

Reproducing Kernels on Vector Bundles

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Abstract

We propose a framework in which reproducing kernels can be defined for Hilbert spaces of sections of complex vector bundles. As an example we describe families of Bergman spaces of sections which occur naturally in the study of weighted Bergman and Hardy spaces.

Introduction

A *reproducing kernel space* \mathfrak{H} is a Hilbert space of functions such that evaluation ev_z at any point z is a continuous linear functional on \mathfrak{H} . Then ev_z is given by the scalar product with an element K_z of \mathfrak{H} . The function $K(z, w) := K_w(z)$ is called the *reproducing kernel of \mathfrak{H}* . One of the prime examples of such Hilbert spaces is the Bergman space of square integrable holomorphic functions on a bounded domain in \mathbb{C}^n , where the continuity of point evaluations is a consequence of the Cauchy integral formula.

The theory of reproducing kernels for Hilbert spaces of scalar valued functions can be traced back as far as 1909 and was developed in a systematic way by N. Aronszajn in the forties (cf. [Ar50] which also contains a historical account of the early developments). An extension to vector valued functions has been worked out by R. Kunze in [Ku66]. A systematic account of the vector valued theory which also includes representation theory of semigroups on reproducing kernel spaces can be found in K.-H. Neeb's book [Ne98] which we use as our standard reference.

In the present note we generalize the concept of Hilbert spaces of vector valued functions on a manifold and their reproducing kernels to the case of *Hilbert spaces*

of sections of vector bundles and their reproducing kernels. Such a set-up appears already in the literature (cf. e.g. [Ko68]), but it seems that no systematic account of the relevant definitions and general results in this case has been given so far. We hope that this note helps to fill this gap. The results presented here are not original; most of them carry over directly from the case of vector-valued functions, but some become simpler and more transparent than in the case of functions. This concerns in particular the notion of *invariant* kernels (Chapter 2) and a related theorem, due to S. Kobayashi, on the irreducibility of certain unitary representations (Theorem 2.5).

Our approach is motivated by problems in the harmonic analysis of certain reproducing kernel spaces. As a simple example, consider the *weighted Bergman spaces* $\mathfrak{H}_m(D)$, $m \in \mathbb{N}$, of holomorphic functions f on the unit disc D such that

$$\int_D |f(z)|^2 (1 - |z|^2)^{m-1} dz < \infty.$$

This family of spaces plays an important role in the harmonic analysis of the group $G(D) = \mathrm{SU}(1, 1)$; similar definitions can be made for more general bounded symmetric domains, cf. e.g. [FK94, Ch.XIII]. Now, for $m = 1$ this is just the classical Bergman space which, as is well-known, has a natural invariant interpretation as a space of square-integrable holomorphic top-degree forms. Therefore one may ask whether for $m > 1$ these spaces have similar natural interpretations. The answer is “Yes”: the weighted Bergman spaces defined above are indeed as natural as the Bergman space itself – in Chapter 3 we realize them as spaces of holomorphic forms with values in certain line bundles. Once this realization is obtained, our formulas do no longer depend on the picture of D as a disc or an upper half-plane; in this sense, our description is “invariant.” One should note that function spaces like the weighted Bergman spaces may have several “invariant interpretations,” and it is a matter of the mathematical problem in question to choose the most suitable one. For example, the weighted Bergman spaces may also be considered as being related to the space with parameter $m = 1$ by a *gauge transformation*; see Chapter 4 for definition.

More difficult than the invariant description of Bergman spaces is the invariant description of *Hardy spaces* which form another important class of reproducing kernel spaces. The classical Hardy space $H^2(D)$ is the space of holomorphic functions f on the disc having L^2 -boundary values on the circle in the sense that

$$\|f\|^2 := \sup_{0 < r < 1} \int_{S^1} |f(ru)|^2 du < \infty.$$

The group $G(D)$ has a unitary representation on this space; however, this is not obvious from the definition since the measure du on S^1 is *not* invariant under $G(D)$. In the joint project [BCFH97] we will propose – using the general framework outlined here – an invariant definition of Hardy spaces as spaces of sections of vector

bundles for which the unitary action of $G(D)$ is indeed natural. This is a step leading to a better understanding not only of classical Hardy spaces, but also of a class of “non-classical” Hardy spaces defined in [HOØ91]. Several authors have already studied the relation of these non-classical Hardy spaces with the classical ones, and it is our impression that an invariant formulation will turn out to be very useful in studying this problem.

1 Reproducing Kernels

Let M be a topological space and $p: \mathbf{V} \rightarrow M$ a complex vector bundle. We assume that the fibers \mathbf{V}_z over $z \in M$ are finite dimensional and denote the complex antilinear dual bundle by $q: \mathbf{V}^* \rightarrow M$. This means that the fiber \mathbf{V}_z^* of \mathbf{V}^* consists of the complex antilinear functionals on \mathbf{V}_z . The corresponding evaluation map will be denoted by $\langle \cdot, \cdot \rangle_z: \mathbf{V}_z^* \times \mathbf{V}_z \rightarrow \mathbb{C}$. We will use the canonical identification $\mathbf{V}_z \leftrightarrow (\mathbf{V}_z^*)^*$, $v \mapsto \hat{v}$, given by $\hat{v}(\xi) = \overline{\xi(v)}$, and its global analog $\mathbf{V} \cong \mathbf{V}^{**}$ without further mentioning.

We write $C(M, \mathbf{V})$ for the continuous sections of \mathbf{V} . If M is a manifold and \mathbf{V} a smooth vector bundle, we write $C^\infty(M, \mathbf{V})$ for the smooth sections. Moreover, if M is a complex manifold and \mathbf{V} is a holomorphic vector bundle, then we denote the holomorphic sections of \mathbf{V} by $\mathcal{O}(M, \mathbf{V})$. The point evaluations $f \mapsto f(z)$ will be denoted $\text{ev}_z: C(M, \mathbf{V}) \rightarrow \mathbf{V}_z$.

Definition 1.1. A complex vector subspace $\mathfrak{H} \subseteq C(M, \mathbf{V})$ is called a *Hilbert space of sections* if it carries a Hilbert space structure for which the point evaluations $\text{ev}_z: \mathfrak{H} \rightarrow \mathbf{V}_z$, $f \mapsto f(z)$ are continuous. ■

If we reverse the complex structure on the fibers of \mathbf{V} we write $\overline{\mathbf{V}}$ instead of \mathbf{V} . The fibers of \mathbf{V} will then be denoted by $\overline{\mathbf{V}}_z$. Consider the *exterior tensor product bundle* $\mathbf{V} \boxtimes \overline{\mathbf{V}} \rightarrow M \times M$ with fibers

$$(\mathbf{V} \boxtimes \overline{\mathbf{V}})_{(z,w)} = \mathbf{V}_z \otimes \overline{\mathbf{V}}_w = \text{Hom}(\mathbf{V}_w^*, \mathbf{V}_z).$$

Recall that the dual operator $A^*: \mathbf{V}_z^* \rightarrow \mathbf{V}_w^*$ for a \mathbb{C} -linear operator $A: \mathbf{V}_w \rightarrow \mathbf{V}_z$ is determined by the formula

$$(1.1) \quad \langle \xi, A\eta \rangle_{q(\xi)} = \overline{\langle \eta, A^*\xi \rangle_{q(\eta)}}$$

Now suppose that $\mathfrak{H} \subseteq C(M, \mathbf{V})$ is a Hilbert space of sections. Then the dual map $\text{ev}_z^*: \mathbf{V}_z^* \rightarrow \mathfrak{H}$ is defined by

$$(1.2) \quad (\text{ev}_z^*(\xi) \mid f)_{\mathfrak{H}} = \langle \xi, f(z) \rangle_z$$

for $\xi \in \mathbf{V}_z^*$. Consider the functions

$$(1.3) \quad K_\xi := \text{ev}_z^*(\xi): M \rightarrow \mathbf{V}$$

defined for $\xi \in \mathbf{V}_z^*$, and note that $K_\xi \in C(M, \mathbf{V})$ since we assumed that \mathfrak{H} consisted of continuous sections. We will identify \mathfrak{H} with its antilinear dual \mathfrak{H}^* via $f \mapsto (f | \cdot)_{\mathfrak{H}}$. Then we can define

$$(1.4) \quad K(z, w) = \text{ev}_z \circ \text{ev}_w^* \in \text{Hom}(\mathbf{V}_w^*, \mathbf{V}_z)$$

for all $z, w \in M$. Then for $\xi \in \mathbf{V}_z^*$ and $\eta \in \mathbf{V}_w^*$ we calculate

$$\begin{aligned} \langle \xi, K(z, w)\eta \rangle_z &= \langle \xi, \text{ev}_z \circ \text{ev}_w^* \eta \rangle_z \\ &= (\text{ev}_z^* \xi | \text{ev}_w^* \eta)_{\mathfrak{H}} = \overline{(\text{ev}_w^* \eta | \text{ev}_z^* \xi)_{\mathfrak{H}}} \\ &= \overline{\langle \eta, \text{ev}_w \circ \text{ev}_z^* \xi \rangle_w} = \overline{\langle \eta, K(w, z)\xi \rangle_w} \\ &= \langle \xi, K(w, z)^* \eta \rangle_z, \end{aligned}$$

which implies

$$(1.5) \quad K(z, w) = K(w, z)^*.$$

Moreover,

$$\langle \xi, K(z, w)\eta \rangle_z = \langle \xi, \text{ev}_z \circ \text{ev}_w^* \eta \rangle_z = \langle \xi, \text{ev}_w^* \eta(z) \rangle_z = \langle \xi, K_\eta(z) \rangle_z$$

implies

$$(1.6) \quad K_\eta(z) = K(z, q(\eta))\eta \forall \eta \in \mathbf{V}^*$$

Note that (1.5) and (1.6) imply that K is a continuous section of the bundle $\mathbf{V} \boxtimes \overline{\mathbf{V}}$ over $M \times M$. Finally we observe

$$(1.7) \quad \sum_{j,k=1}^n \langle \xi_k, K(q(\xi_k), q(\xi_j))\xi_j \rangle_{q(\xi_j)} = \left\| \sum_{j=1}^n \text{ev}_{q(\xi_j)}^* \xi_j \right\|_{\mathfrak{H}}^2$$

for $\xi_1, \dots, \xi_n \in \mathbf{V}^*$.

Definition 1.2. A section $K \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$ is called a *positive definite kernel* if for every finite sequence $\xi_1, \dots, \xi_n \in \mathbf{V}^*$ the expression

$$\sum_{j,k=1}^n \langle \xi_k, K(q(\xi_k), q(\xi_j))\xi_j \rangle_{q(\xi_k)}$$

is real and non-negative. ■

Lemma 1.3. *Let $K \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$ be a positive definite kernel for \mathbf{V} . Then*

- (i) $\langle \xi, K(q(\xi), q(\xi))\xi \rangle_{q(\xi)} \geq 0$ for all $\xi \in \mathbf{V}^*$.
- (ii) $K(w, z) = K(z, w)^* \in \text{Hom}(\mathbf{V}_z^*, \mathbf{V}_w)$ for all $z, w \in M$.

Proof. The first part is an immediate consequence of the definitions. For the second we note that

$$\begin{aligned} & \langle \xi, K(q(\xi), q(\xi))\xi \rangle_{q(\xi)} + \langle \xi, K(q(\xi), q(\eta))\eta \rangle_{q(\xi)} + \\ & + \langle \eta, K(q(\eta), q(\xi))\xi \rangle_{q(\eta)} + \langle \eta, K(q(\eta), q(\eta))\eta \rangle_{q(\eta)} \end{aligned}$$

is non-negative for $\xi, \eta \in \mathbf{V}^*$. Using (i), this implies

$$\langle \xi, K(q(\xi), q(\eta))\eta \rangle_{q(\xi)} + \langle \eta, K(q(\eta), q(\xi))\xi \rangle_{q(\eta)} \in \mathbb{R}.$$

Therefore

$$\text{Im} \langle \xi, K(q(\xi), q(\eta))\eta \rangle_{q(\xi)} = -\text{Im} \langle \eta, K(q(\eta), q(\xi))\xi \rangle_{q(\eta)} = \text{Im} \langle \xi, K(q(\xi), q(\eta))^* \eta \rangle_{q(\xi)}.$$

Replacing ξ by $i\xi$ now implies the claim. \blacksquare

Theorem 1.4. *Let M be a topological space and $p: \mathbf{V} \rightarrow M$ a complex vector bundle. Suppose that $K \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$. Then the following statements are equivalent:*

- (1) K is a positive definite kernel for \mathbf{V} .
- (2) There exists a Hilbert space $\mathfrak{H} \subseteq C(M, \mathbf{V})$ such that $\text{ev}_z|_{\mathfrak{H}}: \mathfrak{H} \rightarrow \mathbf{V}_z$ is continuous and

$$K(z, w) = \text{ev}_z \circ \text{ev}_w^* \in \text{Hom}(\mathbf{V}_w^*, \mathbf{V}_z)$$

for all $z, w \in M$.

Proof. The implication “(2) \Rightarrow (1)” follows from (1.7). For the converse we define $K_\eta \in C(M, \mathbf{V})$ for $\eta \in \mathbf{V}^*$ via the formula (1.6) and set

$$\mathfrak{H}_K^0 := \text{span}\{K_\eta \in C(M, \mathbf{V}) \mid \eta \in \mathbf{V}^*\}.$$

For $f = \sum_k K_{\eta_k}$ and $g = \sum_j K_{\xi_j}$, let

$$\begin{aligned} (f \mid g)_K &:= \sum_{j,k} \langle \eta_k, (K(q(\eta_k), q(\xi_j))\xi_j) \rangle_{q(\eta_k)} \\ &= \sum_k \langle \eta_k, g \circ q(\eta_k) \rangle_{q(\eta_k)} \\ &= \sum_j \overline{\langle \xi_j, f \circ q(\xi_j) \rangle_{q(\xi_j)}}. \end{aligned}$$

These equations show that $(\cdot | \cdot)$ is well defined on \mathfrak{H}_K^0 . Then (1) implies that the form $(\cdot | \cdot)_K$ is positive semidefinite. In particular one has

$$(1.8) \quad (K_\eta | K_\xi)_K = \overline{\langle \xi, K(q(\xi), q(\eta))\eta \rangle_{q(\xi)}} = \langle \eta, K(q(\eta), q(\xi))\xi \rangle_{q(\eta)}$$

for all $\xi, \eta \in \mathbf{V}^*$ and

$$(1.9) \quad (K_\xi | f)_K = \langle \xi, f \circ q(\xi) \rangle_{q(\xi)}$$

for all $f \in \mathfrak{H}_K^0$ and $\xi \in \mathbf{V}^*$. Using Cauchy-Schwarz and letting $\xi \in \mathbf{V}^*$ vary, one now finds that $(f | f)_K = 0$ implies $f = 0$ in \mathfrak{H}_K^0 whence \mathfrak{H}_K^0 is a pre-Hilbert space. Its completion \mathfrak{H}_K is embedded into $C(M, \mathbf{V})$ via

$$\langle \xi, f(q(\xi)) \rangle_{q(\xi)} = (K_\xi | f)_K$$

for all $\xi \in \mathbf{V}^*$ and $f \in \mathfrak{H}_K$. Now fix $\eta, \xi \in \mathbf{V}$ with $q(\eta) = w$ and $q(\xi) = z$ and consider the dual map $\text{ev}_w^*: \mathbf{V}_w^* \rightarrow \mathfrak{H}_K$. Then the calculation

$$\begin{aligned} \langle \xi, \text{ev}_z \circ \text{ev}_w^* \eta \rangle_z &= \langle \xi, \text{ev}_w^* \eta(z) \rangle_z = (K_\xi | \text{ev}_w^* \eta)_K = \overline{\langle \eta, K_\xi(w) \rangle_w} \\ &= \overline{\langle \eta, K(w, z)\xi \rangle_w} = \langle \xi, K(z, w)\eta \rangle_z \end{aligned}$$

yields (1.4). ■

The equation (1.9) is called the *reproducing property* of the kernel K . The Hilbert space \mathfrak{H}_K is called the *reproducing kernel Hilbert space* associated to the kernel K , and K is called the *reproducing kernel* of \mathfrak{H}_K . Theorem 1.4 shows that any positive definite kernel can be viewed as the reproducing kernel of a Hilbert space of sections. Therefore will call such a kernel simply a reproducing kernel. The argument given in [Ne98, Lemma I.5] shows that for any reproducing kernel Hilbert space $\mathfrak{H} \subseteq C(M, \mathbf{V})$ with reproducing kernel $K(z, w) = \text{ev}_z \circ \text{ev}_w^*$ we have $\mathfrak{H} = \mathfrak{H}_K$.

Remark 1.5. (cf. [Ne98, Prop. I.7]).

- (i) Suppose we choose a family of norms $z \mapsto \|\cdot\|_z$ on the fibers of \mathbf{V} such that $v \mapsto \|v\|_{p(v)}$ is continuous. Then the mapping $\mathfrak{H}_K \rightarrow C(M, \mathbf{V})$ is continuous with respect to the topology of uniform convergence on all subsets of M on which the map $z \mapsto \|K(z, z)\|_{\mathbf{V}_z \otimes \overline{\mathbf{V}}_z}$ is bounded.
- (ii) Suppose that M is a complex manifold. We denote the manifold M , when equipped with the opposite complex structure, by \overline{M} . If a reproducing kernel $K: M \times M \rightarrow \mathbf{V} \boxtimes \overline{\mathbf{V}}$ is holomorphic in the first variable, then the space \mathfrak{H}_K consists of holomorphic sections of \mathbf{V} and K is holomorphic when viewed as a map $K: M \times \overline{M} \rightarrow \mathbf{V} \boxtimes \overline{\mathbf{V}}$. ■

Proposition 1.6. *Let M be a topological space, $p: \mathbf{V} \rightarrow M$ a vector bundle, and $K \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$ a reproducing kernel. Suppose that $N \subseteq M$ is a dense subspace and $K_N := K|_{N \times N}$. Then $K_N \in C(N \times N, \mathbf{V}|_N \boxtimes \overline{\mathbf{V}}|_N)$ is a reproducing kernel, and the restriction to N yields a Hilbert space isomorphism $\mathfrak{H}_K \rightarrow \mathfrak{H}_{K_N}$.*

Proof. Consider the subspace

$$\tilde{\mathfrak{H}}_K := \text{span}\{K_\xi \mid \xi \in q^{-1}(N)\}$$

of \mathfrak{H}_K . Formula (1.8) implies that the map $\xi \mapsto K_\xi, \mathbf{V} \rightarrow \mathfrak{H}_K$ is continuous. Therefore $\tilde{\mathfrak{H}}_K$ is dense in \mathfrak{H}_K . Again from (1.8) we see that the restriction to N gives a surjective isometry $\tilde{\mathfrak{H}}_K \rightarrow \mathfrak{H}_{K_N}^0$. This implies the claim. ■

Remark 1.7. Let M be a topological space, $p: \mathbf{V} \rightarrow M$ a vector bundle, and \mathfrak{H} a complex Hilbert space. Consider the infinite dimensional bundle $\text{Hom}(\mathbf{V}^*, \mathfrak{H})$ which can be given a natural topology. Suppose that ϕ is a continuous section of this bundle. Then, using the methods described in [Ne98, Thm. I.11] it is easy to show that

(i) The formula

$$K(z, w) = \phi(z)^* \phi(w)$$

defines a positive definite kernel $K \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$.

(ii) The map

$$\Phi: \mathfrak{H} \rightarrow C(M, \mathbf{V}), \quad \Phi(f)(z) = \phi(z)^* f$$

vanishes on the orthogonal complement of $\text{span}\{\phi(q(\xi))\xi \mid \xi \in \mathbf{V}^*\}$ in \mathfrak{H} .

(iii) Φ induces a Hilbert space isomorphism

$$\overline{\text{span}}\{\phi(q(\xi))\xi \mid \xi \in \mathbf{V}^*\} \rightarrow \mathfrak{H}_K,$$

where $\overline{\text{span}}$ denotes the closure of the linear span. ■

Remark 1.8. Let M be a topological space, $p: \mathbf{V} \rightarrow M$ a vector bundle, and \mathfrak{H} a complex Hilbert space.

(i) Suppose that $K_1, K_2 \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$ are two reproducing kernels. Apply Remark 1.7 to the Hilbert space $\mathfrak{H} = \mathfrak{H}_{K_1+K_2}$ and to the section ϕ of $\text{Hom}(\mathbf{V}^*, \mathfrak{H})$ defined by

$$\phi(q(\xi))\xi = K_{1,\xi} \oplus K_{2,\xi} \in \mathfrak{H}.$$

Then

$$\mathfrak{H}_{K_1+K_2} = \mathfrak{H}_{K_1} + \mathfrak{H}_{K_2} \subseteq C(M, \mathbf{V}).$$

- (ii) Suppose that we have two bundles $p_j: \mathbf{V}_j \rightarrow M$, $j = 1, 2$ together with two reproducing kernels $K_j \in C(M \times M, \mathbf{V}_j \boxtimes \overline{\mathbf{V}}_j)$. Set $\mathbf{V} := \mathbf{V}_1 \otimes \mathbf{V}_2$ and define a new kernel $K_1 \otimes K_2 \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$ via

$$(K_1 \otimes K_2)(z, w) := K_1(z, w) \otimes K_2(z, w).$$

Apply Remark 1.7 to the Hilbert space $\mathfrak{H} = \mathfrak{H}_{K_1 \otimes K_2}$ and the section ϕ of $\text{Hom}(\mathbf{V}^*, \mathfrak{H})$ defined by

$$\phi(q(\xi_1 \otimes \xi_2))(\xi_1 \otimes \xi_2) = K_{1, \xi_1} \otimes K_{2, \xi_2} \in \mathfrak{H}.$$

Then one obtains a surjective map

$$\mathfrak{H}_{K_1 \otimes K_2} \rightarrow \mathfrak{H}_{K_1} \otimes \mathfrak{H}_{K_2} \subseteq C(M, \mathbf{V}). \quad \blacksquare$$

2 Semigroup Representations

Now suppose that S is a semigroup that acts from the right on \mathbf{V}^* by *vector bundle morphisms*. This means that S also acts on M from the right by continuous maps and we have

- (i) $p(\xi \cdot s) = p(\xi) \cdot s$ for all $\xi \in \mathbf{V}^*$ and $s \in S$.
- (ii) $s_z: \mathbf{V}_z^* \rightarrow \mathbf{V}_{z \cdot s}^*$, $\xi \mapsto \xi \cdot s$ is linear.

Then the dual maps $s_z^*: \mathbf{V}_{z \cdot s} \rightarrow \mathbf{V}_z$ yield a left S -action on $C(M, \mathbf{V})$ defined by

$$(2.1) \quad (s \cdot f)(z) := (s_z)^* \circ f(z \cdot s)$$

for $z \in M$, $f \in C(M, \mathbf{V})$, and $s \in S$.

In the following we assume in addition that $(S, *)$ is an *involutive semigroup*, i.e., that $S \rightarrow S$, $s \mapsto s^*$ is an involutive antiautomorphism of S . Recall the concept of a *Hermitian representation* of S on a pre-Hilbert space \mathfrak{H}^0 from [Ne98, Def. II.3.3]: The vector space $B_0(\mathfrak{H}^0)$ of linear operators $A: \mathfrak{H}^0 \rightarrow \mathfrak{H}^0$ for which a formal adjoint exists is an involutive semigroup, and a Hermitian representation of S on \mathfrak{H}^0 is a semigroup homomorphism $\pi: S \rightarrow B_0(\mathfrak{H}^0)$ preserving the involutions, i.e. $\pi(s^*) = \pi(s)^*$. In the special case where $S = G$ is a group and the involution is the group inversion, a Hermitian representation is the same as a *unitary representation*.

Theorem 2.1. *Let M be a topological space and $p: \mathbf{V} \rightarrow M$ a complex vector bundle. Suppose that $\mathfrak{H}_K \subseteq C(M, \mathbf{V})$ is a reproducing kernel space. Further let $(S, *)$ be an involutive semigroup acting from the right on \mathbf{V}^* by vector bundle morphisms. Then the following are equivalent.*

- (1) $(s_z)^* \circ K(z \cdot s, w) = K(z, w \cdot s^*) \circ (s^*)_w$ for all $z, w \in M$ and $s \in S$.
- (2) \mathfrak{H}_K^0 is invariant under the left action $f \mapsto s \cdot f$ of S on $C(M, \mathbf{V})$, and this action defines a Hermitian representation of S on \mathfrak{H}_K^0 .

Proof. Suppose that (1) holds. Then we have

$$\begin{aligned}
s.K_\xi(z) &= (s_z)^* \circ K_\xi(z.s) \\
&= (s_z)^* \circ K(z.s, q(\xi))\xi \\
&= K(z, q(\xi).s^*) \circ (s^*)_{q(\xi)}(\xi) \\
&= K(z, q(\xi.s^*)) \circ (\xi.s^*) \\
&= K_{\xi.s^*}(z)
\end{aligned}$$

so that $s.K_\xi = K_{\xi.s^*}$ which implies the S -invariance of \mathfrak{H}_K^0 . Similarly we calculate

$$\begin{aligned}
(s.K_\xi | K_\eta)_K &= (K_{\xi.s^*} | K_\eta)_K \\
&= \langle \eta, K(q(\eta), q(\xi.s^*))(\xi.s^*) \rangle_{q(\eta)} \\
&= \langle \eta, (s_{q(\eta)})^* \circ K(q(\eta).s, q(\xi))(\xi) \rangle_{q(\eta)} \\
&= \langle s_{q(\eta)}\eta, K(q(\eta).s, q(\xi))(\xi) \rangle_{q(\eta).s} \\
&= \langle \eta.s, K(q(\eta.s), q(\xi))(\xi) \rangle_{q(\eta.s)} \\
&= (K_\xi | K_{\eta.s})_K \\
&= (K_\xi | s^*.K_\eta)_K
\end{aligned}$$

which shows that the representation is Hermitian. Rearranging these calculations and assuming (2) we find that $(s.K_\xi | K_\eta)_K = (K_\xi | s^*.K_\eta)_K$ implies first $s.K_\xi = K_{\xi.s^*}$ and then

$$\langle \eta, (s_{q(\eta)})^* \circ K(q(\eta).s, q(\xi))(\xi) \rangle_{q(\eta)} = \langle \eta, K(q(\eta), q(\xi.s^*)) \circ s_{q(\xi)}(\xi) \rangle_{q(\eta)}.$$

But this proves (1). ■

If under the hypotheses of Theorem 2.1 the positive definite kernel K satisfies the equivalent conditions (1) and (2) we call it an S -invariant kernel and denote the representation of S on \mathfrak{H}_K^0 by π_K .

Corollary 2.2. *If in the situation of Theorem 2.1 the semigroup is a group and the involution is the group inversion, then π_K is a unitary representation of S .* ■

Note that in the situation of Corollary 2.2 property 2.1(1) takes the form

$$(2.2) \quad K(z.g, w.g) = ((g_z)^*)^{-1} \circ K(z, w) \circ (g_w)^{-1}$$

for $g \in G$. This means that K is a G -invariant section in the usual sense.

Theorem 2.3. *Let M be a topological space and $p: \mathbf{V} \rightarrow M$ a complex vector bundle. Further let $(S, *)$ be an involutive semigroup acting from the right on \mathbf{V}^* by vector bundle morphisms. Suppose that $K \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$ is an S -invariant positive definite kernel. Then*

$$\|\pi_K(s)\|^2 = \sup \left\{ \frac{\langle \xi \cdot s^* s, K(z \cdot s^* s, z) \xi \rangle_{z \cdot s^* s}}{\langle \xi, K(z, z) \xi \rangle_z} \mid z \in M, \xi \in \mathbf{V}_z, K_\xi \neq 0 \right\}.$$

In particular, the representation extends to a representation of S on \mathfrak{H}_K by bounded operators if and only if for each $s \in S$ this supremum is finite.

Proof. Note first that the S -invariance of K implies

$$\begin{aligned} \|\pi_K(s)K_\xi\|_K^2 &= \|K_{\xi \cdot s^*}\|_K^2 \\ &= \langle \xi \cdot s^*, K(z \cdot s^*, z \cdot s^*)(\xi \cdot s^*) \rangle_{z \cdot s^*} \\ &= \langle \xi \cdot s^*, (s_{z \cdot s^*})^* K(z \cdot s^* s, z) \xi \rangle_{z \cdot s^*} \\ &= \langle \xi \cdot s^* s, K(z \cdot s^*, z \cdot s^*) \xi \rangle_{z \cdot s^* s} \end{aligned}$$

for $\xi \in \mathbf{V}_z^*$. Since the set $\{K_\xi \mid \xi \in \mathbf{V}^*\}$ is S -invariant and total in \mathfrak{H}_K , [Ne98, Lemma. II.3.8] now implies the claim. \blacksquare

Remark 2.4. (cf. [Ne98]) Let M be a topological space and $p: \mathbf{V} \rightarrow M$ a complex vector bundle. The set $\mathcal{P}(\mathbf{V})$ of positive definite kernels $K \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$ is a convex cone in $C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$. Using the methods of [Ne98, Section II.4] one can show that in the situation of Corollary 2.2 the representation π_K is irreducible if and only if K is an extremal ray in the cone $\mathcal{P}(\mathbf{V})^S$ of S -invariant kernels. In fact, with a little care one can write down such a statement also for the more general situation of Theorem 2.3. \blacksquare

We present a version of S. Kobayashi's Theorem on the irreducibility of unitary representations on holomorphic vector bundles (cf. [Ko68] and [Ne94, Thm. 6.9]):

Theorem 2.5. *Let M be a connected complex manifold and $p: \mathbf{V} \rightarrow M$ a holomorphic vector bundle. Further let G be a group acting on \mathbf{V}^* by holomorphic bundle automorphisms such that*

- (a) G acts transitively on M .
- (b) There exists a point $z \in M$ such that the stabilizer G_z of z in G acts irreducibly in \mathbf{V}_z .

Then we have the following conclusions:

- (i) Up to scalar multiples there is at most one G -invariant reproducing kernel with values in \mathbf{V} .
- (ii) There exists at most one non-zero Hilbert space of sections in $\mathcal{O}(M, \mathbf{V})$ on which G acts unitarily.
- (iii) If such a space exists, it is irreducible.

Proof. Let $K_1, K_2 \in \mathcal{O}(M \times \overline{M}, \mathbf{V} \boxtimes \overline{\mathbf{V}})$ be non-zero G -invariant positive definite kernels. Then (a) implies that $K_i(z, z) \neq 0$, and the G -invariance gives

$$(h_z)^* \circ K_i(z, z) = K_i(z, z) \circ (h^{-1})_z: \mathbf{V}_z^* \rightarrow \mathbf{V}_z,$$

i.e., both $K_1(z, z)$ and $K_2(z, z)$ intertwine the irreducible dual G_z -actions on \mathbf{V}_z and \mathbf{V}_z^* . Now Schur's Lemma shows that there exists a $0 \neq \lambda \in \mathbb{C}$ such that $\lambda K_1(z, z) = K_2(z, z)$. Again the G -invariance together with $z.G = M$ gives

$$\lambda K_1(w, w) = K_2(w, w) \quad \forall w \in M.$$

Finally, the holomorphy of λK_1 and K_2 on $M \times \overline{M}$ shows that $\lambda K_1 = K_2$. But then the spaces \mathfrak{H}_{K_1} and \mathfrak{H}_{K_2} agree. This proves (i) and (ii).

To show the last part, suppose that \mathfrak{H} is a Hilbert space of sections on which G acts unitarily. If $\mathfrak{H}' \subseteq \mathfrak{H}$ is a closed G -invariant subspace, then (ii) implies that \mathfrak{H}' is either zero or equal to \mathfrak{H} . ■

3 Bergman Spaces

In this section M is a complex manifold of dimension n and $\mathbf{H} \rightarrow M$ a holomorphic *Hermitian* vector bundle. As in Remark 1.5, we denote the manifold M , when equipped with the opposite complex structure, by \overline{M} . Moreover, when we view M as a *real* manifold with almost complex structure, we write $M_{\mathbb{R}}$ instead of M . Then $\overline{\mathbf{H}} \rightarrow \overline{M}$ is the holomorphic vector bundle $\mathbf{H} \rightarrow M$ equipped with the opposite complex structure.

We denote the *holomorphic* tangent bundle by TM and its complex dual by T^*M . A *holomorphic p -form with values in \mathbf{H}* is by definition (cf. [We73]) a holomorphic section of the bundle

$$(3.1) \quad \bigwedge^p T^*M \otimes \mathbf{H} = \text{Hom}\left(\bigwedge^p TM, \mathbf{H}\right)$$

over M . We write $\Omega^p(M, \mathbf{H})$ for the space of holomorphic p -forms and denote the bundle $\bigwedge^n T^*M \otimes \mathbf{H} \rightarrow M$ by $\mathfrak{p}: \mathbf{V} \rightarrow M$.

Given $v, v' \in \mathbf{H}_z$ and $\alpha, \alpha' \in \bigwedge^n T^*M_z$, we define a scalar valued (n, n) -form on $TM_{\mathbb{R}}$ by

$$(3.2) \quad \langle \alpha \otimes v, \alpha' \otimes v' \rangle = (v \mid v')_{\mathbf{H}_z} (\alpha \wedge \overline{\alpha'}).$$

In this way we obtain a sesquilinear map

$$(3.3) \quad \langle \cdot, \cdot \rangle: \Omega^n(M, \mathbf{H}) \times \Omega^n(M, \mathbf{H}) \rightarrow \mathcal{E}^{(n, n)}(M_{\mathbb{R}}),$$

where $\mathcal{E}^{(n,n)}(M_{\mathbb{R}})$ denotes the differential forms of type (n, n) on the almost complex manifold $M_{\mathbb{R}}$. With these definitions

$$\mathcal{B}^2(M, \mathbf{H}) := \{\omega \in \Omega^n(M, \mathbf{H}) \mid \int_{M_{\mathbb{R}}} \langle \omega, \omega \rangle < \infty\},$$

is called the *Bergman space of square integrable sections* (cf. [Ko68, p. 639]). In case \mathbf{H} is just the trivial line bundle with the canonical pairing, this space is known as the *Bergman space of M* and denoted by $\mathcal{B}^2(M)$. As for the classical Bergman space, it can be shown that $\mathcal{B}^2(M, \mathbf{H})$ is a Hilbert subspace of $\Omega^n(M, \mathbf{H})$ with respect to the inner product

$$(3.4) \quad (\omega \mid \omega')_{\mathcal{B}} := \int_{M_{\mathbb{R}}} \langle \omega, \omega' \rangle$$

(cf. [Ch90, p.334]). Moreover, the point evaluations $\text{ev}_z: \mathcal{B}^2(M, \mathbf{H}) \rightarrow \mathbf{V}_z$ are continuous, so that Theorem 1.4 yields a reproducing kernel

$$K \in \Omega^{2n}(M \times \overline{M}, \mathbf{H} \boxtimes \overline{\mathbf{H}})$$

given by

$$K(z, w) = \text{ev}_z \circ \text{ev}_w^* \in \text{Hom}(\mathbf{V}_w^*, \mathbf{V}_z).$$

To see the reproducing property fix $\xi \in \mathbf{V}_z^*$. Then

$$(3.5) \quad \langle \xi, f(z) \rangle_z = (K(\cdot, z)\xi \mid f(\cdot))_{\mathcal{B}} = \int_{M_{\mathbb{R}}} \langle K(w, z)\xi, f(w) \rangle$$

for $z \in M$ and $f \in \mathcal{B}^2(M, \mathbf{H})$.

Example 3.1. (“Weighted Bergman spaces”) The reproducing kernel $K = K^{\mathcal{B}}$ of the Bergman space $\mathcal{B}^2(M)$ is called the *Bergman kernel*. Let us assume that the Bergman space is non-trivial and that $K(z, z)$ defines a nowhere vanishing (n, n) -form on $TM_{\mathbb{R}}$. Then the canonical line bundle on M and its tensor-powers carry canonical structures of Hermitian line bundles, and hence Bergman spaces with values in these bundles are canonically defined. In fact, if $\alpha, \beta \in \Omega^n(M)$, then there exists a function f such that $(\alpha \wedge \overline{\beta})(z) = f(z)K(z, z)$, and clearly $(\alpha, \beta) \mapsto f$ defines a Hermitian structure on the canonical bundle. This Hermitian structure induces a Hermitian structure on the tensor powers of the canonical bundle $\Lambda^n T^*M$, where $n = \dim_{\mathbb{C}} M$. If $M = D$ is the unit disc (or, more generally, a bounded symmetric domain), then the Bergman spaces $\mathcal{B}^2(M, (\Lambda^n T^*M)^k)$ for $k \in \mathbb{N}$ form an important series of *weighted Bergman spaces* (cf. [FK94, p.262]) which are known to be non-trivial. ■

Now suppose that $\phi: \mathbf{H} \rightarrow \mathbf{H}$ is a holomorphic bundle map for which the linear maps $\phi_z: \mathbf{H}_z \rightarrow \mathbf{H}_{\phi(z)}$, $v \mapsto \phi(v)$ are all invertible. Then ϕ acts on the holomorphic sections via

$$(3.6) \quad (\phi^* \sigma)(z) = (\phi_z)^{-1} \circ \sigma \circ \phi(z)$$

for $\sigma \in \mathcal{O}(M, \mathbf{H})$. Then we also have an action on $\Omega^n(M, \mathbf{H})$ via

$$(3.7) \quad \phi^*(\alpha \otimes \sigma) := \phi^* \alpha \otimes \phi^* \sigma,$$

where $\phi^* \alpha$ is the usual pullback of forms. One easily checks that we have

$$(3.8) \quad (\phi_1 \circ \phi_2)^* = \phi_2^* \circ \phi_1^*.$$

If the maps ϕ_z , in addition, are all *isometries* for the Hermitian structure, then the function

$$(3.9) \quad M \rightarrow \mathbb{C}, z \mapsto (\sigma_1 | \sigma_2)(z) := (\sigma_1(z) | \sigma_2(z))_{\mathbf{H}_z}$$

with $\sigma_1, \sigma_2 \in \mathcal{O}(M, \mathbf{H})$ satisfies

$$(3.10) \quad (\phi^* \sigma_1 | \phi^* \sigma_2) = (\sigma_1 | \sigma_2) \circ \phi.$$

But then we obtain

$$(3.11) \quad \phi^* \langle \omega, \omega' \rangle = \langle \phi^* \omega, \phi^* \omega' \rangle,$$

for $\omega, \omega' \in \Omega^n(M, \mathbf{H})$ and hence

$$(3.12) \quad \int_{M_{\mathbb{R}}} \langle \phi^* \omega, \phi^* \omega' \rangle = \int_{M_{\mathbb{R}}} \phi^* \langle \omega, \omega' \rangle = \int_{\phi(M_{\mathbb{R}})} \langle \omega, \omega' \rangle.$$

As a special case we obtain the following theorem:

Theorem 3.2. *Let M be a complex manifold and $p: \mathbf{H} \rightarrow M$ a holomorphic Hermitian vector bundle. Suppose that an involutive semigroup $(S, *)$ acts on \mathbf{H} by bundle maps with contractive fiber maps. Denote the resulting bundle maps by $\tau_s: \mathbf{H} \rightarrow \mathbf{H}$. Then $f \mapsto s.\omega := \tau_s^* \omega$ for $\omega \in \mathcal{B}^2(M, \mathbf{H})$ defines a contractive representation of S on $\mathcal{B}^2(M, \mathbf{H})$.*

Proof. We actually have a semigroup action, since

$$s_1.(s_2.\omega) = \tau_{s_1}^*(\tau_{s_2}^*\omega) = (\tau_{s_2} \circ \tau_{s_1})^* \omega = \tau_{(s_1 s_2)}^* \omega.$$

Finally we calculate

$$\|s.\omega\|_{\mathcal{B}}^2 = \int_{M_{\mathbb{R}}} \langle s.\omega, s.\omega \rangle \leq \int_{M_{\mathbb{R}}} \langle \tau_{s^*}^* \omega, \tau_{s^*}^* \omega \rangle = \int_{\tau_s(M_{\mathbb{R}})} \langle \omega, \omega \rangle \leq \int_{M_{\mathbb{R}}} \langle \omega, \omega \rangle = \|\omega\|_{\mathcal{B}}^2$$

which proves the claim. ■

Corollary 3.3. *Let M be a complex manifold and $p: \mathbf{H} \rightarrow M$ a holomorphic Hermitian vector bundle. Suppose that a group G acts on \mathbf{H} by automorphisms of the Hermitian vector bundle structure. Denote the resulting bundle automorphism by $\tau_g: \mathbf{H} \rightarrow \mathbf{H}$. Then $f \mapsto g \cdot f := \tau_{g^{-1}}^* f$ for $f \in \mathcal{B}^2(M, \mathbf{H})$ defines a unitary representation of G on $\mathcal{B}^2(M, \mathbf{H})$. ■*

Example 3.4. (“Holomorphic discrete series”) If \mathbf{H} is the trivial line bundle, then the group G of all holomorphic diffeomorphisms of M acts canonically on \mathbf{H} , and Corollary 3.3 shows that this leads to a unitary representation of G in $\mathcal{B}^2(M)$. According to Corollary 2.2, the Bergman kernel K is therefore G -invariant; in particular, $K(z, z)$ defines a G -invariant (n, n) -form on M . If this form is nowhere vanishing, then the Hermitian structure on the canonical bundle and its tensor-powers defined in the Example 3.1 is again G -invariant. Applying again Corollary 3.3, we see that G acts also unitarily on the Bergman spaces with values in the tensor powers of the canonical bundle. If $M = D$ is a bounded symmetric domain, then the unitary representations of $G = G(D)$ on the weighted Bergman spaces contribute to the *holomorphic discrete series* of $G(D)$. It is known that $G(D)$ acts transitively on D , and Theorem 2.5 shows that the representations thus obtained are irreducible. The complexification $G_{\mathbb{C}}$ of G contains a subsemigroup with non-empty interior and G as group of units which map D into itself. This semigroup carries a natural involution and hence Theorem 3.2 shows it acts on all of the weighted Bergman spaces by contractions. ■

4 Gauge transformations

The definition of the Bergman space and its inner product depends via (3.2) explicitly on the choice of the Hermitian structure on \mathbf{H} . In this section we calculate how a change of metric effects the reproducing kernel.

Given a complex vector bundle $p: \mathbf{H} \rightarrow M$ let $GL(\mathbf{H}) \subseteq \text{End}(\mathbf{H})$ be the bundle over M whose fibers are given by $GL(\mathbf{H}_z)$. We call the continuous sections of $GL(\mathbf{H})$ *gauge transformations* of \mathbf{H} . A gauge transformation can be interpreted as a bundle automorphism of \mathbf{H} covering the identity of M .

To each gauge transformation we associate a linear bijection $A^\sharp: C(M, \mathbf{H}) \rightarrow C(M, \mathbf{H})$ via

$$(4.1) \quad (A^\sharp f)(z) = A(z)f(z) \quad \forall f \in C(M, \mathbf{H}), z \in M.$$

The continuous sections of $\mathrm{GL}(\mathbf{H})$ act on the Hermitian structures as follows: Let $A \in C(M, \mathrm{GL}(\mathbf{H}))$ be a gauge transformation and $z \mapsto (\cdot | \cdot)_{\mathbf{H}_z}$ a Hermitian structure on \mathbf{H} , we define a Hermitian structure $z \mapsto (\cdot | \cdot)_{\mathbf{H}_z^A}$ on \mathbf{H} via

$$(4.2) \quad (v | v')_{\mathbf{H}_z^A} := (A(z)v | A(z)v')_{\mathbf{H}_z}$$

for $v, v' \in \mathbf{H}_z = \mathbf{H}_z^A$.

In the following we fix a Hermitian vector bundle $p: \mathbf{H} \rightarrow M$, i.e., we fix a Hermitian structure on \mathbf{H} , and denote the Hermitian vector bundle obtained from \mathbf{H} via (4.2) by \mathbf{H}^A .

We take up the situation of Section 3. In particular, M is assumed to be a complex manifold of dimension n and \mathbf{V} is the bundle of holomorphic \mathbf{H} -valued n -forms. Fix a holomorphic gauge transformation $A \in \mathcal{O}(M, \mathrm{GL}(\mathbf{H}))$. The gauge transformation A of \mathbf{H} induces a gauge transformation of \mathbf{V} via $\mathrm{id} \otimes A(z)$ which we also denote by A . Thus we also have an associated map $A^\sharp: \Omega^n(M, \mathbf{H}) \rightarrow \Omega^n(M, \mathbf{H})$ given by

$$(A^\sharp \omega)(z) = A(z)\omega(z) \quad \forall \omega \in \Omega^n(M, \mathbf{H}), z \in M.$$

We set

$$(4.3) \quad \langle \alpha \otimes v, \alpha' \otimes v' \rangle_A = (v | v')_{\mathbf{H}_z^A} (\alpha \wedge \overline{\alpha'})$$

and

$$(4.4) \quad \langle \cdot, \cdot \rangle_A: \Omega^n(M, \mathbf{H}^A) \times \Omega^n(M, \mathbf{H}^A) \rightarrow \mathcal{E}^{(n,n)}(M_{\mathbb{R}}),$$

cf. (3.2) and (3.3). Then we obtain

$$(4.5) \quad \langle \omega, \omega' \rangle_A = \langle A^\sharp \omega, A^\sharp \omega' \rangle.$$

This proves the following theorem:

Theorem 4.1. *Let M be a complex manifold and $p: \mathbf{H} \rightarrow M$ a holomorphic Hermitian vector bundle. Suppose that $A \in \mathcal{O}(M, \mathrm{GL}(\mathbf{H}))$ is a holomorphic gauge transformation. Then the map $A^\sharp: \Omega^n(M, \mathbf{H}) \rightarrow \Omega^n(M, \mathbf{H})$ induces a Hilbert space isomorphism*

$$A^\sharp: \mathcal{B}^2(M, \mathbf{H}^A) \rightarrow \mathcal{B}^2(M, \mathbf{H}),$$

and the reproducing kernel K^A of $\mathcal{B}^2(M, \mathbf{H}^A)$ is given in terms of the reproducing kernel K of $\mathcal{B}^2(M, \mathbf{H})$ by the formula

$$K(z, w) = A(z) \circ K^A(z, w) \circ A(w)^*.$$

Proof. Only the last statement remains to be shown. Let $f \in \mathcal{B}^A := \mathcal{B}(M, \mathbf{H}^A)$. Then, using (3.5) we calculate

$$\begin{aligned}
(K(\cdot, z)\xi \mid A^\sharp f)_\mathcal{B} &= \langle \xi, A^\sharp f(z) \rangle_{\mathbf{H}_z} \\
&= \langle \xi, A(z)f(z) \rangle_{\mathbf{H}_z} \\
&= \langle A^*(z)\xi, f(z) \rangle_{\mathbf{H}_z} \\
&= (K^A(\cdot, z)A^*(z)\xi \mid f)_{\mathcal{B}^A} \\
&= (A^\sharp(K^A(\cdot, z)A^*(z)\xi) \mid A^\sharp f)_\mathcal{B}
\end{aligned}$$

so that $K(\cdot, z)\xi = A^\sharp(K^A(\cdot, z)A^*(z)\xi)$ which implies the claim. ■

References

- [Ar50] Aronszajn, N., “Theory of reproducing kernels”, *Trans. AMS* **68** (1950), 338–404.
- [BCFH97] Bertram, W., M. Chadli, J. Faraut, and J. Hilgert, “On a class of Bergman and Hardy spaces”, in preparation.
- [Ch90] Chabat, B., *Introduction à l’analyse complexe 2. Fonction de plusieurs variable*, Editions MIR, Moscow, 1990.
- [FK94] Faraut, J., and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.
- [HOØ91] Hilgert, J., G. Ólafsson, and B. Ørsted, “Hardy spaces on affine symmetric spaces”, *J. reine angew. Math.* 415, p. 189-218, 1991.
- [Ko68] Kobayashi, S., “Irreducibility of certain unitary representation”, *J. Math. Soc. Japan* **20** (1968), 638–642.
- [Ku66] Kunze, R., “Positive definite operator valued kernels and unitary representations”, in B.R. Gelbaum Ed., *Proc. Conf. Functional Anal.*, Irvine, 1966.
- [Ne94] Neeb, K.-H., “Realization of general unitary highest weight representations”, Preprint, 1994.
- [Ne98] Neeb, K.-H., *Holomorphy and Convexity in Lie Theory*, de Gruyter, to appear.
- [We73] Wells, R.O., *Differential Analysis on Complex Manifolds*, Prentice Hall, 1973.