Algebraic structures of Makarevič spaces. I.

Wolfgang BERTRAM
Technische Universität Clausthal
Institut für Mathematik
Erzstraße 1
D - 38678 Clausthal-Zellerfeld

Abstract. — We generalize features of bounded symmetric domains to the *Makarevič spaces* introduced in [Be96b]: we associate a generalized *Bergman operator* to such a space and describe the invariant pseudo-metric and the invariant measure on the space by means of this family of operators (Th.2.4.1). Moreover, the space itself can be characterized essentially as the domain in a Jordan algebra where the generalized Bergman operator is non-degenerate (Th.2.1.1). We give an application of these results to the theory of *compactly causal symmetric spaces*: we describe explicitly the complex domain Ξ associated to such a space (Th.3.3.5). This result will be important for the analysis of the Hardy space on Ξ.

0. Introduction

0.1 Makarevič spaces. — In [Be96b] we have defined *Makarevič spaces*: these are symmetric spaces $X = G/H$ which have a special realization, namely they can be realized as open symmetric orbits in the *conformal compactification of a (semi-simple) Jordan algebra.* Such spaces have been classified by B.O. Makarevič ([Ma73]); examples are (*cf.* [Be96b, section 2.2]): the groups $Gl(n, \mathbb{F})$ (where $\mathbb{F}$ is the field of real or complex numbers or the skew-field of quaternions), the groups $U(p, q)$, $Sp(n, \mathbb{R})$, $SO(2n)$ and $SO^*(2n)$, furthermore symmetric cones, their non-convex analogues, and Hermitian and pseudo-Hermitian spaces of tube type.

We develop in this work an algebraic theory for these spaces which is modelled on the approach to Hermitian symmetric spaces by Jordan theory. Our basic references are the lecture notes by Loos [Lo77] and the book by Satake [Sa80]. However, our theory is self-contained because the use of our Liouville theorem ([Be96a]) simplifies our approach compared to the axiomatic viewpoint of Loos and Satake. In particular, we will not use the axiomatic theory of Jordan triple systems although there is a close relation between Makarevič spaces and Jordan triple systems which we are going to explain in [Be96c].

0.2 The Bergman operator associated to a Makarevič space. Any bounded symmetric domain D can be equipped whith its *Bergman metric* and becomes thus a *Riemannian symmetric space*. When the domain is *circled* (i.e. stable by the rotations $e^{i\varphi}$), the Bergman metric can be written in the form $h_z(u,v) = h_0(B_D(z,z)^{-1}u,v)$, where z is in D and u,v are in the vector space V in which D lies and which is canonically identyfied with the tangent space of D at z. It is known that then the *Bergman operator* B_D is given by a polynomial

$$B_D : V \times V \to End(V),$$

and there is a simple formula for this polynomial in terms of the *Jordan triple product associated to* D, see [Lo77]. Furthermore, one can describe D algebraically by means of this polynomial; namely D is the connected component of 0 of the set

$$\{x \in V |\ \det B_D(x,x) \neq 0\};$$

it is also the zero-component of the set where $B_D(x,x)$ is positive definite. It is furthermore known that the points where $B_D(x,x)$ is non-degenerate but indefinite belong to *pseudo-Hermitian symmetric spaces* which are a sort of "indefinite metric versions of D".

We will generalize these facts for Makarevič spaces. For these general spaces we cannot carry over directly the definition of the Bergman metric of a bounded symmetric domain since no analogue of the Bergman space is in view. For this reason we go the other way round: there is a natural polynomial generalizing the polynomial B_D mentioned above, and we will see that due to its very pleasant transformation properties we can describe our spaces as the set of points where this polynomial has non-degenerate values. More precisely: the Makarevič space $X = G/H$ is realized in the *conformal compactification* V^c of some real or complex *(semi-simple) Jordan algebra* V, and it is labelled by a certain endomorphism α of V; we write $X = X^{(\alpha)}$, *cf.* [Be96b]. For example, bounded symmetric domains are obtained when V is complex and α is a conjugation of V with respect to a Euclidean real form. In section 1.3 we define a polynomial

$$B_\alpha : V \times V \to End(V)$$

and show that the G-orbit of a point $x \in V$ is open in V^c if and only if $B_\alpha(x,x)$ is non-degenerate (Th.2.1.1). This condition thus describes the points of $X^{(\alpha)}$ which do not ly "at infinity", i.e. not in $V^c \setminus V$:

$$(X^{(\alpha)} \cap V)_0 = \{x \in V |\ \det B_\alpha(x,x) \neq 0\}_0,$$

where the subscript 0 means "connected component of 0". We also get a formula for the invariant pseudo-metric $h^{(\alpha)}$ of $X^{(\alpha)}$:

$$h_x^{(\alpha)}(u,v) = h_0^{(\alpha)}(B_\alpha(x,x)^{-1}u,v); \qquad h_0^{(\alpha)}(u,v) = \operatorname{tr} L(u\alpha(v)),$$

where $L(w)$ is the operator of left multiplication by w in the Jordan algebra V (Th.2.4.1). From this one deduces that the density of the invariant measure on $X^{(\alpha)}$ with respect to the Lebesgue measure on V is given by the function

$$|\det B_\alpha(x,x)|^{-1/2}.$$

In the case when α is an involution of V, there exists also a *Cayley-transformed* realization of $X^{(\alpha)}$, and we can describe it in a similar way. The Cayley-transformed spaces turn out to be *tube domains* (Cor.2.1.3).

The tools needed to obtain these results are developed in the first chapter which we call "Algebraic differentiation". They are resumed in the formulas 1.3 (1) – (13) which are algebraic identities involving conformal transformations, the generalized Bergman operator and the quadratic representation. These formulas are a natural further development of the approach to the conformal group by the generalized LIOUVILLE-theorem [Be96a]: there we showed that for any conformal map ϕ, the differential $D\phi$ defines a *polynomial* χ_ϕ by the formula

$$\chi_\phi(x) = (D\phi^{-1}(x))^{-1}.$$

From this we deduced that conformal maps are actually rational, and we can differentiate conformal maps by *algebraic* methods. We already used this fact in [Be96a, Th.2.2.1] where we introduced an analogue of the Bergman operator in an even more general context. The formulas given in section 1.3 come out of the fact that in the Jordan algebra case the general Bergman operator from [Be96a] has additional symmetry properties related to the symmetry of the conformal Lie algebra given by the Jordan inverse $j(x) = x^{-1}$; the formula

$$\chi_j(x) = -P(x)$$

(where P is the *quadratic representation* of the Jordan algebra) is therefore of fundamental importance. In the case of a bounded symmetric domain our formulas are known by the work of Loos [Lo77], and the general formulas could have been derived in a similar way from the theory of *Jordan triple-systems*. However, our proofs are much simpler because we have, by the generalized LIOUVILLE-theorem, the conformal group at our disposition from the beginning on, whereas the axiomatic theory of Jordan triple-systems reaches this stage only much later.

We wish to emphasize that the methods we use are essentially *algebraic* in nature. We have in mind their generalization to other base fields than the fields of real or complex numbers and also to the case of arbitrary, non semi-simple Jordan algebras.

0.3 Compactly causal symmetric spaces and complex domains of semigroup actions. — Besides Hermitian and pseudo-Hermitian symmetric spaces, another interesting class of Makarevič spaces is given by the class of *causal symmetric spaces* we have described in [Be96b, Th.2.4.1]. These fall into two subclasses: the spaces which are *compactly causal* and those which are *non-compactly causal*; the former can be written as $X^{(-\alpha)}$ and the latter as $X^{(\alpha)}$, where α is an involution of a Euclidean Jordan algebra V. It may happen that $X^{(\alpha)}$ and $X^{(-\alpha)}$ are isomorphic to each other (namely when α is the automorphism interchanging factors in a direct product $V \times V$; see [Be96b, Ex.2.2.10]); such spaces are called of *Cayley type*, and they have already been investigated by Jordan methods, see [Cha95].

To a compactly causal symmetric space $X = G/H$, Hilgert, Ólafsson and Ørsted associated a domain $\Xi = G\exp(iC) \cdot x_0$ in the complexification $X_{\mathbb{C}}$ of X on which a semigroup $G\exp(iC)$ acts holomorphically, see [HOØ91]. For the special case of Cayley-type spaces M.Chadli ([Cha95]) described this domain by Jordan methods and Ólafsson and Ørsted ([OØ96]) by methods based on representation theory, and in other special cases K.Koufany and B.Ørsted ([KØ95]) gave similar descriptions. We generalize these results in the following way: let $X^{(-\alpha)}$ be a Makarevič space which is compactly causal; it is realized in the conformal compactification V^c of the Euclidean Jordan algebra V. By definition, Ξ is a domain in $X_{\mathbb{C}}$; it contains the space X in its boundary and there locally looks like the tube domain $T_\Omega = V + i\Omega \subset V_{\mathbb{C}}$, where Ω is the symmetric cone associated to V (Lemma 3.3.1). In other terms, Ξ can locally, near the "boundary" $X^{(-\alpha)}$, be described as

$$\Xi = X_{\mathbb{C}}^{(-\alpha)} \cap T_\Omega.$$

Our main result (Th. 3.3.4) is, that this description is *globally* true. If we combine this result with the algebraic description of $X_{\mathbb{C}}^{(-\alpha)}$ by means of the associated Bergmann operator $B_{-\alpha}$, we obtain (Th. 3.3.5)

$$\Xi = \{z \in T_\Omega |\ \det B_{-\alpha}(z, z) \neq 0\}.$$

Another geometrically important realization of this domain is given by the *Cayley transform* C which is a birational map transforming the tube T_Ω onto the disc $D = C(T_\Omega)$. In this realization, our result reads

$$C(\Xi) = \{z \in D |\ \det P(z + \alpha(z)) \neq 0\},$$

where P is the quadratic representation of the Jordan algebra. These results confirm and precise conjectures stated in [HOØ91, Remark 5.1]. However, as pointed out in [Be96b, section 0.5], one should keep in mind that these results hold only for spaces which can be realized as Makarevič spaces; there are other compactly causal symmetric spaces (the group $SU(p,q)$, for example) which do not have such a realization. Correspondingly, the spaces having a Makarevič space-realization seem to be much better accessible to harmonic analysis. The results and methods developed here will allow an explicit harmonic analysis of the Hardy spaces associated to these spaces (introduced in [HOØ91]).

Acknowledgements. The author would like to thank the Mittag-Leffler institute for hospitality during the program "Harmonic Analysis on Lie groups" when this work was carried out, and to Jacques Faraut, Gestur Ólafsson and Bent Ørsted for helpful discussions.

1. Algebraic Differentiation

1.0 Jordan algebras. — We will use notations related to Jordan algebras as in [Be96a] and in [Be96b]: the *structure group* of a Jordan algebra V (with unit element e) is denoted by $Str(V)$; it can be defined as the group of elements $g \in Gl(V)$ such that jgj again belongs to $Gl(V)$ where $j(x) = x^{-1}$ is a birational map, the *Jordan inverse.* The orbit $\Omega := Str(V)_0 \cdot e$ is open in V and carries the structure of a symmetric space; it is a *symmetric cone* if V is *Euclidean* (in this case we use most notations as in [FK94].)

The *conformal* or *Kantor-Koecher-Tits group* $Co(V)$ of the Jordan algebra V is the group of birational maps generated by the translations τ_v by vectors $v \in V$, elements of $Str(V)$ and the Jordan inverse j. The Lie algebra of $Co(V)$ is denoted by $\mathfrak{co}(V)$; it is a graded Lie algebra of (quadratic) polynomial vector fiels on V, sometimes also called the *Kantor-Koecher-Tits algebra* of V. The *conformal compactification of* V is an open dense imbedding $V \hookrightarrow V^c$ where V^c is the quotient space $Co(V)/P$ with respect to the (parabolic) subgroup P generated by $Str(V)$ and $j\tau_V j$ and the imbedding is given by $v \mapsto \tau_v P$. The characterization of $Co(V)$ as the group of $Str(V)$-conformal local diffeomorphisms ([Be96a]) is the starting point of the present work.

1.1 The "automorphy factors" associated to the conformal group. — Let V be a semi-simple Jordan algebra (having no ideal isomorphic to $\mathbb{R}$ or $\mathbb{C}$) and $G := Str(V) \subset Gl(V)$. A *conformal map of* V is a locally defined diffeomorphism ϕ such that the differential $D\phi(x)$ of

ϕ at x belongs to G for all x where ϕ is defined. By [Be96a, 2.1.3],

$$\chi_\phi : V \mapsto End(V), \quad x \mapsto (D\phi^{-1}(x))^{-1}$$

extends from the domain of ϕ^{-1} to a (quadratic) polynomial on V; following [Sa80] we will call it an *automorphy factor*. For example, if $\phi = \tau_v$ (translation by $v \in V$), then $\chi_{\tau_v}(x) = id_V$; if $\phi = g \in G$, then $\chi_g(x) = g$. A very important example is given by $\phi = j$, where $j(x) = x^{-1}$ is the *Jordan inverse*: then j is conformal, and

$$\chi_j(x) = -P(x) \quad (i.e.,\ Dj(x)^{-1} = -P(x)),$$

where $P(x)w = 2x(xw) - x^2 w$ denotes the *quadratic representation* of V, see [FK94, Prop II.3.3]. Since the conformal group $Co(V)$ is generated by the previously mentioned transformations, the following lemma permits us to calculate all automorphy factors:

1.1.1 Lemma. — *The automorphy factors satisfy the* cocycle relations

$$\chi_{\phi\circ\psi}(x) = \chi_\phi(x) \cdot \chi_\psi(\phi^{-1}(x)),$$
$$\chi_{\phi^{-1}}(x) = (\chi_\phi(\phi^{-1}(x))^{-1}.$$

Proof. — Chain rule. ∎

Remark. The automorphy factors can be defined in the more general context of any *Lie algebra of finite type*, see [Be96a, 2.1.3].

1.2 The "Bergman operator" associated to a Jordan algebra. The automorphy factors associated to the abelian subgroup $j\tau_V j$ of $Co(V)$ define the *Bergman operator* B of the Jordan algebra V:

$$B(x,v) := (D(j\tau_{-v}j\tau_x)(0))^{-1} = (D(j\tau_{-v}j)(x))^{-1}$$

$$= \chi_{j\tau_v j}(x) = \chi_j(x) \cdot \chi_{\tau_v j}(j(x)) = P(x)P(j(x) - v);$$

in the last line we used the cocyle relations. The function $B(x,v)$ is polynomial (quadratic) in x because $B(\cdot, v)$ is an automorphy factor. We will show now that $B(x,v)$ is also polynomial (quadratic) in v. In fact, this is a special case of the more general considerations in [Be96a, section 2.3]. Let us briefly recall the argument used there: from the relation $\frac{d}{dt}|_{t=0} j(tv + j(x)) = -P(x)v$ we see that the infinitesimal generator of the one-parameter group $j\tau_{-tv}j$ is the homogeneous quadratic vector field $\xi(x) := (j_* v)(x) = -P(x)v$; hence we will write $j\tau_{-v}j = \exp(j_* v)$, and then

$$B(x,v) = \chi_{\exp(-j_* v)}(x) = (D(\exp(j_* v))(x))^{-1}.$$

In order to give an explicit formula for this expression, we define $(u \,\square\, v) \cdot x := \frac{1}{2}[v, j_* u](x) = \frac{1}{2}((v(ux) - u(vx) + (vu)x)$.

1.2.1 Proposition. — *The function B extends to a polynomial in both variables, given by*

$$B(x,y) = id - 2x \,\square\, y + P(x)P(y).$$

Proof. — Recall that the natural action of a diffeomorphisms ϕ on the space of vector fields on V is given by the formula

$$(\phi_* \eta)(x) = (D\phi)(\phi^{-1}(x)) \cdot \eta(\phi^{-1}(x)),$$

and for $\phi = \exp \xi$ with $\xi \in \mathfrak{co}(V)$ we have

$$\chi_{\exp \xi}(x) v = ((\exp \xi)_* v)(x) = (e^{ad(\xi)} v)(x).$$

(*cf.* [Be96a, section 2.2]). Following the calculation given in [Be96a] we obtain

$$\begin{aligned} B(x,y)v &= ((\exp(j_* y))_* v)(x) = (e^{ad(j_* y)} \cdot v)(x) \\ &= v + [j_* y, v](x) + \frac{1}{2}[j_* y, [j_* y, v]](x) \\ &= v - 2(v \,\square\, y)(x) + [v \,\square\, y, j_* y](x) \\ &= (id - 2x \,\square\, y + P(x)P(y)) \cdot v. \end{aligned}$$

The last equality is obtained by observing that $[v \,\square\, y, j_* y] = -j_*[y \,\square\, v, y] = j_*(P(y)v)$. ◨

We will hardly use the explicit formula just proved. More important are the symmetry relations of the function B and its behaviour under the conformal group:

1.2.2 Lemma. — *(i) For all $\phi \in Co(V)$ with $\phi(0) \in V$ and $j\phi j(0) \in V$,*

$$D(j\phi^{-1} j)(0) = j(D\phi(0))^{-1} j.$$

(ii) If $\det B(x,y) \neq 0$, then

$$jB(y,x)^{-1} j = B(x,y),$$

and $B(x,y) \in Str(V)$.

Proof. — (*i*) By [Be96, Th.2.1.4], we can decompose any $\phi \in Co(V)$ with $\phi(0) \in V$ uniquely as $\phi = vgn$ with $v = \tau_{\phi(0)} \in V$, $g = D\phi(0) \in Str(V)$ and $n = j\tau_{-(j\phi^{-1}j)(0)} j \in jVj$. Then if $j\phi j(0) \in V$,

$$j\phi^{-1} j = jn^{-1} j \, jg^{-1} j \, jv^{-1} j,$$

and by uniqueness of this decomposition, the middle factor, being an element of $Str(V)$, is equal to $D(j\phi^{-1}j)(0)$. This was to be shown.

(ii) Using the definition of B and (i),

$$jB(y,x)^{-1}j = jD(j\tau_{-x}j\tau_y)(0)j = (D(j\tau_{-y}j\tau_x)(0))^{-1} = B(x,y).$$

The last claim should be seen as a consequence of [Be96a, Th.2.3.1], but it can also be deduced immediately from the formula just proved because $Str(V) = \{g \in Gl(V) |\, jgj \in Gl(V)\}$. ∎

Remark that part (i) also implies that

$$jP(x)^{-1}j = -jD(j\tau_x)(0)j = -(D(j\tau_{-x})(0))^{-1} = P(x).$$

From this and part (ii) we then get another formula for $B(x,y)$:

$$B(x,y) = jB(y,x)^{-1}j = j(P(y)P(j(y)-x))^{-1}j = P(j(y)-x)P(y).$$

1.2.3 Proposition. — *The Bergman operator has the following behaviour under the action of the conformal group: for all $\phi \in Co(V)$ and $z, w \in V$ such that $\phi(z) \in V$ and $j\phi j(w) \in V$,*

$$B\big(\phi(z), j\phi j(w)\big) = D\phi(z) \circ B(z,w) \circ j\big(D(j\phi j)(w)\big)^{-1} j.$$

Proof. — If $\psi \in Co(V)$ and $x \in V$ such that $\psi(x) \in V$, then we have the decomposition $\psi\tau_x = \tau_{\psi(x)}\, D\psi(x)\, n$ with $n \in N = jVj$ [Be96a], and hence

$$\tau_{\psi(x)} = \psi\, \tau_x\, n^{-1}\, (D\psi(x))^{-1}.$$

Using this expression, we obtain

$$\begin{aligned} B\big(\phi(z), j\phi j(w)\big)^{-1} &= D\big(j\tau^{-1}_{j\phi j(w)} j\tau_{\phi(z)}\big)(0) \\ &= D\big(j\, D(j\phi j)(w)\, n'\tau_{-w} j\phi^{-1} j\, j\phi\tau_z n^{-1}(D\phi(z))^{-1}\big)(0) \\ &= jD(j\phi j)(w)j\ D\big(jn'j\, j\tau_{-w} j\tau_z n^{-1}\big)(0)\ (D\phi(z))^{-1} \\ &= jD(j\phi j)(w)j\ D(j\tau_{-w} j\tau_z)(0)\ (D\phi(z))^{-1}. \end{aligned}$$

Taking inverses on both sides yields, by definition of $B(z,w)$, the claim. (In the last equality we used that $jn'j$, coming from the left, and n^{-1}, coming from the right, do not contribute to the differential at the origin.) ∎

1.2.4 Corollary. — (i) *For all $\phi \in Co(V)$ such that $\phi(0) \in V$, $j\phi j(0) \in V$,*

$$B\big(\phi(0), j\phi j(0)\big) = D\phi(0) \circ D(\phi^{-1})(0).$$

(ii) For all ϕ in the fixed point group $Co(V)^{j_*}$ of the involution $j_*(g) = jgj$,

$$B\big(\phi(z), \phi(w)\big) = D\phi(z) \circ B(z,w) \circ j(D\phi(w))^{-1} j.$$

(iii) If $\phi \in Co(V)^{j_*}$ is such that $(-id)\phi(-id) = \phi^{-1}$, then

$$B\big(\phi(0), \phi(0)\big) = ((D\phi)(0))^2.$$

Proof. — The first two claims follow immediately from the proposition. For the last one, observe that $(D\phi^{-1})(0) = D((-id)\phi(-id))(0) = D\phi(0)$. □

In the special case $\phi = \tau_x$ or $\phi = j\tau_x j$, Prop. 1.2.3 yields identities for the B-operators which correspond to the identities $JP\,33$ and $JP\,34$ in [Lo77]. We now prove an analogue of [Lo77, Th.8.11] which describes the conformal group by generators and relations; it can be seen as an "integration" of the defining relation $(JT2)$ of a Jordan triple-system. We first need a lemma:

1.2.5 Lemma. — *Let $x, y \in V$. Then*

$$(j\tau_y j)(x) \in V \Leftrightarrow \det B(y, -x) \neq 0.$$

Proof. — For any conformal map ϕ and $x \in V$,

$$\phi^{-1}(x) \in V \Leftrightarrow \det(\chi_\phi(x)) \neq 0$$

because $\phi^{-1}(x) = (\chi_\phi(x))^{-1} \cdot \lambda_\phi(x)$ with a polynomial λ_ϕ, see [Be96a, Prop.2.1.3]. The lemma follows by taking $\phi = j\tau_{-y}j$ and using the definition of B. □

1.2.6 Proposition. — *If $\det B(u,v) \neq 0$, then*

$$j\tau_{-(j\tau_{-u}j)(v)}j \circ \tau_{-u} \circ j\tau_v j \circ \tau_{(j\tau_{-v}j)(u)} = B(u,v).$$

Proof. — We first notice that the left hand side is well defined: the translations appearing in the formula exist by the Lemma just proved. Let us denote the left hand side by ϕ, $\phi \in Co(V)$. Then $\phi(0) = 0$; in fact this equation is equivalent to $(j\tau_{-v}j)(j\tau_v j)(u) = u$ which is trivially verified. As $\phi(0) = 0$, we can write $\phi = D\phi(0) \circ n$ with $n = j\tau_{-j\phi^{-1}j(0)}j$. Let us show that $n = id_V$, that is, $j\phi^{-1}j(0) = 0$. As above, this equation is seen to be equivalent to $v = (j\tau_u j)(j\tau_{-u} j)(v)$, which is again trivially verified. Finally,

$$(D\phi(0))^{-1} = (D\phi^{-1})(0) = (D(j\tau_{-v} j\tau_u))(0) = B(u,v)^{-1},$$

implying that $\phi = B(u,v)$. □

Specializing the proposition to the case $u = v =: x$ and composing by j we obtain that, if $\det B(x,x) \neq 0$, then

$$B(x,x)j = n \circ \tau_{-x} \circ j \circ \tau_x \circ n^{-1},$$

where $n = j\tau_{-(j\tau_{-x}j)(x)}j$. This means that j and $B(x,x)j$ are conjugate in $Co(V)$.

1.3 The modified Bergman operator. — In this section we assume that $\alpha \in Gl(V)$ is given such that $(j\alpha)^2 = id_V$. In the notation of [Be96b, Prop.2.2.1], $\alpha \in Str(V)^{Jj_*} = \{g \in Str(V) |\, jgj = g^{-1}\}$. We will show that all formulas given in the previous section hold with j replaced by $j\alpha$. The differential of $j\alpha$ is given by

$$D(j\alpha)(x) = Dj(\alpha x)\circ\alpha = -P(\alpha x)^{-1}\circ\alpha = -(\alpha P(x)\alpha)^{-1}\alpha = -(P(x)\alpha)^{-1},$$

$$\text{i.e.}: \quad \chi_{j\alpha}(x) = -P(x) \circ \alpha,$$

where we have used that $P(gx) = jg^{-1}jP(x)g$ for all $g \in Str(V)$. We will write $P_\alpha(x) := P(x) \circ \alpha$. Define the *$\alpha$-modified Bergman operator* by

$$B_\alpha(x,y) := \chi_{j\alpha\tau_y j\alpha}(x) = (D(j\alpha\tau_{-y} j\alpha)(x))^{-1}$$

$$= (D(j\tau_{-\alpha(y)} j\tau_x)(0))^{-1} = B(x, \alpha y),$$

where now $B = B_{id}$. Using the above formula for the differential of $j\alpha$, all proofs in the previous section go through for j replaced by $j\alpha$. We give the resulting formulas; here ϕ always denotes an element of $Co(V)$, arguments of ϕ are always supposed to be chosen such that their image under ϕ belongs to V and $Co(V)^{j_*\alpha_*}$ denotes the fixed point group of the involution $j_*\alpha_*(g) = j\alpha g j\alpha$:

$$
\begin{array}{ll}
(1) & D(j\alpha\,\phi^{-1}\,j\alpha)(0) = j\alpha\,(D\phi(0))^{-1}\,j\alpha, \\
(2) & j\alpha\,B_\alpha(y,x)^{-1}\,j\alpha = B_\alpha(x,y), \\
(3) & j\alpha P_\alpha(x) j\alpha = P_\alpha(x)^{-1} \\
(4) & B_\alpha(x,y) = P_\alpha(x)\,P_\alpha(j\alpha(x)-y) \\
(5) & \qquad\qquad = P_\alpha(j\alpha(y)-x)\,P_\alpha(y) \\
(6) & \qquad\qquad = id - 2x \,\square\, \alpha(y) + P_\alpha(x)P_\alpha(y) \\
(7) & B_\alpha\big(\phi(z), j\alpha\phi j\alpha(w)\big) = D\phi(z)\circ B_\alpha(z,w)\circ j\alpha\,(D(j\alpha\phi j\alpha)(w))^{-1}\,j\alpha, \\
(8) & B_\alpha\big(\phi(0), j\alpha\phi j\alpha(0)\big) = D\phi(0)\,(D\phi^{-1})(0), \\
(9) & \forall \phi \in Co(V)^{j_*\alpha_*}: \\
 & \qquad\qquad B_\alpha(\phi(z),\phi(w)) = D\phi(z)\,B_\alpha(z,w)\,j\alpha(D\phi(w))^{-1}j\alpha, \\
(10) & \forall \phi \in Co(V)^{j_*\alpha_*} \text{ s.t. } (-id)\phi(-id) = \phi^{-1}: \\
 & \qquad\qquad B_\alpha\big(\phi(0),\phi(0)\big) = (D\phi(0))^2, \\
(11) & \det B_\alpha(u,v) \neq 0 \Rightarrow \\
 & \qquad j\alpha\tau_{-(j\alpha\tau_{-u}j\alpha)(v)}j\alpha \,\circ\, \tau_{-u} \,\circ\, j\alpha\tau_v j\alpha \,\circ\, \tau_{(j\alpha\tau_{-v}j\alpha)(u)}v = B_\alpha(u,v), \\
(12) & \det B_\alpha(x,x) \neq 0 \Rightarrow \text{ for } n := j\alpha\tau_{-(j\alpha\tau_{-x}j\alpha)(x)}j\alpha, \\
 & \qquad j\alpha B_\alpha(x,x)^{-1} = n\circ\tau_{-x}\circ j\alpha\circ\tau_x\circ n^{-1}.
\end{array}
$$

Remark. We will see in [Be96c] that P_α and B_α are actually associated to a certain Jordan triple product T_α.

2. Description of the Makarevič spaces $X^{(\alpha)}$

2.1 Description by algebraic equations. — Let us recall from [Be96b, Prop.2.2.1] the definition of the *Makarevič space* $X^{(\alpha)}$: As usual, V denotes a (semi-simple) Jordan algebra, $Co(V)$ its conformal group and V^c its conformal compactification. Let $\alpha \in Str(V)$ be such that $(j\alpha)^2 = id_V$, i.e. $\alpha \in Str(V)^{Jj_*} = \{g \in Str(V) |\, jgj = g^{-1}\}$ (here $J(g) = g^{-1}$ is an anti-automorphism of $Str(V)$); then the conjugation $(j\alpha)_*\phi = j\alpha\phi j\alpha$ defines an involution of $Co(V)$ the fixed point group of which we denote by $Co(V)^{(j\alpha)_*}$. Then ([Be96b, Prop.2.2.1]) the orbit

$$X^{(\alpha)} := Co(V)_0^{(j\alpha)_*} \cdot 0 \subset V^c$$

is open and is a symmetric space having $-id_V$ as geodesic symmetry with respect to the origin 0.

We shall characterize the spaces $X^{(\alpha)}$ essentially as the set of points x for which the Bergmann operator $B_\alpha(x,x)$ is non-degenerate. As mentioned in the introduction, for Hermitian symmetric spaces such a description is well-known. However, our proof is of quite different nature than the known one for Hermitian symmetric spaces - we do not use the exponential map, but the fact that, if $\det B_\alpha(x,x) \neq 0$, then we have an algebraic formula establishing conformal equivalence of the orbit of x whith one of our model spaces, namely with $X^{(\alpha B_\alpha(x,x)^{-1})}$. In Jordan theory a similar phenomenon is known as *mutation*. Our proof, using essentially only "affine", not "semi-simple" arguments, can be adapted to the more general situation of arbitrary Jordan algebras.

2.1.1 Theorem. — *(i) If $x \in V$ is such that $\det B_\alpha(x,x) \neq 0$, then the orbit*

$$Co(V)_0^{(j\alpha)_*} \cdot x \subset V^c$$

is open, and as a homogeneous and symmetric space it is isomorphic to

$$X^{(\alpha B_\alpha(x,x)^{-1})} = Co(V)_0^{(j\alpha B_\alpha(x,x)^{-1})_*} \cdot 0 \subset V^c.$$

(ii) The intersection $X^{(\alpha)} \cap V$ is a union of connected components of the set $\{x \in V | \det B_\alpha(x,x) \neq 0\}$. In particular,

$$(X^{(\alpha)} \cap V)_0 = \{x \in V | \det B_\alpha(x,x) \neq 0\}_0.$$

(iii) If V is complex and α complex-linear, then $X^{(\alpha)}$ is dense in V^c, and

$$X^{(\alpha)} \cap V = \{x \in V | \det B_\alpha(x,x) \neq 0\}.$$

Proof. — (i) If $\det B_\alpha(x,x) \neq 0$, then $n := j\alpha\tau_{-(j\alpha\tau_{-x}j\alpha)(x)}j\alpha$ is a well-defined element of the conformal group (*cf.* Lemma 1.2.5). We have $n\tau_{-x}(x) = n(0) = 0$, and by eqn. 1.3. (12):

$$n\tau_{-x}(Co(V)_0^{(j\alpha)_*} \cdot x) = Co(V)_0^{(n\tau_{-x}j\alpha\tau_x n^{-1})_*} \cdot 0 = Co(V)_0^{(j\alpha B_\alpha(x,x)^{-1})_*} \cdot 0.$$

The proof is finished by showing that this orbit is open. Now, due to 1.3. (2), $(j\alpha B_\alpha(x,x)^{-1})^2 = id_V$; that is $\alpha B_\alpha(x,x)^{-1} \in Str(V)^{Jj_*}$. This means that the assumptions of [Be96b, Prop.2.2.1] are verified, and the orbit $Co(V)_0^{(j\alpha B_\alpha(x,x)^{-1})_*} \cdot 0$ is nothing but the open set $X^{(\alpha B_\alpha(x,x)^{-1})} \subset V^c$.

(ii) The set $\{x \in V | \det B_\alpha(x,x) = 0\}$ is a hypersurface in V, and from the transformation law 1.3 (9) it is clear that no point of this set can belong to an open $Co(V)^{(j\alpha)_*}$-orbit. Conversely, by (i), any point from the complement of this set belongs to an open orbit. This implies the claim.

(iii) is an immediate consequence of (ii) since the complement of an algebraic hypersurface in a complex vector space is connected. $\square$

The Cayley-transformed realization. — If α is an involution of V, then there is a Cayley-transformed realization of $X^{(\alpha)} = G/H$: recall the *real Cayley transform* $R(x) = (x-e)(x+e)^{-1}$ from [Be96b, Lemma 2.1.1], having the property that $RjR^{-1} = -id_V$. The group RGR^{-1} acting on $R(X^{(\alpha)})$ is then given by

$$RGR^{-1} = Co(V)_0^{(Rj\alpha R^{-1})_*} = Co(V)_0^{(-\alpha)_*}.$$

2.1.2 Theorem. — *Let α be an involutive automorphism of V.*
(i) If $x \in V$ is such that $\det P(\alpha(x)+x) \neq 0$, then the orbit

$$Co(V)_0^{(-\alpha)_*} \cdot x \subset V^c$$

is open.

(ii) The intersection $R^{-1}(X^{(\alpha)}) \cap V = (Co(V)_0^{(-\alpha)_*} \cdot e) \cap V$ is a union of connected components of the set $\{x \in V | \det P(\alpha(x)+x) \neq 0\}$. In particular,

$$(R^{-1}(X^{(\alpha)}) \cap V)_e = \{x \in V | \det P(\alpha(x)+x) \neq 0\}_e.$$

(iii) If V is complex and α complex linear, then

$$R^{-1}(X^{(\alpha)}) \cap V = \{x \in V | \det P(\alpha(x)+x) \neq 0\},$$

and this set is dense in V.

Proof. — We have $DR(x) = 2P(x+e)^{-1}$. Using $R(y) = jRj(-j(y))$, we get from 1.3 (7), if $R(x)$ and $R(y)$ are in V,

$$\begin{aligned} B_\alpha(R(x), R(y)) &= DR(x) \circ B_\alpha(x, -j(y)) \circ (j\alpha)_*(D(j\alpha Rj\alpha)(-j(y))^{-1} \\ &= DR(x) \circ B_\alpha(x, -j(y)) \circ \alpha P(y) DR(y)\alpha \\ &= DR(x) \circ P(x+\alpha(y))P(-j(y)) \circ P(y)\, DR(y)\alpha \\ &= -DR(x)\, P(x+\alpha(y))\, DR(y)\alpha, \end{aligned}$$

where we used 1.3. (5) and the fact that $P(x^{-1}) = P(x)^{-1}$ for invertible x.

(i) Let $x \in V$ such that $R(x) \in V$. Then $\det DR(x)^{-1} \neq 0$, and the preceeding calculation shows that the condition $\det P(x+\alpha(x)) \neq 0$

is equivalent to $\det B_\alpha(R(x), R(x)) \neq 0$. By the previous theorem, this property characterizes the points such that the orbit

$$R(Co(V)_0^{(-\alpha)_*} \cdot x) = Co(V)_0^{(j\alpha)_*} \cdot R(x)$$

is open in V^c. All that remains to show is that the condition $R(x) \in V$ is inessential; in other terms we have to show that the zeroes of the function $x \mapsto \det(DR(x)^{-1}) = \frac{1}{2} \det P(e+x)$ belong generically to an open orbit. But this is clear from the fact that this function vanishes at $-e$, and the orbit $Co(V)^{(-\alpha)_*} \cdot (-e) = -Co(V)^{(-\alpha)_*} \cdot e$ is open in V^c.

Now *(ii)* and *(iii)* follow as in the proof of Theorem 2.1.1. $\square$

The statement of the previous theorem can be made more geometrical using the concept of a *tube domain*: let $V = V^+ \oplus V^-$ be the eigenspace decomposition of V with respect to the involution α, and let Ω^+ be the connected component of e of the intersection of V^+ with the open (in general non-convex) cone $\Omega = Str(V)_0 \cdot e \subset V$. Then $\Omega^+ = Str(V^+)_0 \cdot e$ is the open cone associated to the Jordan algebra V^+. The *tube domain associated to the involution* α is defined as

$$V^- + \Omega^+ = \{x + y \,|\, x \in V^-, y \in \Omega^+\}.$$

2.1.3 Corollary. — *Under the assumptions of the previous theorem,*

$$(Co(V)^{(-\alpha)_*} \cdot e \cap V)_e = V^- + \Omega^+,$$

where the subscript e denotes "connected component of e".

Proof. — The condition $\det P(x + \alpha(x)) \neq 0$ is equivalent to $\det P(x^+) \neq 0$ (where $x = x^+ + x^-$, $x^+ \in V^+$, $x^- \in V^-$), and this is equivalent to the fact that x^+ is invertible. But Ω^+ is exactly the component of e of the set of invertible elements in V^+. $\square$

Examples. — 1. An important special case is given by a complex Jordan algebra with a conjugation α; then $V^- = iV^+$, and the tube domain has the form $i(V^+ + i\Omega^+)$; such domains are studied in [FG95].

2. The classical matrix-spaces. Let $V = M(n, \mathbb{F})$, where $\mathbb{F}$ is the field of real or complex numbers or the skew-field of quaternions. The Jordan product is given by $X \cdot Y = \frac{1}{2}(XY + YX)$, the Jordan inverse j is the usual matrix inverse, the quadratic representation given by $P(X)Y = XYX$, and 1.3. (4) implies that

$$B(X, Y)Z = (I - XY)Z(I - YX),$$

where $I = I_n$ is the identity matrix. Then $\det B_\alpha(X,X) = \det(I - X\alpha(X))^n$, and Theorem 2.1.1 describes the space $X^{(\alpha)}$ by the condition

$$\det(I - X\alpha(X)) \neq 0.$$

We may choose α to be the involution "adjoint with respect to a matrix B and an involution ε of $\mathbb{F}$", i.e. $\alpha(X) = (B\varepsilon)X^t(B\varepsilon)^{-1}$ (*cf.* [Be96b, ch.I]); then the stated condition is equivalent to

$$\det(B - XB\varepsilon(X)^t) \neq 0;$$

this special case of Theorem 2.2.1 has been stated in [Be96b, Th.1.7.1]; as mentioned there, for $B = I$ and $\varepsilon(X) = \overline{X}$ we obtain the classical Siegel-space (see [Sa80]). The tube realization is characterized by the condition

$$\det(X + \alpha(X)) \neq 0;$$

this is equivalent to saying that $X^+ = \frac{1}{2}(X + \alpha(X))$ is invertible in the ordinary sense.

3. Hyperboloids. Let $V = \mathbb{R}^n$ equipped with a non-degenerate symmetric bilinear form $(\cdot|\cdot)$, i.e. with an isomorphism $V \to V^*$, $v \mapsto v^* = (\cdot|v)$ such that $v^*(w) = w^*(v)$. We choose a base point $e \in V$ such that $(e|e) = 1$. Then the formula

$$xy := (x|e)y + (y|e)x - (x|y)e$$

defines a Jordan product on V (see [FK94, II.1]). Writing $L(x) = (x|e)id_V + x \otimes e^* - e \otimes x^*$ (where $(v \otimes \mu)(x) := \mu(x)v$ for $v \in V$, $\mu \in V^*$) for the operator of left multiplication by x in V, we obtain

$$P(x) = 2L(x)^2 - L(x^2) = (x|x)id_V + x \otimes (4(x|e)e^* - 2x^*) - 2(x|x)e \otimes e^*.$$

Let α be the orthogonal reflection with respect to $\mathbb{R}e$, i.e. $\alpha = 2e \otimes e^* - id_V$. One immediately verifies that this is an involution of V, and

$$P_\alpha(x) = 2x \otimes x^* - (x|x)id_V$$

since a short calculation shows that indeed $P_\alpha(x) \circ \alpha = P(x)$. From this we obtain $x \,\square\, \alpha x = (x|x)id_V$, and using these formulas we get

$$B_\alpha(x,x) = id_V - 2x \,\square\, \alpha x + (P_\alpha(x))^2 = (1 - (x|x))^2 id_V,$$

$$B_{-\alpha}(x,x) = id_V + 2x \,\square\, \alpha x + (-P_\alpha(x))^2 = (1 + (x|x))^2 id_V.$$

Hence

$$(X^{(\alpha)}\cap V)_0 = \{x \in V \mid (x|x) \neq 1\}_0, \quad (X^{(-\alpha)}\cap V)_0 = \{x \in V \mid (x|x) \neq -1\}_0. \blacksquare$$

Since $P(x + \alpha(x)) = P(2(e|x)e) = 4(e|x)^2 id_V$, the Cayley transformed realization of $X^{(\alpha)}$ is given by the half-plane

$$R^{-1}(X^{(\alpha)}) = \{x \in V \mid (e|x) \neq 0\}_e.$$

Now let the signature of $(\cdot|\cdot)$ be (p, q); then $p \geq 1$ since $(e|e) = 1$. The following is known (see [Ma73]): the conformal group of V is isomorphic to $SO(p+1, q+1)$, and $X^{(\alpha)} \cong SO(p, q+1)/SO(p, q)$, $X^{(-\alpha)} \cong SO(p+1, q)/SO(p, q)$. The following special cases are most interesting:

a) $p = n$, $q = 0$: then $X^{(\alpha)} = \{x \in V \mid ||x|| < 1\} \cong SO(1, n)/SO(n)$ is a real hyperbolic space H^n, and $X^{(-\alpha)} \cong S^n$ is a sphere, the conformal compactification of Euclidean n-space.

b) $p = 1$, $q = n - 1$: in this case V is a Euclidean Jordan algebra, and the associated symmetric cone is the Lorentz cone. Then $X^{(\alpha)} \cong SO(1, n)/SO(1, n-1)$ and $X^{(-\alpha)} \cong SO(2, n-1)/SO(1, n-1)$ are causally flat realizations of relativistic space-time models of constant curvature. The global topology of these spaces cannot easily be read off the flat description.

2.2 The Cartan involution. — Let V be a semi-simple Jordan algebra, $V_{\mathbb{C}}$ its complexification and τ its conjugation with respect to the real form V. Then there exists a conjugation θ of $V_{\mathbb{C}}$ commuting with τ such that the real form $(V_{\mathbb{C}})^\theta$ is a Euclidean Jordan algebra ([FK94, Th.VIII.5.2]). The restriction of θ to V is called a *Cartan involution of* V; if $V = V^+ \oplus V^-$ is the corresponding eigenspace decomposition, then the Euclidean real form of $V_{\mathbb{C}}$ is given by $V^+ \oplus iV^-$.

2.2.1 Lemma. — *The Makarevič space*

$$X^{(-\theta|_V)} = Co(V)_0^{(-j\theta|_V)_*} \cdot 0 \subset V^c$$

is compact and equal to V^c.

Proof. — The Makarevič space $X^{(\theta)}$ is a bounded symmetric domain in $V_{\mathbb{C}}$; in fact, we have seen above that it is the tube domain associated to the Euclidean Jordan algebra $V^+ \oplus iV^-$ in its disc realization. By [Be96b, Prop.2.3.2], the space $X^{(-\theta)}$ is its c-dual; it is a compact symmetric space because the disc is of the non-compact type. As $X^{(-\theta)} \subset (V_{\mathbb{C}})^c$ is an open and compact inclusion in a connected space, we must have equality. Now

$Co(V)_0^{(-j\theta|_V)_*}$ can be considered as the identity component of the τ_*-fixed closed subgroup of the compact group $Co(V_{\mathbb{C}})_0^{(-j\theta)_*}$ (*cf.* [Be96b, Lemma 2.3.1]), and hence is also compact. Therefore $X^{(-\theta|_V)}$ is compact, and by topological reasons it is then equal to V^c. □

Remark. We have so far not yet used the fact that V^c is actually compact; hence this fact may be *proved* by the previous lemma.

2.2.2 Corollary. — *In the situation of the preceeding lemma, the operator $B_{-\theta|V}(x,x)$ is non-degenerate for all $x \in V$. In particular, if V is Euclidean, then*

$$B_{-id_V}(x,x) = id + 2x \,\square\, x + P(x)^2 = P(x)P(x + x^{-1})$$

is non-degenerate for all $x \in V$.

Proof. — By the lemma, every point of $x \in V$ belongs to an open $Co(V)_0^{(-j\theta|_V)_*}$-orbit, and by Theorem 2.1.1, this means that $B_{-\theta|_V}(x,x)$ is non-degenerate. □

2.3 Description of subspaces, and Makarevič spaces of the second kind. — Let us consider a Makarevič space $X^{(\alpha)}$ together with an involution β of the underlying Jordan algebra V such that α and β commute. We are going to describe the $-\beta$-fixed subspace $X^{(\alpha)} \cap V^{-\beta}$ of $X^{(\alpha)} \cap V$. Important examples are given below.

2.3.1 Proposition. — (i) *The orbit of $x \in V^{-\beta}$ under the action of the group*

$$G := (Co(V)^{(-\beta)_*} \cap Co(V)^{(j\alpha)_*})_0$$

is open in $(V^c)^{-\beta}$ if and only if $\det B_\alpha(x,x) \neq 0$. *In particular,*

$$(G \cdot 0 \cap V)_0 = \{x \in V^{-\beta} |\, \det B_\alpha(x,x) \neq 0\}_0.$$

(ii) *We have the following equality of subsets of V^c:*

$$(V^c)_0^{-\beta} = Co(V)_0^{(-\beta)_*} \cdot 0,$$

and the inclusion $V^{-\beta} \subset (V^c)_0^{-\beta}$ is open and dense.

Proof. — (i) We first remark that the compability relation $\alpha\beta = \beta\alpha$ ensures that $(V^c)^{-\beta}$ is indeed stable by G, where β is considered here as a conformal map defined on the whole of V^c. Now the proof can be carried out as in Th.2.1.1 (i) since the transformation $n\tau_{-x}$ used there stabilizes the set $(V^c)^{-\beta}$ if $x \in V^{-\beta}$.

(ii) Let us choose for $\alpha = -\theta$ where θ is a Cartan involution of V wich commutes whith β. Then G is compact and

$$G \cdot 0 = (V^c)_0^{-\beta}$$

by compactness. Furthermore

$$V^{-\beta} = (G \cdot 0) \cap V = \{\phi(0) |\ \phi \in G, \det \chi_{\phi^{-1}}(0) \neq 0\}$$

is an open dense set in $G \cdot 0$. $\square$

It is clear that a similar proposition can be stated for the $+\beta$-fixed subspaces, but they are less interesting as they are nothing but Makarevič spaces in the Jordan sub-algebra V^β. This is not the case for the $-\beta$-fixed subspace as shows the example mentioned above, and therefore we will call the $-\beta$-fixed subspaces Makarevič spaces *of the second kind.* They have been classified in [Ma73, Th.1 and 2] without using Jordan theory and the "flat" realization given here.

Examples. — 1. Unitary groups. Recall from [Be96b] that $U := U(A, \varepsilon, \mathbb{F})$ denotes the linear group preserving the ε-sesquilinear form given by the matrix A on $\mathbb{F}^n$, where ε as an anti-automorphism of the base field $\mathbb{F}$ of real or complex numbers or quaternions. Let $V = M(n, \mathbb{F})$ and $\beta(X) = A\varepsilon X^t (A\varepsilon)^{-1}$ the map "adjoint", supposed to be involutive. Then, by definition, $V^- = Aherm(A, \varepsilon, \mathbb{F})$. The space associated to $\alpha = id_V$ is

$$(X^{(id_V)} \cap V^-)_0 = \{X \in Aherm(A, \varepsilon, \mathbb{F}) |\ \det(I - X^2) \neq 0\}_0,$$

cf. 2.1, example 2. This is of course nothing but the Cayley-transformed realization of the group $U(A, \varepsilon, \mathbb{F})$ described in [Be96b, 1.5]. The c-dual space $U_{\mathbb{C}}/U$ is given by

$$(X^{(-id_V)} \cap V^-)_0 = \{X \in Aherm(A, \varepsilon, \mathbb{F}) |\ \det(I + X^2) \neq 0\}_0.$$

Recall from [Be96b] that, if V^- contains invertible elements, then it can be equipped with a Jordan *algebra* structure such that the corresponding unitary group becomes actually a Makarevič space of the *first* kind. This is the case for the groups $Sp(n, \mathbb{R})$, $U(p, q)$ and some others, but not for the groups $SO(2n + 1)$.

2. Grassmannians. Let $V = Herm(I_n, \varepsilon, \mathbb{F})$ and assume that $\varepsilon(x) = \overline{x}$ is the canonical conjugation of the base field. Then V is a Euclidean Jordan algebra. Let $\beta(X) = I_{p,q} X I_{p,q}$. Then

$$M(p, q, \mathbb{F}) \to V^-, \quad X \mapsto \begin{pmatrix} 0 & X \\ \overline{X}^t & 0 \end{pmatrix}$$

is a vector space isomorphism (and an isomorphism of Jordan triple systems with respect to the natural triple products). Under this isomorphism, the space associated to $\alpha = id_V$ is

$$X^{(id_V)} \cap V^- = \{X \in M(p,q,\mathbb{F}) |\, I - XX^* >> 0\} \cong U(p,q;\mathbb{F})/(U(p,\mathbb{F}) \times U(q,\mathbb{F});$$

this is the non-compact dual of the space $X^{(-id_V)} \cap (V^c)_0^{-\beta}$ which is isomorphic to the Grassmannian $G_{p,p+q}(\mathbb{F})$. The special case $p = 1$ yields the projective space $\mathbb{PF}^{n+1}$ and its non-compact dual, the hyperbolic space $H^n(\mathbb{F})$. However, the "conformal structure" on $H^n(\mathbb{R})$ considered here is not the one considered in 2.1, example 3 since the conformal groups are not the same; correspondingly, the global topological nature of the compact dual is not the same in these examples.

3. Bounded symmetric domains. Let V be a complex Jordan algebra, α be a conjugation with respect to a Euclidean real form and β a complex linear involution commuting with α. Then

$$X^{(\alpha)} \cap V^{-\beta}$$

is a bounded symmetric domain which is not of tube type in general. We will discuss this example in Proposition 3.1.1.

2.4 The invariant (pseudo-)metric of $X^{(\alpha)}$. — As usual, V denotes a semi-simple Jordan algebra and α is an element of $Str(V)^{Jj_*}$. By definition of semi-simplicity, the symmetric bilinear form $\langle u, v\rangle := \mathrm{tr}(u \,\square\, v) = \mathrm{tr}\, L(uv)$ is then non-degenerate, and we have for all $g \in Str(V)$ and $u, v \in V$,

$$\langle gu, v\rangle = \langle u, (jgj)^{-1} v\rangle.$$

In particular, α is self-adjoint. Hence the form

$$\langle u, v\rangle_\alpha = \mathrm{tr}(u \,\square_\alpha v) = \mathrm{tr}\, L(u\alpha(v)) = \langle u, \alpha v\rangle$$

is also non-degenerate and symmetric. Furthermore, for all $g \in Str(V)$ and $u, v \in V$,

$$\langle gu, v\rangle_\alpha = \langle u, (j\alpha g j\alpha)^{-1} v\rangle_\alpha.$$

In particular, $\langle \cdot, \cdot \rangle_\alpha$ is invariant by $Str(V)^{(j\alpha)_*}$, which is the stabilizer of the base point 0 in $X^{(\alpha)}$. Hence we can transport this form by the group $Co(V)^{(j\alpha)_*}$ to any point of $X^{(\alpha)}$ and obtain thus an invariant pseudo-metric on $X^{(\alpha)}$. The important point is that we can define this pseudo-metric by a rational expression.

2.4.1 Theorem. — *Let us define, for $z \in V$ such that $\det B_\alpha(z,z) \neq 0$,*

$$h_z(u,v) := \langle B_\alpha(z,z)^{-1} u, v\rangle_\alpha.$$

Then $z \mapsto h_z$ defines a pseudo-metric which is $Co(V)^{(j\alpha)_*}$-invariant in the following sense: if $\phi \in Co(V)^{(j\alpha)_*}$ is such that $\phi(z) \in V$,

$$(\phi_*^{-1} h)_z(u,v) := h_{\phi(z)}\big(D\phi(z)\cdot u, D\phi(z)\cdot v\big) = h_z(u,v).$$

The pseudo-metric h can be continued in a $Co(V)^{(j\alpha)_*}$-invariant way onto the union of open $Co(V)^{(j\alpha)_*}$-orbits in V^c.

Proof. — We have already remarked that h_0 is non-degenerate and symmetric. Hence, if $B_\alpha(z,z)$ is non-singular, h_z is also non-degenerate. Let us show that h_z is symmetric:

$$h_z(u,v) = \langle u, (j\alpha B_\alpha(z,z) j\alpha) v\rangle_\alpha = \langle B_\alpha(z,z)^{-1} v, u\rangle_\alpha = h_z(v,u)$$

where we have used 1.3 (2). In order to prove the invariance, let $\phi \in Co(V)^{(j\alpha)_*}$ such that $\phi(z) \in V$. Then

$$\begin{aligned}(\phi_*^{-1} h)_z(u,v) &= \langle B_\alpha\big(\phi(z),\phi(z)\big)^{-1} D\phi(z)\cdot u, D\phi(z)\cdot v\rangle_\alpha \\ &= \langle j\alpha\, D\phi(z)\, j\alpha\, B_\alpha(z,z)^{-1}\cdot u, D\phi(z)\cdot v\rangle_\alpha \qquad \text{(by 1.3.(9))} \\ &= \langle B_\alpha(z,z)^{-1} u, v\rangle_\alpha = h_z(u,v).\end{aligned}$$

Now, if $z \in V^c$ belongs to an open $Co(V)^{(j\alpha)_*}$-orbit U, then $U \cap V \neq \emptyset$ because V is dense in V^c. For any $x \in U\cap V$, $\det B_\alpha(x,x) \neq 0$ (otherwise the orbit of x were not open.) Because h_x is, by the preceeding calculation, invariant under the action of the stabilizer of x, we can transport h_x to z, thus extendig the pseudo-metric from $U\cap V$ to the whole of U. ∎

The volume element of a pseudo-Riemannian manifold with metric tensor $g_{ik}(x)$ is given by $\sqrt{|\det(g_{ik}(x))|}$. Here we have $(g_{ik}(x)) = (B_\alpha(x,x)^{-1})$. This proves

2.4.2 Corollary. — *The $Co(V)^{(j\alpha)_*}$-invariant measure on $X^{(\alpha)}\cap V$ is given by the density*

$$|\det B_\alpha(x,x)|^{-1/2}$$

with respect to the Lebesgue measure of V.

∎

In the tube-realization we have similarly:

2.4.3 Proposition. — *Let α be an involution of V. The invariant pseudo-metric on the domain $R(X^{(\alpha)}) = -Co(V)_0^{(-\alpha)_*}\cdot e$ is given by*

$$g_x(u,v) = \langle P(x+\alpha x)^{-1} u, v\rangle,$$

and the invariant measure by the density

$$|\det P(x+\alpha(x))|^{-1/2}$$

with respect to the Lebesgue measure of V.

Proof. — All we have to do is to calculate the forward-transport R_*h of the pseudo-metric h from the previous theorem by the real Cayley-transform R:

$$\begin{aligned}(R_*h)_x(u,v) &= h_{R^{-1}(x)}(DR(x)u, DR(x)v) \\ &= \langle j\alpha DR(x)^{-1} j\alpha \circ B_\alpha(R^{-1}(x), R^{-1}(x))^{-1} \circ DR(x)u, v\rangle_\alpha \\ &= \langle P(x+\alpha(x))^{-1}u, v\rangle,\end{aligned}$$

where in the last line we have used that $j\alpha DR(x)^{-1} j\alpha = \alpha DR(x)\alpha$ and the calculation given in the proof of Prop. 2.1.2. □

3. Description of domains of holomorphic semi-group actions

3.1 Decompositions of bounded symmetric domains. — Let D be a bounded symmetric domain of tube type: V is a Euclidean Jordan algebra with associated symmetric cone Ω; let $V_{\mathbb{C}}$ be the complexification of V and τ be the corresponding complex conjugation. Then $T_\Omega = V + i\Omega$ is a tube domain which is equivalent by the Cayley transform C to the disc

$$D = X^{(\tau)} = Co(V_{\mathbb{C}})_0^{(j\tau)_*} \cdot 0.$$

We will use the notation $G := G(D) = Co(V_{\mathbb{C}})_0^{(j\tau)_*}$. Let α be an involutive automorphism of V; by the same letter we denote its complex linear continuation as an automorphism of $V_{\mathbb{C}}$. We write $V_{\mathbb{C}}^{\pm}$ for the ± 1-eigenspaces of α and consider the bounded domains

$$D^{\pm} := D \cap V_{\mathbb{C}}^{\pm} \subset V_{\mathbb{C}}^{\pm}.$$

3.1.1 Proposition. — (*i*) *The space D^+ is a bounded symmetric domain of tube type in the complex vector space $V_{\mathbb{C}}^+$. It is homogeneous under the action of the group*

$$G^+ := (Co(V_{\mathbb{C}})^{\alpha_*} \cap Co(V_{\mathbb{C}})^{(j\tau)_*})_0.$$

(*ii*) *The space D^- is a bounded symmetric domain (in general not of tube type) in the complex vector space $V_{\mathbb{C}}^-$. It is homogeneous under the action of the group*

$$G^- := (Co(V_{\mathbb{C}})^{(-\alpha)_*} \cap Co(V_{\mathbb{C}})^{(j\tau)_*})_0.$$

(iii) The disc D decomposes as a fiber space over D^+ with fibers isomorphic to D^- (and also the other way round). More precisely,

$$D = G^- \cdot D^+ = \bigcup_{z \in D^+} G^- \cdot z,$$

where the union is a disjoint union of orbits, each of which is isomorphic to D^- as a homogeneous and symmetric space.

Proof. — The disc D is a Riemannian symmetric space of non-compact type; i.e. $D = G/K$, where K is the fixed point group of a Cartan-involution. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding decomposition of the Lie algebra; here we have $\mathfrak{p} = \{v + j_* \tau v | \, v \in V_{\mathbb{C}}\}$. As for any Riemannian symmetric space of the non-compact type, the exponential map $\mathrm{Exp} : \mathfrak{p} \to D$ is a diffeomorphism. The involution α_* commutes with the given Cartan-involution; we write $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ for the decompostion into ± 1-eigenspaces of $\mathfrak{p}$ w.r. to the involution α_*.

(i) It is clear that $D^+ = \mathrm{Exp}(\mathfrak{p}^+)$; because $\exp(\mathfrak{p}^+) \subset G^+$, this group is transitive on D^+. Clearly D^+ is bounded and symmetric, and it is of tube type because V^+ is a Jordan algebra.

(ii) We apply the same arguments as above; but D^- will in general not be of tube type because the Jordan triple system V^- will in general not be associated to a Jordan algebra.

(iii) By a theorem of Mostow (see [Lo69, Prop.IV.2.5]), the map

$$\mathfrak{p}^- \times \mathfrak{p}^+ \to D, \quad (X, Y) \mapsto \exp(X) \exp(Y) \cdot 0$$

(where exp is the exponential map of G) is a diffeomorphism. Let us show that this decomposition, $D = \exp(\mathfrak{p}^-) \exp(\mathfrak{p}^+) \cdot 0$, is nothing but the claim: first, $D^+ = \exp(\mathfrak{p}^+) \cdot 0$ (as this is a Riemannian symmetric space of the non-compact type), and second, if $z \in D^+$, $z = \exp(Y) \cdot 0$ with $Y \in \mathfrak{p}^+$, then $D^- \to G^- \cdot z$, $x \mapsto \exp(Y) \cdot x$ is a diffeomorphism. $\square$

Looking at the classification of bounded symmetric domains (see [Lo77]), we easily verify that every bounded symmetric domain can be realized as a "subdisc" D^- in a "big disc" of tube type, but this realization is not unique. (For example, $\{X \in Sym(2n, \mathbb{C}) | \, I - XX^* >> 0\}$ can be realized in a -1-eigenspace in $M(2n, \mathbb{C})$ and in $Sym(2n, \mathbb{C}) \times Sym(2n, \mathbb{C})$.) One may remark that these realizations are special cases of Makarevič spaces *of the second kind* as defined in section 2.3. In [Be96c] we will present a deeper study of such realizations.

3.2 Compactly and non-compactly causal symmetric spaces. Recall from [Be96b] that every Makarevič space associated to a *Euclidean*

Jordan algebra V has an invariant *causal structure*: for these Jordan algebras the open cone $\Omega = Str(V)_0 \cdot e$ is *convex*, and the constant cone field on V given by Ω is invariant under $Co(V)_0$. Hence any Makarevič space associated to V inherits this invariant causal structure. As we have seen in [Be96b, Prop.2.4.3], any Makarevič space associated to V is isomorphic either to $X^{(\alpha)}$ or to $X^{(-\alpha)}$, where α is an involutive automorphism of V. Recall that $X^{(-id_V)}$ is compact and $X^{(id_V)}$ is of non-compact type.

Let $X = X^{(\pm\alpha)}$. In the usual notation related to *causal symmetric spaces*, one writes $X = G/H$ with associated decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ of the Lie algebra $\mathfrak{g}$ of G and is then interested in $Ad(H)$-invariant regular cones $C \subset \mathfrak{q}$. One usually chooses a Cartan-involution of G compatible with the involution seperating H and writes $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the associated Cartan decomposition of $\mathfrak{g}$. Then one distinguishes the cases $C \cap \mathfrak{k} \neq \emptyset$ and $C \cap \mathfrak{p} \neq \emptyset$, calling the former structure *compactly causal* and the latter *non-compactly causal.*

In our realization of $X = X^{(\pm\alpha)}$ we have $G = Co(V)_0^{(j\alpha)_*}$, $G_0^{(-id_V)_*} \subset H \subset G^{(-id_V)_*}$, and the decomposition of $\mathfrak{g}$ is given by (writing v ($v \in V$) for the constant vector field $\xi(x) = v$ on V)

$$\mathfrak{h} = \mathfrak{str}(V)^{\alpha_*}, \qquad \mathfrak{q} = \mathfrak{q}^{(\pm\alpha)} = \{v \pm j_*\alpha v |\, v \in V\},$$

where $(j_*w)(x) = -P(x)w$ ($w \in V$), *cf.* [Be96b, Prop.2.2.1]. Via the evaluation map at the origin, $\xi \mapsto \xi(0)$, the space $\mathfrak{q}$ is identified with V by $v \pm j_*\alpha v \mapsto v$. Hence the $Ad(H)$-invariant cone in $\mathfrak{q}^{(\pm\alpha)}$ which we consider is defined by

$$C := C^{(\pm\alpha)} := \{v \pm j_*\alpha v |\, v \in \Omega\}.$$

Let us show that $C^{(-\alpha)}$ is compactly causal and $C^{(\alpha)}$ is non-compactly causal: the Cartan involution of G is given by $(-j)_*$ (see section 2.2), and one deduces that

$$\mathfrak{k} = \mathfrak{k}^{(\pm\alpha)} = Der(V)^{\alpha_*} \oplus \{v - j_*v |\, v \in V^{\mp}\}, \quad \mathfrak{p} = \mathfrak{p}^{(\pm\alpha)} = L(V^-) \oplus \{v + j_*v |\, v \in V^{\pm}\},$$

where $V = V^+ \oplus V^-$ is the eigenspace decomposition of V with respect to α. We now remark that $\Omega \cap V^+ \neq \emptyset$ because this set contains the unit element e, and $\Omega \cap V^- = \emptyset$: if this set contained an element x, then $-x = \alpha(x)$ would also be contained in it because Ω is stable by automorphisms; but since Ω is an open pointed cone, this leads to a contradiction. From this observation we now easily deduce that $C^{(-\alpha)} \cap \mathfrak{k} \neq \emptyset$, $C^{(-\alpha)} \cap \mathfrak{p} = \emptyset$, $C^{(\alpha)} \cap \mathfrak{p} \neq \emptyset$ and $C^{(\alpha)} \cap \mathfrak{k} = \emptyset$.

3.3 The complex domain Ξ associated to a compactly causal symmetric space. — Let $X = X^{(-\alpha)}$ be a compactly causal symmetric space which can be realized as a Makarevič space associated to the Euclidean Jordan algebra V, let $G = Co(V)_0^{(-j\alpha)_*}$ and $C = C^{(-\alpha)} \subset \mathfrak{q}$. Let $\Gamma(C)$ be the subset $G\exp(iC)$ of $G_{\mathbb{C}}$, where exp denotes the exponential map of $G_{\mathbb{C}}$; this is actually a semi-group, but we will not use this fact in the sequal. We want to describe the subset

$$\Xi := \Xi^{(-\alpha)} := \Gamma(C) \cdot 0 = G\exp(i\overline{C}) \cdot 0$$

of the complexified space $X_{\mathbb{C}} = X_{\mathbb{C}}^{(-\alpha)}$. The closure of Ξ in $X_{\mathbb{C}}$ is given by $\overline{\Xi} = \Gamma(\overline{C}) \cdot 0 = G\exp(i\overline{C}) \cdot 0$; it contains X in its border because $0 \in \overline{C}$. Let us show that locally at a point $x \in X$ the domain Ξ looks like the tube domain $T_\Omega = V + i\Omega$:

3.3.1 Lemma. — *Let $\xi \in iC$, i.e. $\xi = i(v - j_*\alpha v)$ with $v \in \Omega$. Then for any $x \in V$,*

$$\xi(x) \in i\Omega.$$

Proof. — We have to show that for all $x \in V$, the vector

$$\xi(x) = i(v - (j_*\alpha v)(x)) = i(v + P(x)\alpha v)$$

belongs to $i\Omega$. First remark that $v \in \Omega$, $\alpha v \in \Omega$ (because α is an automorphism), second that $P(V) \cdot \Omega = \overline{\Omega}$ (which follows by continutity from the fact that $P(x) \cdot \Omega = \Omega$ for invertible $x \in V$, *cf.* [FK94, Prop.III.2.2]), and finally the sum of points in Ω still lies in Ω since Ω is a convex cone; more precisely: if $a \in \Omega$ and $b \in \overline{\Omega}$, then $a + b \in \Omega$. $\square$

One may remark here that for $X^{(\alpha)}$ similar arguments fail because Ω is not stable by taking the difference of its elements. The lemma can be interpreted in the following way: the vector fields $\xi \in iC$ are *sub-tangent* for the (closed) tube $\overline{T}_\Omega$ at points $x \in V$. This means by definition that, for x on the boundary ∂T_Ω, the vector $\xi(x)$ is never pointed into an outward direction of T_Ω. A formal definition is: there is a sequence $(x_n) \subset \overline{T}_\Omega$ and a sequence $(r_n) \subset \mathbb{R}$ such that $r_n \to \infty$ $(n \to \infty)$ and $\xi(x) = \lim_{n\to\infty} r_n(x_n - x)$. The importance of sub-tangent vector fields lies in the fact that their integral curves stay in $\overline{T}_\Omega$ for non-negative time-parameter (Theorem of Bony-Brezis, see [HHL89, Thm.I.5.17]).

3.3.2 Lemma. — *Let $\xi(x) = i(v + P(x)w)$ with $v, w \in \Omega$. Then $\xi(z)$ is sub-tangent for $\overline{T}_\Omega$ at any point z of $\overline{T}_\Omega$. More precisely: for any point z on the boundary ∂T_Ω of T_Ω, the vector $\xi(z)$ is pointed into the interiour of T_Ω.*

Proof. — Let $z = x + iy$ with $x \in V$ and $y \in \Omega$. Then

$$\xi(z) = i(v + P(x + iy)w) = i(v + P(x)w - P(y)w) - 2P(x, y)w.$$

We have to show that the imaginary part of this expression is sub-tangent for $\overline{\Omega}$ at y. As in the proof of the previous lemma, one remarks that $v+P(x)w \in \Omega$; this is a sub-tangent vector for $\overline{\Omega}$ at y. For the remaining term, observe that $-P(y)w = (-y \,\square\, w)(y)$, and since $-y \,\square\, w$ belongs to $\mathfrak{str}(V)$ which is the Lie algebra of $G(\Omega)$ (the group of linear automorphisms of Ω), the vector $(-y \,\square\, w)(y)$ is sub-tangent for $\overline{\Omega}$ at y. Since Ω is convex, the sum of sub-tangent vectors is still sub-tangent, and if one of them is pointed into the interiour, then the same is also true for the sum; this implies the claim. ∎

3.3.3 Corollary. — *Let $\xi(x) = i(v + P(x)w)$ with $v, w \in \Omega$ and let $\exp(it\xi)$ be the one-parameter subgroup of $Co(V_{\mathbb{C}})$ generated by $i\xi$. Then for all $t > 0$,*

$$\exp(it\xi) \cdot \overline{T}_\Omega \subset T_\Omega,$$

Proof. — Let $z \in \overline{T}_\Omega$. By the previous lemma and the theorem of Bony-Brezis (cited above), $\exp(it\xi) \cdot z \in \overline{T}_\Omega$ for small $t \geq 0$, and even $\exp(it\xi) \cdot z \in T_\Omega$ because $\xi(z)$ is everywhere pointed into the interiour of T_Ω. In order to prove the Corollary we only have to show that this statment is true for all $t > 0$. This can be proved using the bounded realization $D = C(T_\Omega)$ of T_Ω: then $\overline{D}$ is compact and we may find $t_0 > 0$ such that $\exp(C_*(it_0\xi)) \cdot \overline{D} \subset D$, and hence the whole semi-group generated by $C_*(it_0\xi)$ is a semi-group of contractions of D. ∎

Clearly the Corolloray implies that

$$G \exp(iC) \subset \{g \in G_{\mathbb{C}} | g \cdot \overline{T}_\Omega \subset T_\Omega\},$$

and hence

$$\Xi = G \exp(iC) \cdot 0 \subset (X_{\mathbb{C}} \cap T_\Omega).$$

We will show now that we have actually equality. In other terms, we will prove a transitivity result.

3.3.4 Theorem. — *The domain Ξ associated to the non-compactly causal symmetric space $X^{(-\alpha)}$ is given by*

$$\Xi = X_{\mathbb{C}}^{(-\alpha)} \cap T_\Omega,$$

where $T_\Omega = V + i\Omega$ is the tube domain associated to the Euclidean Jordan algebra V in which $X^{(-\alpha)}$ is realized as a Makarevič-space. By the Cayley-transform $C(x) = (ix - e)(ix + e)^{-1}$, we have equivalently

$$C(\Xi^{(-\alpha)}) = (Co(V_{\mathbb{C}})_0^{(-\alpha)_*} \cdot ie) \cap D,$$

where $D = C(T_\Omega) = Co(V_{\mathbb{C}})_0^{(j\tau)_*} \cdot 0$ *is the disc realization of* T_Ω.

Proof. — a) Let us show that the two realizations are equivalent by the Cayley transform: $C(T_\Omega) = D$, $C(X_{\mathbb{C}}^{(-\alpha)}) = Co(V_{\mathbb{C}})_0^{(-\alpha)_*} \cdot ie$, hence $C(X_{\mathbb{C}} \cap T_\Omega) = (Co(V_{\mathbb{C}})_0^{(-\alpha)_*} \cdot ie) \cap D$.

b) Let us prove first the claim in the special case $\alpha = id_V$, i.e. the case where X is compact and $X = V^c$; then $C(X)$ is the Shilov-boundary of D. In this case $Co(V_{\mathbb{C}})_0^{(-id)_*} = Str(V_{\mathbb{C}})_0$ is a group acting by linear maps, and we have to show that

$$C(\Xi) = (Str(V_{\mathbb{C}})_0 \cdot ie) \cap D.$$

Now, $Str(V_{\mathbb{C}}) \cdot ie \cong U_{\mathbb{C}}/K_{\mathbb{C}}$ is the complexification of the compact symmetric space $C(X) = U \cdot ie \cong U/K$ where $U = Str(V_{\mathbb{C}})_0^{(j\tau)_*}$ (compact real form of $Str(V_{\mathbb{C}})$). We have the well-known polar decomposition of $U_{\mathbb{C}}/K_{\mathbb{C}}$ (coming from the Cartan decomposition $U_{\mathbb{C}} = U \exp(i\mathfrak{u})$): $U_{\mathbb{C}}/K_{\mathbb{C}} = U \exp(i\mathfrak{q}) K_{\mathbb{C}}$ (no uniqueness), where $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ is the Lie algebra-decomposition associated to U/K. Consider now $z \in Str(V_{\mathbb{C}}) \cdot ie \cap D$ and decompose it as $z = u \exp X \cdot ie$ with $u \in U$ and $X \in \mathfrak{p} := i\mathfrak{q}$. Then $\mathfrak{p} = L(V)$, i.e. there is $v \in V$ such that $\exp X = e^{L(v)}$. Recall now the description of D as the unit ball for the spectral norm $|w| := ||w \,\square\, \overline{w}||^{1/2}$, see [FK94, p. 198]. As the spectral norm is invariant by U, the condition $z \in D$ is equivalent to

$$1 > |e^{L(v)} \cdot ie| = |e^v|.$$

This is equivalent to the condition that v has only negative eigenvalues, which by [FK94, Th.III.2.1] is equivalent to the condition that $v \in -\Omega$, i.e. $z = ue^{-L(v)} \cdot ie$ with $u \in U$ and $v \in \Omega$. Since $Ue^{-\Omega}$ is the Cayley-transformed realization of the semi-group $G \exp(iC)$, this proves the claim in the special case $\alpha = id_V$.

c) We now prove the claim for a general involution α of V. As remarked just before stating the Theorem, only the inclusion "$\supset$" remains to be shown. This problem will be reduced to the special case $\alpha = id_V$ by means of the fibration of D by its "subdiscs" D^+ and D^- stated in Prop. 3.1.1. Recall that τ is the conjugation of $V_{\mathbb{C}}$ with respect to the Euclidean real form V, and by the Cayley-tranform $C = iRi$ we have $C\tau C^{-1} = j\tau$ and $C(-j\alpha)C^{-1} = -\alpha$, see [Be96b, Lemma 2.1.1]. Observe that the Cayley-transformed group

$$CGC^{-1} = C(Co(V_{\mathbb{C}})^{(-j\alpha)_*} \cap Co(V_{\mathbb{C}})^{\tau_*})_0 C^{-1} = (Co(V_{\mathbb{C}})^{(-\alpha)_*} \cap Co(V_{\mathbb{C}})^{(j\tau)_*})_0$$

acts transitively on the subdisc D^-, see Prop. 3.1.1 (*ii*).

Now let $z \in D \cap Co(V_{\mathbb{C}})^{(-\alpha)_*} \cdot ie$. By Prop.3.1.1 (*iii*) we can find $\phi \in CGC^{-1}$ such that $z' := \phi^{-1}(z) \in D^+$. But then, by part b), we can find $u \in U = (CGC^{-1})_0^{\alpha_*}$ and $v \in \Omega^+ = \Omega \cap V^+$ such that $z' = ue^{-L(v)} \cdot ie$, whence $z = \phi ue^{-L(v)} \cdot ie$. Transforming back by the Cayley-transform, we obtain $C^{-1}(z) \in G \exp(iC) \cdot 0$ which had to be shown. We even obtain the more specific result that we can write $C^{-1}(z) = g \exp(i(v - j_* v)) \cdot 0$ with $v \in V^+$, i.e.

$$\Xi = G \exp i(C \cap \mathfrak{k}) \cdot 0.$$

□

Remark: Our proof is inspired by the treatment of the special case of *Cayley-type spaces* (see example below) in [Cha95]; see also [OØ96]. By combining the preceeding result with Theorems 2.1.1 and 2.1.3 we get the following:

3.3.5 Theorem. — *In the situation of the preceeding theorem,*

$$\Xi = \{z \in T_\Omega | \det B_{-\alpha}(z, z) \neq 0\},$$

$$C(\Xi) = \{z \in D | \det P(z + \alpha(z)) \neq 0\}.$$

If $V_{\mathbb{C}}^- + \Omega_{\mathbb{C}}^+$ *denotes the tube domain associated to the involution* $\alpha_{\mathbb{C}}$ *defined in section 2.1, then we have also*

$$C(\Xi) = D \cap (V_{\mathbb{C}}^- + \Omega_{\mathbb{C}}^+).$$

Proof. — Since α is a $\mathbb{C}$-linear involution of $V_{\mathbb{C}}$, the descriptions 2.1.1 (*iii*), resp 2.1.2 (*iii*) and 2.1.3 hold for $X_{\mathbb{C}}^{(\alpha)}$ and for $X_{\mathbb{C}}^{(-\alpha)} = iX_{\mathbb{C}}^{(\alpha)}$, respectively for the Cayley-transformed realizations. □

As mentioned in the introduction, these results confirm conjectures stated in [HOØ91, remark 5.1]. Moreover, they imply that $X^{(-\alpha)}$ is *dense* in V^c, and hence it makes sense to call V^c the *causal compactification of* X, a notion introduced in [OØ96]. The following result has also been obtained by F. Betten [Bet96] using only Lie-theoretic methods (double-coset decompositions by Matsuki).

3.3.6 Theorem. — *Let* α *be an involution of the Euclidean Jordan algebra* V. *Then the Makarevič space* $X^{(-\alpha)}$ *is dense in* V^c, *and*

$$X^{(-\alpha)} \cap V = \{x \in V | \det B_{-\alpha}(x, x) \neq 0\}.$$

If $\Sigma = C(V^c)$ *denotes the Shilov-boundary of the disc* $D = C(T_\Omega)$, *then*

$$C(X^{(-\alpha)}) = \{z \in \Sigma | \det P(z + \alpha(z)) \neq 0\}$$

$$= \{z \in V_{\mathbb{C}} |\ \det(P(z)P(z+\alpha(z))) \neq 0,\ z\overline{z} = e\}.$$

Proof. — The closure of the domain $\Xi = X_{\mathbb{C}} \cap T_\Omega$ in $X_{\mathbb{C}}$ is given by $\overline{\Xi} = X_{\mathbb{C}} \cap \overline{T}_\Omega$. From the preceeding theorem it is clear that the set $\{x \in V |\ \det B_{-\alpha}(x,x) \neq 0\}$ is contained in $\overline{\Xi}$. Let x be an element of this set. We have to show that there is $g \in G$ such that $x = g \cdot 0$. By [HOØ91, Lemma 1.3], the closure $\overline{\Xi}$ is given by $\overline{\Xi} = G \exp(i\overline{C}) \cdot 0$, hence we can write $x = g \exp(iY) \cdot 0$ with $Y \in \overline{C}$. But if $Y \neq 0$, then $\exp(iY) \cdot 0$ does not belong to the real Jordan-algebra V, leading to a contradiction, hence $Y = 0$ and $x = g \cdot 0$ which had to be shown. The other equalities are obtained by Cayley-transform and the description of Σ as the set of points $z \in V_{\mathbb{C}}$ such that $z\overline{z} = e$. ∎

Examples. — 1. The Cayley-type spaces: here α is the involution of a direct product $V = V_1 \times V_1$ of Euclidean Jordan algebras given by $\alpha(x,y) = (y,x)$. The condition $\det P(z+\alpha(z)) \neq 0$ is equivalent to $z = (x,y)$ with $x+y$ invertible in V_1, hence

$$C(\Xi) = \{(x,y) \in D \times D |\ \det P(x+y) \neq 0\},$$

$$C(X) = \{(x,y) \in \Sigma \times \Sigma |\ \det P(x+y) \neq 0.\}$$

The realization used in [Cha95] is related to this one by a "rotation of angle $\frac{\pi}{2}$", $J(x,y) = (-y,x)$ (see [Be96b, Ex.2.2.10]).

2. The group case $X^{(-\alpha)} = Sp(n,\mathbb{R})$: here $V = Sym(2n,\mathbb{R})$ and $\alpha(X) = FXF^{-1}$ where $F = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, see [Be96b, section 1.5 B]. Hence

$$C(\Xi) = \{Z \in Sym(2n,\mathbb{C}) |\ \det(Z - FZF) \neq 0,\ ZZ^* << I\},$$

$$C(X) = \{Z \in Sym(2n,\mathbb{C}) |\ \det(Z - FZF) \neq 0,\ ZZ^* = I\}.$$

3. The group case $X^{(-\alpha)} = U(p,q)$: here $V = Herm(p+q,\mathbb{C})$ and $\alpha(X) = I_{p,q} X I_{p,q}$, see [Be96b, section 1.5 B]. Hence

$$C(\Xi) = \{Z \in M(n,\mathbb{C}) |\ \det(Z + I_{p,q} Z I_{p,q}) \neq 0,\ ZZ^* << I\},$$

$$C(X) = \{Z \in M(n,\mathbb{C}) |\ \det(Z + I_{p,q} Z I_{p,q}) \neq 0,\ ZZ^* = I\}.$$

4. The space $X^{(-\alpha)} = SO^*(2n)/SO(n,\mathbb{C})$: here $V = Herm(n,\mathbb{C})$ and $\alpha(X) = X^t$, see [Be96b, Prop.1.5.2]. Hence

$$C(\Xi) = \{Z \in M(n,\mathbb{C}) |\ \det(Z + Z^t) \neq 0,\ ZZ^* << I\},$$

$$C(X) = \{Z \in M(n,\mathbb{C}) |\ \det(Z + Z^t) \neq 0,\ ZZ^* = I\}.$$

5. The space $X^{(-\alpha)} = Sp(2n,\mathbb{R})/Sp(n,\mathbb{C})$: here $V = Herm(2n,\mathbb{C})$ and $\alpha(X) = FX^tF^{-1}$ with F as in example 2, see [Be96b, Prop.1.5.2]. Hence

$$C(\Xi) = \{Z \in M(2n,\mathbb{C}| \det(Z - FZ^tF) \neq 0,\ ZZ^* << I\},$$

$$C(X) = \{Z \in M(2n,\mathbb{C}| \det(Z - FZ^tF) \neq 0,\ ZZ^* = I\}.$$

6. Hyperboloids. Here V and α are as in 2.1, example 3 b). Then

$$\Xi = \{z \in V + i\Omega |\, (z|z) \neq -1\},$$

where Ω is the Lorentz-cone in V.

References

[Be96a] W. BERTRAM, Un théorème de Liouville pour les algèbres de Jordan, Bulletin de la Soc. Math. Française 124 (1996) p. 299-327.

[Be96b] W. BERTRAM, On some causal and conformal groups, preprint of the Mittag-Leffler institute, 1996.

[Be96c] W. BERTRAM, Algebraic structures of Makarevič spaces. II. (in preparation)

[Bet96] F. BETTEN, Kausale Kompaktifizierung kompakt-kausaler Räume, thesis, Göttingen 1996.

[Cha95] M. CHADLI, Domaine complexe associé à un espace symétrique de type Cayley, CRASP, t. 321, Série I, p. 1157 - 1162, 1995.

[FG95] J. FARAUT and S. GINDIKIN, Pseudo-Hermitian Spaces of Tube Type, preprint 1995.

[FK94] J. FARAUT and A. KORANYI, Analysis on symmetric Cones, Oxford University Press 1994.

[HHL89] J. HILGERT, K.H. HOFFMANN and J.D. LAWSON, Lie groups, convex cones, and Semi-groups. Oxford University Press, 1989.

[HOØ91] J. HILGERT, G. ÓLAFSSON and B. ØRSTED, Hardy spaces on affine symmetric spaces, J. reine angew. Math. 415, p. 189-218, 1991.

[KØ95] K. KOUFANY and B. ØRSTED, Function spaces on the Olshanski semi-group and Gelfand Gindikin-program, preprint 1995.

[Lo69] O. LOOS, Symmetric Spaces I, Benjamin, New York 1969.

[Lo77] O. LOOS, Bounded symmetric domains and Jordan pairs, Lecture Notes, Irvine 1977.

[Ma73] B.O. MAKAREVIČ, Open symmetric orbits of reductive groups in symmetric R-spaces, Math. USSR Sbornik 20 (1973), No 3, p. 406-418.

[OØ96] G. ÓLAFSSON and B. ØRSTED, Causal compactifications and Hardy Spaces. Preprint of the Mittag-Leffler institute, 1996.

[Sa80] I.SATAKE, Algebraic Structures of symmetric domains, Iwanami Shoten, Princeton 1980.