Z) = \pi(1)$, it follows that $\pi_*(1 - Z) = 0$. Thus the ideal generated by $1 - Z$, which is $P^-(\mathbb{K}[G]) = \mathbb{K}[G]^-$, belongs to the kernel of π_*. On the other hand, the basis $(h + Zh)_{h \in H}$, is sent to the basis $\pi_*(h + Zh) = 2\pi_*(h)$ of $\mathbb{K}[H]$, so the restriction of π_* to $\mathbb{K}[G]^+$ defines a bijection $\mathbb{K}[G]^+ \to \mathbb{K}[H]$, whence $\mathbb{K}[G]^-$ is the kernel of π_*, and π_* induces an isomorphism of $\mathbb{K}[G]^+$ onto $\mathbb{K}[G^0]$.

(2) We have already noticed that the decompositions define superalgebras. To prove the remaining statements, we decompose further $H = H_0 \sqcup H_1$ into even and odd elements, whence $G = H_0 \sqcup ZH_0 \sqcup H_1 \sqcup ZH_1$. Since α is the identity on G_0, we get $\mathbb{K}[G_0] \subset \mathbb{A}_0$. Also, $P^+ \mathbb{K}[H_1] \subset \mathbb{A}_0$, since, if $g \in H_1$, then $\alpha_*(\delta_g + Z_* \delta_g) = \delta_{Zg} + \delta_g = \delta_g + Z_* \delta_g$. It follows that $\mathbb{A}_1 = P^- \mathbb{K}[H_1] \subset \mathbb{K}[G_1]$. $\qquad \square$

4.2. **Graded and ungraded tensor product.** If A, B are sets, the free module with basis $A \sqcup B$ is $\mathbb{K}[A \sqcup B] = \mathbb{K}[A] \oplus \mathbb{K}[B]$, and the one with basis $A \times B$ is $\mathbb{K}[A \times B] = \mathbb{K}[A] \otimes \mathbb{K}[B]$. The tensor product of two associative algebras is again an associative algebra, for the "usual" product $a \otimes b \cdot a' \otimes b' = aa' \otimes bb'$. If $\mathbb{A}, \mathbb{B}$ are $\mathbb{Z}/2\mathbb{Z}$-*graded*, then their *graded tensor product* is the tensor product, as module, together with bilinear product

(4.5) $$a \otimes b \cdot a' \otimes b' = (-1)^{|b| \, |a'|} aa' \otimes bb',$$

where a, b, a', b' are assumed to be *homogeneous* elements. If $\mathbb{A}$ and $\mathbb{B}$ are associative, then this is again an *associative* algebra (cf. [BtD], p. 56).

Theorem 4.2. *Let (G_i, d_i, Z_i), $i = 1, 2$, two centrally $\mathbb{Z}/2\mathbb{Z}$-graded groups, and $G = G_1 \widehat{\times}_Z G_2$. Then the algebras $\mathbb{K}[G]^{\pm}$ are given in terms of ungraded, resp. graded tensor products of algebras by those of G_i, $i = 1, 2$, as*

$$\mathbb{K}[G]^+ = \mathbb{K}[G_1]^+ \otimes \mathbb{K}[G_2]^+, \qquad \mathbb{K}[G]^- = \mathbb{K}[G_1]^- \widehat{\otimes} \mathbb{K}[G_2]^-.$$

In particular, it follows that for $G = Q(\mathbf{t})$, the ideal $\mathbb{A}^-$ is a Clifford algebra: for $\mathbf{t} \in \{1, Z\}^n$, define $\underline{\mathrm{Cl}}(\mathbf{t})$ to be the Clifford algebra of $\mathbb{K}^n$ for the diagonal quadratic form with coefficient 1 if $t_i = 1$ and -1 if $t_i = Z$. Then

$$\mathbb{K}[Q(\mathbf{t})]^+ \cong \mathbb{K}[\mathcal{P}(\mathsf{n})], \qquad \mathbb{K}[Q(\mathbf{t})]^- \cong \underline{\mathrm{Cl}}(\mathbf{t}).$$

In particular, $\mathbb{K}[Q_{p,q}]^- \cong \mathrm{Cl}_{p,q}(\mathbb{K})$.

Proof. Choose H in G, and H_i in G_i, as in the preceding proofs. As above, elements $P^\pm(\delta_h) = \frac{1}{2}(\delta_h \pm Z_* \delta_{h_i})$, $h \in H$, form a basis in $\mathbb{K}[G]^\pm$, and likewise such elements with $h_i \in H_i$ form a basis in $\mathbb{K}[G_i]^\pm$. We compute, in $\mathbb{K}[G]$,

$$\delta_{h_1} \otimes \delta_{h_2} \cdot \delta_{h'_1} \otimes \delta_{h'_2} = \delta_{[h_1,h_2]} \cdot \delta_{[h'_1,h'_2]}$$
$$= Z^{d(h_2)d(h'_1)} * \delta_{[h_1 h'_1, h_2 h'_2]} = Z^{d(h_2)d(h'_1)} * \delta_{h_1 h'_1} \otimes \delta_{h_2 h'_2},$$

and project with P^- onto $\mathbb{K}[G]^-$. Since Z projects to -1, this gives

$$P^-(\delta_{h_1}) \otimes P^-(\delta_{h_2}) \cdot P^-(\delta_{h'_1}) \otimes P^-(\delta_{h'_2}) =$$
$$(-1)^{d(h_2)d(h'_1)} P^-(\delta_{h_1}) P^-(\delta_{h'_1}) \otimes P^-(\delta_{h_2}) P^-(\delta_{h'_2}),$$

so the algebra product is given by the graded tensor product. On $\mathbb{K}[G]^+$, since Z projects to 1, there is no sign change, and we get the usual tensor product.

The statement on the Clifford agebras follows from the preceding statement by induction, using Relation (0.1) for Clifford algebras. (Cf. the arguments given in loc.cit., [BtD], where this is used to prove that the dimension of the Clifford algeba $\mathrm{Cl}(\mathbb{K}^n)$ is 2^n. On the other hand, knowing the dimension of the Clifford algebra, the proof could also be given by noticing that the generators of $\mathbb{A}^-$ satisfy exactly the same relations as the generators of the Clifford algebra $\mathrm{Cl}(\mathbf{t})$.) $\square$

Example 4.1 (Group algebra of Q and of D_4). The preceding arguments can be used to show, in a transparent way, that the Clifford algebra $\mathrm{Cl}_{2,0}$ is isomorphic to $M_2(\mathbb{K})$, and that $\mathrm{Cl}_{0,2}$ is the usual quaternion algebra. Namely, by the preceding results, the group $Q_{p,q}$ is (isomorphic to) the group generated by the basis elements e_i in the Clifford algebra $\mathrm{Cl}_{p,q}(\mathbb{K}) \cong \mathbb{K}[Q_{p,q}]^-$. For instance, the quaternion group $G = Q$ if often defined as group of matrices $\{\pm 1, \pm i, \pm i, \pm k\} \subset \mathbb{H}^* \subset \mathrm{Gl}(2, \mathbb{C})$. Now, modulo -1, these matrices furnish a basis of $\mathbb{H}$, the elements multiply as they should in the group, and the matrix $Z = -1$ acts as it should on $\mathbb{K}[G]^-$ (by the scalar -1): so we can conclude that $\mathbb{K}[G]^- \cong \mathbb{H}$ as algebra, whence $\mathrm{Cl}_{0,2}(\mathbb{R}) \cong \mathbb{H}$. The same argument works for the group $G = D_4$, which we realize as matrix group $\{\pm 1, \pm I, \pm J, \pm K\} \subset \mathrm{Gl}(2, \mathbb{Z})$. Modulo -1, the four matrices give a basis of $M_2(\mathbb{K})$, and $Z = -1$ acts by -1, whence $\mathbb{K}[G]^- \cong M_2(\mathbb{K})$.

Remark 4.1. The fact that Clifford algebras are parts of group algebras, or images of them, is known (cf. [A, AM, S84]), and we hope the framework of graded groups proposed here clarifies those approaches. For instance, the abstract ingredients of the *classification of Clifford algebras* are quite neatly featured in the abstract approach (see Appendix B.3, cf. also [S82] and [L], p.53, for this issue).

4.3. **Harmonic analysis of the group algebra.**
The group $G = Q(\mathbf{t})$ acts on $\mathbb{K}[G]$ in the usual way (left-regular representation). This representation decomposes into sub-representations (ideals) $\mathbb{K}[G] = \mathbb{K}[G]^+ \oplus \mathbb{K}[G]^-$. We'll decompose it further. Since $\mathbb{K}$ is just a ring and not a field, we do not speak about decomposition into "irreducible" modules, but rather determine a basis of the space of *class functions* (which in case of $\mathbb{K} = \mathbb{C}$ correspond to the irreducible characters). Recall that a *class function* is just an element $f \in \mathbb{K}[G]$ commuting with every $g \in G$, that is, an element of the center of $\mathbb{K}[G]$.

Since $\mathbb{K}[G]^+ \cong \mathbb{K}[\mathcal{P}(n)]$, and the group is abelian and each element is of order at most two, a complete decomposition of this algebra is easy:

Proposition 4.3. *The group characters of $\mathcal{P}(n)$ are the maps, for $A \in \mathcal{P}(n)$,*
$$\chi_A : \mathcal{P}(n) \to \{\pm 1\}, \quad B \mapsto \chi_A(B) := (-1)^{|A \cap B|}.$$
Thus $\mathbb{K}[\mathcal{P}(n)] \cong \oplus_{A \in \mathcal{P}(n)} \mathbb{K}\chi_A$ is a decomposition into one-dimensional ideals.

Proof. Since $(A \Delta B) \cap C = (A \cap C) \Delta (B \cap C)$, and $|A \Delta B| \equiv |A| + |B| \mod (2)$,
$$\chi_A(B \Delta C) = (-1)^{|A \cap (B \Delta C)|} = (-1)^{|A \cap B|}(-1)^{|A \cap C|} = \chi_A(B)\chi_A(C),$$
and $\chi_A(\emptyset) = (-1)^0 = 1$, so χ_A is a morphism. Since the map
$$\chi : \mathcal{P}(n) \times \mathcal{P}(n) \to \{\pm 1\}, \quad (A, B) \mapsto \chi_A(B) = (-1)^{|A \cap B|}$$
is symmetric, it follows that
$$\chi_\emptyset = 1, \qquad \chi_{A \Delta B}(C) = \chi_A(C)\chi_B(C),$$
so $A \mapsto \chi_A$ is a group morphism from $\mathcal{P}(n)$ to its dual group. Its kernel is trivial: $\chi_A(X) = 1$ for all X means that $|A \cap X|$ is even, for all X, and taking for X the singletons, if follows that A contains no element, so $A = \emptyset$. Thus the morphism $A \mapsto \chi_A$ is injective. Since characters are always linearly independent, the cardinality of $\widehat{H}$ cannot exceed the one of H, and thus the morphism is a bijection of $\mathcal{P}(n)$ onto its dual group. $\qquad\square$

Remark 4.2. Parametrizing the characters as in the proposition, the *character table* of $G = \mathcal{P}(n)$ is the square matrix
$$(\chi_A(C))_{(A,C) \in \mathcal{P}(n)^2} = ((-1)^{|A \cap C|})_{(A,C) \in \mathcal{P}(n)^2}.$$
A specific feature of these matrices is that they are *symmetric*: $\chi_B(C) = \chi_C(B)$. Explicitly, for $n = 1$, $\mathcal{P}(1) = \{\emptyset, 1\}$, resp. $n = 2$, $\mathcal{P}(2) = \{\emptyset, \{1\}, \{2\}, 2\}$, we get
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Theorem 4.4. *Let $\mathbf{t} \in \{1, Z\}^n$ and $G = Q(\mathbf{t})$, and $\mathbb{A} := \mathbb{K}[G]^G$ the algebra of central $\mathbb{K}$-valued functions on G (center of $\mathbb{K}[G]$). Define for $A \in \mathcal{P}(n)$,*
$$E_A^+ := (e_0 + Z) \sum_{B \in \mathcal{P}(n)} (-1)^{|A \cap B|} e_B.$$

Then $\mathbb{K}[G]^+ \subset \mathbb{K}[G]^G$, and $(E_A)^+$ with $A \in \mathcal{P}(n)$ is a basis of $\mathbb{K}[G]^+$ such that (orthogonal idempotents) Moreover,

   (1) Assume n is even. Then a basis of $\mathbb{K}[G]^G$ is given by the elements $(E_A^+)_{A \in \mathcal{P}(n)}$, along with $e_0^- := (1 - Z)e_0$. The element e_0^- is a basis of the center of the Clifford algebra $\mathbb{K}[G]^-$.
   (2) Assume n is odd. Then a basis of $\mathbb{K}[G]^G$ is given by the elements $(E_A^+)_{A \in \mathcal{P}(n)}$, along with $e_0^- := (1 - Z)e_0$ and $e_n^- := (1 - Z)e_n^-$ (the pseudoscalar). The elements (e_0^-, e_n^-) form a basis of the center of the Clifford algebra $\mathbb{K}[G]^-$.

If $\mathbb{K}$ is a field, then each basis element corresponds to an irreducible representation of G, and each irreducible G-representation arises in this way.

Proof. The dimension of $\mathbb{K}[G]^G$ is equal to the number of conjugacy classes of G, since the characteristic functions of conjugacy classes form a basis. By Theorem 3.6, we know the number of conjugacy classes. We have already exhibited 2^n linearly independent central functions; thus one (resp. two) additional independent elements suffice for defining a basis. Since $\mathbb{K}[G]^G$ is Z-invariant, it decomposes into Z-eigenspaces, so we can choose these additional elements in $\mathbb{K}[G]^-$, i.e., in the Clifford algebra. These elements being central, and since $\mathbb{K}[G]^+ * \mathbb{K}[G]^- = 0$, it is necessary and sufficient that these elements belong to the center of $\mathbb{K}[G]^-$. Now, it is obvious that e_0^-, resp. e_{n}^- (for n odd), satisfy these conditions, and for reasons of dimension, there are no other independent elements satisfying them. This proves (1) and (2). If $\mathbb{K}$ is a field, then by the general theory of representations of finite groups (see [FH]) every irreducible G-representation arises in $\mathbb{K}[G]$. The elements E_A^+ define one-dimensional (hence irreducible) representations; the other elements generate representations that may decompose into direct sums of several irreducibles (cf. the following remark). □

Remark 4.3. To get more information, in case n is odd, a further distinction should be made, according to the square of the pseudoscalar being 1 or Z. When all $t_j = 1$, this corresponds to the cases $n \equiv 1, 5$, resp. $n \equiv 3, 7$ modulo 8. This determines whether the two submodules of dimension bigger than one are isomorphic to each other, or not (cf. [BtD], p. 288, see also [FH], Exercise 3.9, p. 30).

Appendix A. Clifford algebras

A.1. **General definitions.** Let $\mathbb{K}$ be a commutative base ring in which 2 is invertible. To every quadratic space (V, q) is associated, in a functorial way, a unital associative algebra $\mathrm{Cl}(V, q)$, its *Clifford algebra*, together with a linear map $j : V \to \mathrm{Cl}(V, q)$, such that

$$\forall v \in V : \quad (j(v))^2 = q(v)1,$$

and which is universal for this property, see e.g., [BtD, L]. We shall be interested in the case $V = \mathbb{K}^n$, together with a quadratic form $q = q_{\mathbf{t}}$, where $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{K}^n$,

$$q_{\mathbf{t}}(x_1, \ldots, x_n) = \sum_{i=1}^n t_i x_i^2,$$

that is, the canonical basis $e_1, \ldots, e_n$ is orthogonal, and $q(e_i) = t_i$. The corresponding Clifford algebra is then denoted by

$$\mathrm{Cl}_{\mathbb{K}}(\mathbf{t}) = \mathrm{Cl}_{\mathbb{K}}(t_1, \ldots, t_n) = \mathrm{Cl}(V, q_{\mathbf{t}}).$$

When $\mathbf{t} = \mathbf{0} = (0, \ldots, 0)$, this is the exterior algebra: $\mathrm{Cl}(V, q_{\mathbf{0}}) = \wedge(\mathbb{K}^n)$. When $\mathbf{t} = (1, \ldots, 1, -1, \ldots, -1)$, with p terms 1 and q terms -1, we write

$$\mathrm{Cl}_{p,q}(\mathbb{K}) := \mathrm{Cl}(\mathbf{t}) = \mathrm{Cl}(1, \ldots, 1, -1, \ldots, -1).$$

Among these algebras, $\mathrm{Cl}(1, \ldots, 1) = \mathrm{Cl}_{n,0}(\mathbb{K})$ plays a special role: it is *anchored*.

Lemma A.1. *For every* $\mathbf{t} \in \mathbb{K}^n$, *the linear map* $\mathbb{K}^n \to \mathbb{K}^n$, $x \mapsto (t_1 x_1, \ldots, t_n x_n)$ *induces an algebra morphism*

$$\mathrm{Cl}_{\mathbb{K}}(t_1^2, \ldots, t_n^2) \to \mathrm{Cl}_{\mathbb{K}}(1, \ldots, 1).$$

Proof. The linear map is a morphism of quadratic spaces from $(\mathbb{K}(t_1^2, \ldots, t_n^2))$ to $(\mathbb{K}, (1, \ldots, 1))$, and induces, by the universal property, an algebra morphism. $\square$

We call *anchor* the morphism defined in the lemma, and say that the Clifford algebra $\mathrm{Cl}_{\mathbb{K}}(t_1^2, \ldots, t_n^2)$ is *anchored*. Of course, when $t_i \in \{\pm 1\}$, then the only anchored Clifford algebra is $\mathrm{Cl}_{\mathbb{K}}(1, \ldots, 1)$. The terminology is motivated by [BeH]: indeed, the anchored Clifford algebras should be the correct "super-analog" of the anchored tangent algebras defined in loc.cit., and "differential calculus" then is the theory describing the contraction for $\mathbf{t} \to 0$ (which in the graded case still remains to be worked out).

A.2. **Clifford basis.** The standard basis $e_1, \ldots, e_n$ of $\mathbb{K}^n$ defines n independent elements in the Clifford algebra $\mathrm{Cl}_{\mathbb{K}}(\mathbf{t})$. For a subset $A = \{a_1, \ldots, a_k\} \subset \mathsf{n}$, with $a_1 < \ldots < a_n$, we define the element

$$e_A := e_{a_1} \cdots e_{a_k} \in \mathrm{Cl}_{\mathbb{K}}(\mathbf{t}).$$

Theorem A.2. *The elements* $(e_A)_{A \in \mathcal{P}(\mathsf{n})}$ *form a $\mathbb{K}$-basis of the Clifford algebra* $\mathrm{Cl}_{\mathbb{K}}(\mathbf{t})$, *called its* Clifford basis. *The Clifford product of two basis elements is given by*

$$e_A e_B = (-1)^{m(A,B)} \prod_{k \in A \cap B} t_k \cdot e_{A \Delta B},$$

where $e_{\emptyset} = 1$, $A \Delta B$ *is the symmetric difference of* $A, B \subset \mathsf{n}$, *and*

$$m(A, B) = |\{(a, b) \in A \times B \mid a > b\}|.$$

Proof. The dimension of the Clifford algebra is 2^n (cf. [BtD]), and the elements e_A generate it, hence form a basis. To compute the product $e_A e_B$, using the Clifford rules

$$e_k^2 = t_k = t_k e_{\emptyset}, \qquad \forall k \neq \ell : e_k e_{\ell} = -e_{\ell} e_k,$$

we see that $e_A e_B$ is a multiple of $e_{A \Delta B}$. Terms with indices $k = \ell \in A \cap B$ give rise to the scalar $\prod_{k \in A \cap B} t_k$. Terms with indices $(k, \ell) \in A \times B$ have to be exchanged when $k > \ell$ (giving rise to a sign change), and are not altered when $k < \ell$ (the arguments are the same as those of Theorem 3.12). $\square$

Cf. page 12 - 14 of [L] for a similar presentation of the preceding result.

APPENDIX B. CLASSIFICATION OF CLIFFORD GROUPS AND ALGEBRAS

B.1. **Graded and ungraded product.** Both the classification of real Clifford algebras and the one of discrete Clifford groups proceeds by decomposing them into *ungraded* tensor products of elementary algebras, resp. products of groups. On the level of algebras, we have, e.g., $\mathrm{Cl}_{\mathbb{R}}(1, -1, -1) \cong M_4(\mathbb{C}) \cong \mathbb{C} \otimes M_2(\mathbb{R}) \otimes M_2(\mathbb{R})$. On the level of groups, recall the ungraded product of groups $G_1 \times_Z G_2$ (quotient of $G_1 \times G_2$ under $(x_1, Z x_2) \sim (Z x_1, x_2)$, written also $G_1 \times_{C_2} G_2$ when $\Gamma = \mathbb{Z}/2\mathbb{Z}$). We'll

see that all groups $Q(\mathbf{t})$ can be decomposed as ungraded product $\times_{C_2}$ of groups of type

$$C_2, \quad C_2 \times C_2, \quad C_4, \quad D = D_4, \quad Q,$$

corresponding to the decomposition of Clifford algebras into tensor products involving

$$\mathbb{R}, \quad \mathbb{R} \oplus \mathbb{R}, \quad \mathbb{C}, \quad M_2(\mathbb{R}), \quad \mathbb{H}.$$

The key lemma for classifying discrete Clifford groups is:

Lemma B.1. *For all $n \geq 3$ and $\mathbf{t} \in \{1, Z\}^n$, there is a group isomorphism*

$$Q(\mathbf{t}) \cong Q(t_1, t_2) \times_Z Q(Zt_1t_2t_3, \ldots, Zt_1t_2t_n).$$

Proof. Elements of $Q(\mathbf{t})$ are either of the form e_A or Ze_A, with $A \in \mathcal{P}(\mathsf{n})$. When $A \subset \{1, 2\}$, we get the 8 elements of the first factor $Q(t_1, t_2) = \{e_0, e_1, e_2, e_{12}, Z, Ze_1, Ze_2, Ze_{12}\}$. Next, let $H \subset Q(\mathbf{t})$ be the subgroup generated by the elements

$$b_3 := e_1e_2e_3, \quad b_4 := e_1e_2e_4, \quad \ldots, \quad b_n := e_1e_2e_n$$

Elements of $Q(t_1, t_2)$ and H commute: indeed, for all $k = 3, \ldots, n$,

$$e_1b_k = e_1e_1e_2e_k = Z^2e_1e_2e_ke_1 = b_ke_1, \qquad e_2b_k = e_2e_1e_2e_k = Z^2e_1e_2e_ke_2 = b_ke_2,$$

hence $Q(t_1, t_2)$ commutes with the generators of H, hence with H. Moreover, H and $Q(t_1, t_2)$ together generate the group $Q(\mathbf{t})$, and therefore H contains at least 2^{n-1} elements. We compute, for $3 \leq k < \ell$ (using $e_i^2 = t_ie_0$)

$$b_k^2 = e_1e_2e_ke_1e_2e_k = Z^3e_1^2e_2^2e_k^2 = Zt_1t_2t_3e_1e_2e_k = Zt_1t_2t_3\, b_k,$$
$$b_kb_\ell = e_1e_2e_ke_1e_2e_\ell = Ze_1e_2e_\ell e_1e_2e_k = Z\, b_\ell b_k.$$

These are the defining relations of the group $Q(Zt_1t_2t_3, \ldots, Zt_1t_2t_n)$, hence, by Theorem 3.4, there is a surjective morphism from $Q(Zt_1t_2t_3, \ldots, Zt_1t_2t_n)$ to H, which must be an isomorphism for reasons of cardinality. $\square$

In particular, the lemma implies for $(t_1, t_2) = (1, 1)$, or $(1, Z)$, or (Z, Z):

$$Q(1, 1, t_3, \ldots, t_n) \cong Q_{2,0} \times_Z Q(Zt_3, \ldots, Zt_n) \cong D_4 \times_Z Q(Zt_3, \ldots, Zt_n),$$
$$Q(1, Z, t_3, \ldots, t_n) \cong Q_{1,1} \times_Z Q(t_3, \ldots, t_n) \cong D_4 \times_Z Q(t_3, \ldots, t_n),$$
$$Q(Z, Z, t_3, \ldots, t_n) \cong Q_{2,0} \times_Z Q(Zt_3, \ldots, Zt_n) \cong Q \times_Z Q(Zt_3, \ldots Zt_n).$$

From this, the first statements of the following corollary immediately follow:

Corollary B.2. *For all $p, q \in \mathbb{N}$,*

$$Q_{p+2,q} \cong Q_{2,0} \times_Z Q_{q,p} \cong D_4 \times_Z Q_{q,p},$$
$$Q_{p+1,q+1} \cong Q_{1,1} \times_Z Q_{p,q} \cong D_4 \times_Z Q_{p,q},$$
$$Q_{p,q+2} \cong Q_{0,2} \times_Z Q_{q,p} \cong Q \times_Z Q_{q,p}.$$

Applying this several times, using notation $G_Z^3 = G \times_Z G \times_Z G$, etc.,

$$Q_{p+4,q} \cong Q_{2,0} \times_Z Q_{0,2} \times_Z Q_{p,q} \cong D_4 \times_Z Q \times_Z Q_{p,q} \cong Q_{p,q+4},$$
$$Q_{p+2,q+2} \cong D_4 \times_Z D_4 \times_Z Q_{p,q} \cong Q \times_Z Q \times_Z Q_{p,q},$$
$$Q_{p+8,q} \cong Q_{p,q+8} \cong Q_{p+4,q+4} \cong (D_4)_Z^4 \times_Z Q_{p,q} \cong Q_Z^4 \times_Z Q_{p,q}.$$

Moreover, for any $\mathbb{Z}/2\mathbb{Z}$-graded group G,

$$G \times_Z Q_{1,0} = G \times_{C_2} (C_2 \times C_2) \cong G \times C_2.$$

Proof. The last statement is obvious from $(C_2 \times C_2)/C_2 = C_2$. □

Remark B.1. In particular, the corollary gives the group isomorphisms

$$Q_{2,2} \cong D_4 \times_Z D_4 \cong Q \times_Z Q,$$
$$Q_{1,2} \cong C_4 \times_Z D \cong C_4 \times_Z Q.$$

On the level of real Clifford algebras, this corresponds to the algebra isomorphisms $\mathbb{H} \otimes \mathbb{H} \cong M(2,\mathbb{R}) \otimes M(2,\mathbb{R})$ (in turn, this algebra is isomorphic to $M(4,\mathbb{R})$) and $\mathbb{H} \otimes \mathbb{C} \cong M(2,\mathbb{C})$.

Notation. To abbreviate, in the following, we are going to write $D := D_4$, $C := C_4$, $V := C_2^2 := C_2 \times C_2$ (the Klein four-group), and since all direct products are of the form $\times_Z$, we omit this sign, and just write it as juxtaposition, and we write $D^2 = DD$, etc. For instance, $DQ = D \times_Z Q$, $D^2 = D \times_Z D$. The isomorphisms from Remark B.1 thus read $D^2 \cong Q^2$ and $DC_4 \cong QC_4$. Note also that $\times C_2 = \times_Z C_2^2$, so, e.g., $DC_2^2 = D \times_Z C_2^2 = D \times C_2$. Using these isomorphisms and notation, every group $Q_{p,q}$ is isomorphic to one in "normal form"

$$C_2, \quad D^k C_2^2, \quad D^k C_4, \quad D^k Q, \quad Q C_2^2, \quad D^k Q C_2^2.$$

Theorem B.3. *The discrete Clifford groups Q_n, resp. $Q_{0,n}$, are given by*

n	0	1	2	3	4	5	6	7	8
Q_n	C_2	C_2^2	D	DC_4	DQ	DQC_2^2	D^2Q	D^3C_4	D^4
$Q_{0,n}$	C_2	C_4	Q	QC_2^2	DQ	D^2C_4	D^3	$D^3C_2^2$	D^4

and we have the periodicity relations:

$$Q_{n+4} \cong DQ\,Q_n, \quad Q_{n+8} \cong D^4 Q_n, \quad Q_{0,n+4} \cong DQ\,Q_{0,n}, \quad Q_{0,n+8} \cong D^4 Q_{0,n}.$$

Proof. Repeated application of Corollary B.2. □

For sake of completeness, we give in Section B.3 the full "periodic table" of the groups $Q_{p,q}$, together with the corresponding classification of Clifford algebras. Due to the relation $Q_{p+1,q+1} = Q_{p,q} \times_C D_4 = D\,Q_{p,q}$, this table is best presented as a triangle (like Pascal's triangle). This triangle is not symmetric with respect to the "main diagonal" $Q_{n,n}$, but it is symmetric with respect to the "secondary diagonal" $Q_{n+1,n} = C_2 \times (D_4)_Z^n = D^n C_2^2$, by the following lemma:

Lemma B.4. *We have an isomorphism of (ungraded) groups:*

$$Q(t_1, \dots, t_n) \cong Q(t_1, Zt_1t_2, \dots, Zt_1t_n)$$

In particular, there is a group isomorphism $Q_{1+p,q} \cong Q_{1+q,p}$.

Proof. The proof is similar to the one of Lemma B.1: define

$$b_1 := e_1, \quad b_2 := e_1e_2 = e_{12}, \quad \dots \quad b_n := e_{1n}.$$

These elements satisfy the relations $b_1^2 = t_1$, and for $\ell > k \geq 2$,

$$b_k^2 = e_1e_ke_1e_k = Zt_1t_k, \quad b_\ell b_k = e_1e_\ell e_1e_k = Z^3 e_1e_ke_1e_\ell = Zb_kb_\ell,$$

so the group generated by $b_1, \ldots, b_n, Z$ is a quotient of $Q(t_1, Zt_1t_2, \ldots, Zt_1t_n)$. On the other hand, since $b_1b_2 = t_1e_2$, etc., these elements generate $Q(\mathbf{t})$, so for reasons of cardinality, $Q(\mathbf{t}) \cong Q(t_1, Zt_1t_2, \ldots, Zt_1t_n)$. (Another proof is by starting from the isomorphism $Q_2 \cong Q_{1,1}$, and to note that it induces the claimed symmetry.) $\square$

B.2. **The even part.** Recall that the even part G_0 of a $\mathbb{Z}/2\mathbb{Z}$-graded group $G = G_0 \sqcup G_1$ is a (normal) subgroup. If G is finite and the grading non-trivial, then $|G_0| = \frac{1}{2}|G|$. The element Z belongs to G_0, but the grading of G_0 is *a priori* trivial.

Theorem B.5. *Let $n \geq 2$ and $\mathbf{t} \in \{1, Z\}^n$. The even part of $Q(\mathbf{t})$ is isomorphic to*

$$Q(\mathbf{t})_0 \cong Q(Zt_1t_2, \ldots, Zt_1t_n).$$

It follows that

$$(Q_n)_0 \cong Q_{0,n-1} \cong (Q_{0,n})_0, \quad (Q_{1+p,q})_0 \cong Q_{q,p}, \quad (Q_{p,1+q})_0 \cong Q_{p,q}.$$

Explicitly, the even part of Q_n, resp. $Q_{0,n}$, is given by

n	0	1	2	3	4	5	6	7	8
Q_n	C_2	C_2^2	D	DC_4	DQ	DQC_2^2	D^2Q	D^3C_4	D^4
$Q_{0,n}$	C_2	C_4	Q	QC_2^2	DQ	D^2C_4	D^3	$D^3C_2^2$	D^4
$(Q_n)_0 \cong (Q_{0,n})_0$	C_2	C_2	C_4	Q	QC_2^2	DQ	D^2C_4	D^3	$D^3C_2^2$

Proof. For $k = 2, \ldots, n$, define as above the even element $b_k := e_1e_k$. As noted above, $(b_k)^2 = Zt_1t_k$, and, for $i \neq j$,

$$b_ib_j = e_1e_ie_1e_j = Z^2e_1^2e_je_i = Z^3e_1e_je_1e_i = Zb_jb_i,$$

so we get the desired relations for the generators $b_2, \ldots, b_n$ of $Q(\mathbf{t})_0$. For reasons of cardinality we conclude that $Q(\mathbf{t})_0 \cong Q(Zt_1t_2, \ldots, Zt_1t_n)$. Using Theorem B.3, we get the table for the even parts. $\square$

Remark B.2. As said above, *a priori*, the even part is ungraded; but since it is again a Clifford group, it "remembers" the grading of the preceding extensions, and so does $((Q_n)_0)_0$, until we reach C_0, which is ungraded: the table from the theorem gives us a chain of even parts

$$C_2 = (C_4)_0 = ((Q)_0)_0 = (((QC_2^2)_0)_0)_0 = \ldots = ((((((D^4)_0) \ldots)_0.$$

B.3. **The periodic tables of discrete Clifford groups and of Clifford algebras.** Finally, here are the tables classifying discrete Clifford groups and Clifford algebras. We present them in a form akin to Pascal's triangle: for the groups,

$$
\begin{array}{ccccc}
 & & Q_0 & & \\
 & Q_{1,0} & & Q_{0,1} & \\
Q_{2,0} & & Q_{1,1} & & Q_{0,2} \\
 & & \cdots & &
\end{array}
$$

and for the algebras,

$$
\begin{array}{ccccc}
 & & \mathbb{K} & & \\
 & \mathrm{Cl}_{1,0}(\mathbb{K}) & & \mathrm{Cl}_{0,1}(\mathbb{K}) & \\
\mathrm{Cl}_{2,0}(\mathbb{K}) & & \mathrm{Cl}_{1,1}(\mathbb{K}) & & \mathrm{Cl}_{0,2}(\mathbb{K}) \\
 & & \cdots & &
\end{array}
$$

For the following tables, cf. also [S82, S84, L].

Table of groups $Q_{p,q}$ ($p+q$ runs vertically and $p-q$ runs horizontally, notation $V = C_2 \times C_2$, $C = C_4$, $D = D_4$, see above). Note that the column labelled 1 is a symmetry axis of both tables (cf. Lemma B.4).

	8	7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	Cardinality
0									C_2									2
1								V		C								4
2							D		D		Q							8
3						DC		DV		DC		QV						16
4					DQ		D^2		D^2		DQ		DQ					32
5				DQV		D^2C		D^2V		D^2C		DQV		D^2C				64
6			D^2Q		D^2Q		D^3		D^3		D^2Q		D^2Q		D^3			128
7		D^3C		D^2QV		D^3C		D^3V		D^3C		D^2QV		D^3C		D^3V		256
8	D^4		D^3Q		D^3Q		D^4		D^4		D^3Q		D^3Q		D^4		D^4	512

Remark. For very detailed information on the four groups of cardinality 16 listed in line 3 of the table, see [D], page 529 for the group DV, page 530 for DC, and page 531 for QV. (You can recognize them in loc.cit. by looking at the structure of the center, containing the derived group – always of type C_2 – and the quotient with respect to this C_2 – always of type C_2^3). E.g., Aut(DC) is exactly the hyperoctahedral group of order 48, while Aut((QV)) is bigger, of cardinality 192, and Aut(DV) is smaller, of cardinality 32. For the groups $D^2 = D \times_Z D$ and $DQ = D \times_Z Q$ (line 4), cf. loc. cit., p. 626, 627.

Table of classification of Clifford algebras $\mathrm{Cl}_{p,q}(\mathbb{K})$ (notation $\mathbb{C}_\mathbb{K} = \mathbb{K} \oplus i\mathbb{K}$, $\mathbb{H}_\mathbb{K} = \mathbb{K} \oplus i\mathbb{K} \oplus j\mathbb{K} \oplus k\mathbb{K}$, see above): ß

	8	7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8
0									$\mathbb{K}$								
1								$\mathbb{K}^2$		$\mathbb{C}_\mathbb{K}$							
2							$M_2(\mathbb{K})$		$M_2(\mathbb{K})$		$\mathbb{H}_\mathbb{K}$						
3						$M_2(\mathbb{C}_\mathbb{K})$		$M_2(\mathbb{K})^2$		$M_2(\mathbb{C}_\mathbb{K})$		$\mathbb{H}_\mathbb{K}^2$					
4					$M_2(\mathbb{H}_\mathbb{K})$		$M_4(\mathbb{K})$		$M_4(\mathbb{K})$		$M_2(\mathbb{H}_\mathbb{K})$		$M_2(\mathbb{H}_\mathbb{K})$				
5				$M_2(\mathbb{H}_\mathbb{K})^2$		$M_4(\mathbb{C}_\mathbb{K})$		$M_4(\mathbb{K})^2$		$M_4(\mathbb{C}_\mathbb{K})$		$M_2(\mathbb{H}_\mathbb{K})^2$		$M_4(\mathbb{C}_\mathbb{K})$			
6			$M_4(\mathbb{H}_\mathbb{K})$		$M_4(\mathbb{H}_\mathbb{K})$		$M_8(\mathbb{K})$		$M_8(\mathbb{K})$		$M_4(\mathbb{H}_\mathbb{K})$		$M_4(\mathbb{H}_\mathbb{K})$		$M_8(\mathbb{K})$		
7		$M_8(\mathbb{C}_\mathbb{K})$		$M_4(\mathbb{H}_\mathbb{K})^2$		$M_8(\mathbb{C}_\mathbb{K})$		$M_8(\mathbb{K})^2$		$M_8(\mathbb{C}_K)$		$M_4(\mathbb{H}_\mathbb{K})^2$		$M_8(\mathbb{C}_\mathbb{K})$		$M_8(\mathbb{K})^2$	
8	$M_{16}(\mathbb{K})$		$M_8(\mathbb{H}_\mathbb{K})$		$M_8(\mathbb{H}_\mathbb{K})$		$M_{16}(\mathbb{K})$		$M_{16}(\mathbb{K})$		$M_8(\mathbb{H}_\mathbb{K})$		$M_8(\mathbb{H}_\mathbb{K})$		$M_{16}(\mathbb{K})$		$M_{16}(\mathbb{K})$

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