

From linear algebra via affine algebra to projective algebra

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Abstract. We introduce an algebraic formalism, called “affine algebra”, which corresponds to affine geometry over a field or ring \mathbb{K} in a similar way as linear algebra corresponds to affine geometry with respect to a fixed base point. In a second step, we describe projective geometry over \mathbb{K} by a similar formalism, called “projective algebra”. We observe that this formalism not only applies to ordinary projective geometry, but also to several other geometries such as, e.g., Grassmannian geometry, Lagrangian geometry and conformal geometry. These are examples of *generalized projective geometries* (see [Be02] for the axiomatic definition and general theory). The corresponding *generalized polar geometries* give rise to certain “symmetric spaces over \mathbb{K} ” generalizing the symmetric spaces known from the real case; we give here some important examples of this construction.

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0. Introduction

0.1. Synthetic versus analytic geometry. In this work we present an approach to affine and projective geometry over a field or ring \mathbb{K} which is in a sense situated in between of the approaches usually considered “analytic”, respectively “synthetic” or “axiomatic”. It is analytic in the sense that we accept the base field or ring \mathbb{K} to be given *a priori*, and it is axiomatic in the sense that we describe affine and projective spaces by intrinsic properties and do not define them, as usual in analytic geometry, by some construction using linear algebra over \mathbb{K} . The intrinsic properties we are interested in are *algebraic identities* which in a sense encode the whole geometry. The basic idea is simple: the algebraic laws of affine geometry are obtained by taking the algebraic identities of a vector space (or a \mathbb{K} -module) and forgetting the base

point, and similarly for projective geometry (see below). Of course, there are several ways to do this; the interesting problem is then to understand the algebraic nature of these identities and to find a most “natural” set of identities defining affine geometry. Pushing this idea one step further to projective geometry, we arrive naturally at two identities (PG1) and (PG2) which we call “fundamental identities of projective geometry”. In [Be02] it is shown that they are indeed fundamental in the sense that it is possible to use them as starting point of an axiomatic theory of “generalized projective geometries”. The remarkable feature of the identities (PG1) and (PG2) is that their algebraic structure is equivalent to the foundational identities of *Jordan algebraic structures* (see [Be02]). In fact, at the origin of our research in this domain was precisely the problem to find geometric structures corresponding to *Jordan algebras*, *-triple systems* and *-pairs* and in this way to develop a geometric approach to Jordan theory. This was partially achieved in [Be00] for the real finite-dimensional case; the approach presented here uses entirely different methods and is much more general. However, the motivation by the real finite-dimensional case is still visible in the present work – cf. Chapters VI and XIII of [Be00]. Let us now describe in some more detail what we mean by the terms “affine algebra” and “projective algebra”.

0.2. Affine algebra. In linear algebra, the origin is distinguished among all points in several ways. Wishing to remove this distinction, we may simply consider at the same time *all* additions $+_x$ and *all* multiplications r_x by scalars $r \in \mathbb{K}$ with respect to the new origin x in a vector space V : the expressions of these operations with respect to the “old” operations (without index) are

$$y +_x z = y - x + z, \quad r_x(y) = x + r(y - x) = (1 - r)x + ry. \quad (0.1)$$

In other words, we are interested in the following “product maps”:

$$\begin{aligned} \rho : V \times V \times V &\rightarrow V, & (y, x, z) &\mapsto \rho(y, x, z) = y +_x z, \\ \pi_r : V \times V &\rightarrow V, & (x, y) &\mapsto \pi_r(x, y) = r_x(y). \end{aligned} \quad (0.2)$$

The axioms of a vector space and hence also of an affine space over \mathbb{K} can be formulated entirely in terms of the maps ρ and π_r , $r \in \mathbb{K}$. There are many ways to choose the system of axioms – in Chapter 1 we propose a system of four axioms called (Af1) – (Af4) that is particularly simple and has a “geometric flavor”: we assume that \mathbb{K} is a field of characteristic different from 2 and note that the vector addition can be recovered from the product maps because $\pi_{\frac{1}{2}}(x, y) = \frac{x+y}{2}$ is just the midpoint of x and y and thus

$$x + y = 2\pi_{\frac{1}{2}}(x, y) = \pi_2(0, \pi_{\frac{1}{2}}(x, y)). \quad (0.3)$$

Thus the map ρ is no longer needed, and we obtain an equivalence of categories between affine spaces and sets V equipped with product maps $(\pi_r)_{r \in \mathbb{K}}$ satisfying (Af1) – (Af4) (Theorem 1.1). In any case, no matter what special choice of axioms we make, all categorial notions are completely natural in this approach, and in our opinion the presentation of all standard topics in linear algebra related to affine geometry over \mathbb{K} is now much more pleasant – we give a short outline in Chapter 1, with the emphasis on the *universal space*, which is, in the usual approach, according to M. Berger ([B94, p.67]), a “rather technical chapter” and thus very well suited to illustrate the naturality of our method. Another illustration will be given in Chapter 5 where we consider in a similar way affine spaces equipped with a quadratic form: the affine point of view leads to a natural construction generalizing the *conformal compactification* of Euclidean space (Theorem 5.5).

0.3. Projective algebra. In order to describe projective geometry by a similar approach, we have to introduce one more variable: the additional variable is taken from the dual projective space $X' = P(W^*)$ of hyperplanes in the given projective space $X = P(W)$, where for simplicity

we assume here $W \cong \mathbb{K}^{n+1}$ to be finite-dimensional over a field \mathbb{K} of characteristic different from 2 (in the main text we deal with the case of a base ring). Every hyperplane $[\lambda] \in X'$ defines a natural affine structure on its complement $V_{[\lambda]} = X \setminus [\ker \lambda]$; if $\pi_r^{[\lambda]}$ is the corresponding multiplication map in the sense explained above, then, putting these together, we get a ternary map

$$\mu_r : X \times X' \times X \supset D \rightarrow X, \quad ([x], [\lambda], [z]) \mapsto \pi_r^{[\lambda]}([x], [z]), \quad (0.4)$$

where the (Zariski-dense) subset D is defined by the conditions $\lambda(x) \neq 0$, $\lambda(z) \neq 0$. Dually, we get a ternary map μ'_r defined on a subset of $X' \times X \times X'$. It is easy to derive explicit formulae for these maps (Section 2.4); however, our aim is not to work with these formulae but with their algebraic properties. By the very definition it is clear that, if the middle argument $[\lambda]$ is fixed, the resulting partial map satisfies the identities (Af1) – (Af4). There are two more identities, which we call the “fundamental identities of projective geometry”, denoted by (PG1) and (PG2). In [Be02] we give the following conceptual interpretation of these identities: they describe the behaviour of “left, right and middle multiplications” with respect to the whole structure: the ternary maps μ, μ' give rise, by fixing two arguments, to partial maps defined via

$$\begin{aligned} \mu(x, a, z) &= L_{x,a}(z) = M_{x,z}(a) = R_{a,z}(x), \\ \mu'(a, x, b) &= L_{a,x}(b) = M_{a,b}(x) = R_{x,b}(a); \end{aligned} \quad (0.5)$$

in our case, left and right multiplications $L_{x,a}, R_{a,x}$ are operators on X (related to each other just by changing the scalar r to $1 - r$), whereas the middle multiplication $M_{x,y}$ maps X' to X . Now (PG1) says essentially that $L_{x,a}$ is an “adjoint operator” of $L_{a,x}$ and (PG2) that $M_{x,y}$ is an “adjoint operator” of $M_{y,x}$. Explicitly written down, these are identities in 7 variables which we prove here by elementary linear algebra for the classical geometries: in Chapters 2, 4 and 6 we show that the identities (PG1) and (PG2) hold in the case of *Grassmannian geometry* $X = \text{Gras}_{p,q}(\mathbb{K})$, $X' = \text{Gras}_{q,p}(\mathbb{K})$, in the case of a *Lagrangian geometry* (X the space of maximal isotropic subspaces of a certain form, and $X' = X$) and in the case of *conformal geometry* given by a projective quadric X with $X' = X$; a natural approach to the latter case is given via a “universal space associated to a quadratic affine space” (Chapter 5).

0.4. Polar algebra and symmetric spaces. Maybe even more interesting than the projective geometries are the *polar geometries*, that is, the projective geometries (X, X') together with an identification of the partners X and X' given by a *polarity* $p : X \rightarrow X'$. The interesting new feature (which was in fact one of the starting points of the author’s research in this domain, see [Be00]) is that to any generalized polar geometry in this sense we can associate a *symmetric space*; this construction is very general and defines finite or infinite-dimensional symmetric spaces over fields or rings with the only restriction that 2 is invertible in \mathbb{K} . These spaces are *algebraic over* \mathbb{K} in the finite-dimensional case over a field. The general idea is explained in Chapter 4 of [Be02]; here we describe some typical examples such as the projective spaces themselves: if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then the projective spaces are well-known to be (compact) symmetric spaces, $\mathbb{R}P^n = O(n+1)/(O(n) \times O(1))$, resp. $\mathbb{C}P^n = U(n+1)/(U(n) \times U(1))$ (see e.g. [Hel78]). For other base fields such as $\mathbb{K} = \mathbb{Q}$ such a description is in general no longer possible. However, our formalism still permits to describe spaces such as $\mathbb{Q}P^n$ as symmetric spaces in a sense close to the approach of Loos [Lo69]. They are in general no longer homogeneous, but in [Be02] it is shown that Jordan- and hence Lie-theoretic methods still apply; if one is interested in analysis on such spaces, this is certainly a major progress and a strong argument for using Jordan algebraic methods since Lie theory alone does not suffice here. The list of examples of symmetric spaces to which our methods apply is very long – in fact, in the simple finite-dimensional case over $\mathbb{K} = \mathbb{R}$ a classification is possible (due to work of E. Neher); it reveals the surprising fact that *all classical and about half of the exceptional symmetric spaces* are given by our construction (see [Be00]). Therefore one may conjecture that also in the general case over a base field or ring,

possibly infinite-dimensional, an important class of “symmetric spaces over \mathbb{K} ” is given by the methods introduced in this work.

Organization of the paper. The contents of the paper is as follows:

1. Affine algebra
2. Projective algebra: the Grassmannian case
3. Projective algebra + polarity = polar algebra
4. Projective algebra: the Lagrangian case
5. Affine metric algebra
6. Projective algebra: the conformal case

In [Be02] we develop the axiomatic theory of generalized projective and polar geometries and, in particular, their equivalence with *Jordan pairs* and *Jordan triple systems*. In the present work we treat the “classical geometries” in an elementary way; they correspond to the classical series of Jordan pairs (cf. e.g. [Lo77] or [Sa80]): rectangular matrices (Grassmannian geometries), (skew-)symmetric and (skew-)Hermitian matrices (Lagrangian geometries) and the so-called spin-factors (conformal quadrics). There exist also exceptional geometries; they are not considered in this work – see [BeNe03] for a “universal”, but still elementary model covering all cases. The present work is independent of [Be02] and is addressed not only to specialists but also to the general mathematician who may find it interesting to see fundamental mathematical structures presented in a new form.

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1. Affine algebra

Theorem 1.1. *The category of affine spaces over a field \mathbb{K} of characteristic different from 2 is equivalent to the category of sets V equipped with a family π_r , $r \in \mathbb{K}$, of “product maps”*

$$\pi_r : V \times V \rightarrow V, \quad (x, y) \mapsto \pi_r(x, y) =: r_x(y)$$

satisfying the following properties (Af1) - (Af4):

(Af1) *The map $r \mapsto r_x$ is a homomorphism of the unit group \mathbb{K}^\times into the group of bijections of V fixing x , that is,*

$$\pi_1(x, y) = y, \quad \pi_r(x, \pi_s(x, y)) = \pi_{rs}(x, y), \quad \pi_r(x, x) = x.$$

(Af2) *For all $r \in \mathbb{K}$, the map r_x is an endomorphism of π_s , $s \in \mathbb{K}$:*

$$\pi_r(x, \pi_s(y, z)) = \pi_s(\pi_r(x, y), \pi_r(x, z));$$

for $r \neq 0$, this can also be written $r_x s_y r_x^{-1} = s_{r_x y}$.

(Af3) *The “barycentric condition”: $r_x y = (1 - r)_y x$, that is*

$$\pi_r(x, y) = \pi_{1-r}(y, x).$$

(Af4) *The group generated by the $r_x r_y^{-1}$ ($r \in \mathbb{K}^\times$, $x, y \in V$) is abelian, that is, for all $r, s \in \mathbb{K}^\times$,*

$$r_x r_y^{-1} s_u s_v^{-1} = s_u s_v^{-1} r_x r_y^{-1}.$$

More precisely, in every affine space over \mathbb{K} , the maps $\pi_r(x, y) := (1 - r)x + ry$, $r \in \mathbb{K}$ satisfy (Af1)–(Af4). Conversely, if product maps with the properties (Af1) – (Af4) are given and $o \in V$ is an arbitrary point, then

$$\begin{aligned} x + y &:= x +_o y := \pi_2(o, \pi_{\frac{1}{2}}(x, y)) = 2_o 2_x^{-1}(y), \\ rx &:= \pi_r(o, x) = r_o(x) \end{aligned}$$

defines on V the structure of a vector space over \mathbb{K} with zero vector o , and this construction is inverse to the preceding one. Affine maps $g : V \rightarrow V'$ in the usual sense are precisely the homomorphisms of product maps, that is, maps $g : V \rightarrow V'$ such that $g\pi_r(x, y) = \pi'_r(gx, gy)$ for all $x, y \in V$, $r \in \mathbb{K}$.

Proof. Assume first that V is a vector space over \mathbb{K} and define π_r as in the theorem. Then (Af1) and (Af3) are immediate from the definition; for (Af2) note that $g\pi_s(y, z) = \pi_s(gy, gz)$ for every affine map g . Since \mathbb{K} is commutative, $g := r_x$ ($r \in \mathbb{K}$) is affine, and we get (Af2). Finally, in order to prove (Af4), we use the relation $r_x s_y(z) = (1 - r)x + r(1 - s)y + rsz$ which shows that

$$r_x r_y^{-1} = \tau_{(1-r)x + (r-1)y} = \tau_{(1-r)(x-y)}$$

is a translation and hence commutes with all $s_x s_y^{-1}$, whence (Af4). (Note that (Af2) replaces in a sense the law of associativity and (Af3) replaces the law of commutativity.)

The converse implication is proved by checking the vector space axioms for the structure defined in the theorem. This is straightforward – except for the second distributive law $(r + s)x = rx + sx$ which we prove now: if $s = 0$, this is clear. If $s \neq 0$, we get, using (Af1), (Af2) and (Af3),

$$\begin{aligned} rx + sx &= 2_o 2_{r_o(x)}^{-1} s_o(x) = 2_o r_o 2_x^{-1} r_o^{-1} s_o(x) \\ &= (2r)_o 2_x^{-1} \left(\frac{s}{r}\right)_o(x) = (2r)_o 2_x^{-1} \left(1 - \frac{s}{r}\right)_o(x) \\ &= (2r)_o \left(\frac{1}{2} - \frac{s}{2r}\right)_o(x) = (2r)_o \left(\frac{1}{2} + \frac{s}{2r}\right)_o(x) = (r + s)_o(x) = (r + s)x. \end{aligned}$$

The proof of the remaining statements is again straightforward. ■

As usual, the vector space $(V, +_o, \cdot_o)$ is called the *linearization* or *vectorization* of V with respect to the base point o . Let us give some short comments on generalizations of the preceding theorem:

- (a) If \mathbb{K} is assumed to be a *skew-field* with $2 \neq 0$, then everything goes through if we require (Af2) to hold only if $rs = sr$ in \mathbb{K} .
- (b) If \mathbb{K} is assumed to be a *ring with unit 1* and such that $2 \in \mathbb{K}^\times$, then $(V, +, \cdot)$ has all properties of a *module over \mathbb{K}* with the exception of the second distributive law – our proof of $(r + s)x = rx + sx$ holds only for invertible r . We can add the second distributive law by brute force: incorporating all additive groups of vectorizations in a single map, we define

$$\rho : V \times V \times V \rightarrow V, \quad (x, a, y) \mapsto 2_a 2_x^{-1}(y) = x +_a y = x +_o y -_o a.$$

Now we require in addition to (Af1) – (Af4) the following property (Af5) to hold:

$$(Af5) \quad \pi_{r+s}(x, y) = \rho(\pi_r(x, y), x, \pi_s(x, y))$$

which just says that $(r + s)y = ry + sy$ holds with respect to any point x .

- (c) The case of *characteristic 2*: in this case the vector addition cannot be recovered from the major dilatations in a reasonable way. We have to add a new structural feature, namely

a ternary map $\rho : V \times V \times V \rightarrow V$, $(x, a, y) \mapsto \rho(x, a, y)$ such that for fixed a , $\rho(\cdot, a, \cdot)$ defines an abelian group with neutral element a ; such that all r_x ($r \in \mathbb{K}$, $x \in V$) are endomorphisms of ρ in the obvious sense, and finally such that (Af5) holds. It is trivial that then V with $x + y := \rho(x, o, y)$, $rx := \pi_r(o, x)$ is a \mathbb{K} -module.

For geometry, it is a gain rather than a loss that the axiomatics has to be changed according to the cases (a), (b), (c) since these distinctions correspond to important differences in the geometry of the spaces. The preceding discussion shows that one can develop the whole theory of affine spaces over \mathbb{K} starting with the axioms (Af1) - (Af4), possibly complemented by (Af5). As mentioned in the introduction, this approach has some advantages compared with the usual definition via a simply transitive vector space action. In the following we give a short outline of how one can develop the theory, with the aim to introduce the *universal space* which naturally leads to our next topic, projective geometry. Since the proofs are elementary, we leave them as an easy exercise to the reader or give some short comments. For simplicity, we assume in the following that \mathbb{K} is a commutative ring in which 2 is invertible.

1.2. *Affine subspaces* are subsets closed under all π_r 's. These are precisely the usual affine subspaces.

1.3. We say that the *ratio* of $(a, b, \pi_r(a, b))$ is r . Then a map is affine if and only if it preserves collinearity and all ratios.

1.4. Let M be set and W an affine space over \mathbb{K} . Then the set $\text{Fun}(M, W)$ of maps from M to W becomes an affine space by defining

$$(\pi_r(f, g))(x) := \pi_r(f(x), g(x)),$$

i.e. by requiring that all evaluation maps are homomorphisms. If M also is an affine space, then the space $\text{Aff}(M, W)$ given by all affine maps from M to W is an affine subspace of $\text{Fun}(M, W)$. In particular, we get the spaces $\text{Aff}(\mathbb{K}, V)$ and $\text{Aff}(V, \mathbb{K})$ of “geodesics”, respectively of “affine functions”.

1.5. The *space of complements* is an example that will be important later on: let $E \subset V$ be an affine subspace and put

$$C_{E, V} := \{f \in \text{Aff}(V, V) \mid f^2 = f, \text{im}(f) = E\}.$$

One verifies that $C_{E, V}$ is closed under all π_r 's, i.e. it is an affine subspace. (Here the commutativity of \mathbb{K} is used in a crucial way!) If we fix a base point $o \in E$, then $C_{E, V}$ can be interpreted as the *space of complementary submodules of E* by identifying f with $\ker(f)$. Affinely, $C_{E, V}$ is interpreted as the set of equivalence classes of maximal affine subspaces that intersect E transversally.

1.6. The affine spaces $\text{Aff}(V, V)$ and $\text{Fun}(V, V)$ are equipped with a natural base point, namely the identity map id_V . Thus they carry natural \mathbb{K} -module structures with zero vector id_V . Define

$$\begin{aligned} \text{Tran}(V) &:= \{r_x r_y^{-1} \mid x, y \in V, r \in \mathbb{K}^*\}, \\ \text{Dil}(V) &:= \text{Tran}(V) \cup \{r_x \mid x \in V, r \in \mathbb{K}\}. \end{aligned}$$

One shows that $\text{Tran}(V)$ and $\text{Dil}(V)$ are affine subspaces of $\text{Aff}(V, V)$ containing the identity and hence are submodules in a canonical way. If \mathbb{K} is a field, then we have the additional description

$$\text{Dil}(V) = \{r_x \circ s_y \mid x, y \in V; r, s \in \mathbb{K}\};$$

however, in the case of a general base ring, the composition of two dilatations is in general no longer a dilatation.

1.7. (The universal space.) For any $r \in \mathbb{K}$, the map

$$u_r : V \rightarrow \text{Dil}(V), \quad x \mapsto r_x$$

is affine. It is injective if $1 - r$ is invertible in \mathbb{K} . This is the case for $r = 0$; the imbedding u_0 is called the *canonical imbedding of V into its universal space*. (If \mathbb{K} is a skew-field, everything goes through except that $\text{Dil}(V)$ is no longer contained in $\text{Aff}(V, V)$.)

1.8. The construction of the universal space is functorial in the sense that to any affine map $\alpha : V \rightarrow W$ we can associate linear maps $T\alpha : \text{Tran}(V) \rightarrow \text{Tran}(W)$, $\tilde{T}\alpha : \text{Dil}(V) \rightarrow \text{Dil}(W)$ which are uniquely determined by a natural condition of compatibility with α .

2. Projective algebra: the Grassmannian case

2.1. In the following, unless otherwise stated, \mathbb{K} is a commutative ring with unit 1 and $2 \in \mathbb{K}^\times$. Assume L and H are \mathbb{K} -modules and let $W := L \oplus H$. The *complemented Grassmannian variety (of type L and co-type H)* is by definition the set of all \mathbb{K} -submodules of W that are isomorphic to L and admit a complement that is isomorphic to H :

$$\text{Gras}_L(W; H) := \{E \subset W \mid E \cong L, \exists F \cong H : W = E \oplus F\}.$$

The pair

$$(X, X') = (\text{Gras}_L(W; H), \text{Gras}_H(W; L))$$

is called the *complemented Grassmannian geometry of type (L, H)* . If $L \cong \mathbb{K}^p$ and $H \cong \mathbb{K}^q$, then we write also $\text{Gras}_{p,q}(\mathbb{K})$ for the corresponding Grassmannian geometry X , and if moreover $p = 1$, then X is the *projective space* $\mathbb{K}\mathbb{P}^{q+1}$ and X' its dual *space of hyperplanes* (see the article “Geometry over rings” by F.D. Veldkamp in [Bu95]). In the general case, a pair $(E, F) \in X \times X'$ is called *remote* if F is a complement of E , and we denote by

$$M := \{(E, F) \in X \times X' \mid W = E \oplus F\} \tag{2.1}$$

the set of remote pairs. For fixed elements $E \in X$ and $F \in X'$, we denote the sets of elements that are remote to F , resp. to E , by

$$\begin{aligned} V_F &:= \{G \in X \mid (G, F) \in M\}, \\ V'_E &:= \{K \in X' \mid (E, K) \in M\}. \end{aligned} \tag{2.2}$$

By our definitions, the sets V_F with $F \in X'$ cover X , and dually. In case $(X, X') = (\text{Gras}_{1,q}(\mathbb{K}), \text{Gras}_{q,1}(\mathbb{K}))$ is an ordinary projective geometry over a field, these are just the complements of “hyperplanes at infinity”. In this case, we say that a pair (E, F) is *incident* if it is not remote, i.e. if $E \subset F$. In the general case we do not define the notion of incidence – as has been remarked by Veldkamp, Faulkner and others, as soon as we leave the domain of projective geometry over fields, the notion of remoteness plays a more important role than the notion of non-incidence (cf. [F96] or the article by Veldkamp in [Bu95]).

2.2. For fixed $F \in X'$, the set V_F is precisely the set of all submodules that are complementary to F (by assumption, F admits at least one complement that is isomorphic to L ; but then all complements are isomorphic to L , hence the set of complements is a subset of X). Therefore, as explained in Section 1.5, V_F carries a natural structure of an affine space over \mathbb{K} . Put in another way: for fixed $(E, F) \in M$, V_F carries a natural \mathbb{K} -module structure with zero vector E , and

dually. (Therefore we will sometimes denote elements of M by (o, o') .) As in the preceding chapter, we will put all these structures (for a fixed scalar $r \in \mathbb{K}$) together: for $(E, F) \in M$ and $G \in V_F$ let

$$\mu_r(E, F, G) := r_{E, F}(G) := \pi_r(E, G) = rG \quad (2.3)$$

where the last two expressions are taken in the \mathbb{K} -module V_F with zero vector E . In other words, we consider the ternary map

$$\mu_r : X \times X' \times X \supset D \rightarrow X, \quad (E, F, G) \mapsto \mu_r(E, F, G), \quad (2.4)$$

where

$$D = \{(E, F, G) \in X \times X' \times X \mid (E, F) \in M, (G, F) \in M\}, \quad (2.5)$$

and dually we define a map

$$\mu'_r : X' \times X \times X' \supset D' \rightarrow X', \quad (C, E, F) \mapsto \mu'_r(C, E, F) =: r_{C, E}(F). \quad (2.6)$$

The maps (μ_r, μ'_r) , $r \in \mathbb{K}$, are called the *structure maps of the geometry* (X, X') .

2.3. In order to investigate algebraic properties of the structure maps, it will be useful to have an “explicit formula” (in the spirit of Formula (0.1)) for the structure maps of a Grassmann geometry. Elements of X can be described by (the image of) injections $x : L \rightarrow W$ modulo the equivalence relation given by

$$x \sim y :\Leftrightarrow \exists g \in \text{GL}(L) : y = x \circ g,$$

and elements of X' can be described by (the kernel of) surjections $a : W \rightarrow L$ modulo the equivalence relation

$$a \sim b :\Leftrightarrow \exists g \in \text{GL}(L) : b = g \circ a;$$

the equivalence classes are denoted by $[x]$, respectively by $[a]$. In the sequel, for simplicity of notation, we will write ga etc. instead of $g \circ a$ etc. Then a pair $([x], [a])$ belongs to M if, and only if, $ax : L \rightarrow L$ is a bijection. (Cf. also [Lo94] for the preceding description of Grassmannians.)

2.4. The structure maps are now given by the following explicit formulae:

$$\mu_r([x], [a], [z]) = [(1-r)x(ax)^{-1} + rz(az)^{-1}], \quad (2.7)$$

$$\mu'_r([a], [x], [b]) = [(1-r)(ax)^{-1}a + r(bx)^{-1}b]. \quad (2.8)$$

In fact, the right hand side of (2.7) is well-defined, as is seen by replacing a by ga (resp. x by xg or z by zg). Affinizing with respect to a means to normalize x and z such that $ax = az = \text{id}_L$. Thus in the affine picture with respect to a we have $\mu_r([x], [a], [z]) = [(1-r)x + rz]$ as desired. The other equation is proved in the same way. In case $X = \text{Gras}_{1,q}(\mathbb{K})$ is a projective space, these formulae can be simplified since in this case the group $\text{GL}(L) = \text{GL}(\mathbb{K}) = \mathbb{K}^*$ is commutative: multiplying by $(ax)(az)$ from the left in the first formula, we get

$$\mu_r([x], [a], [z]) = [(1-r)xaz + rzaax], \quad (2.9)$$

and dually.

2.5. Next we want to find algebraic identities for the maps μ_r in the spirit of the preceding chapter. It is clear by the very definition of μ_r that, for a fixed $[\lambda] \in X'$, the partial maps $\mu_r(\cdot, [\lambda], \cdot)$ satisfy the algebraic identities (Af1) – (Af5) stated for the maps π_r in the preceding

chapter. There are two more remarkable identities; we will call them the *fundamental equations of projective geometry*, (PG1) and (PG2).

2.6. The group $G := \text{PGL}(W)$ acts on X and on X' by forward transport of sets; in our operator realization it acts on X by $g.x = g \circ x$ and on X' by $g'.a = a \circ g^{-1}$. From Formula (2.7) we get the equivariance of the multiplication maps with respect to the projective group:

$$g\mu_r([x], g^t[a], [y]) = \mu_r(g[x], [a], g[y]) \quad (2.10)$$

with $g^t[a] = [a \circ g]$, and dually. Geometrically, (2.10) means that the restriction

$$V_{g^t[a]} \rightarrow V_{[a]}, \quad [z] \mapsto g.[z] \quad (2.11)$$

is an affine map. Next, we note that for $s \in \mathbb{K}^\times$ the map $s_{[x],[a]} : X \supset V_{[a]} \rightarrow X$ is induced by the linear map $g = s \text{id}_W + (1-s)x(ax)^{-1}a$. Indeed, this is seen by rewriting (2.7) and (2.8) in the following form:

$$\begin{aligned} \mu_r([x], [a], [z]) &= [(1-r)x(ax)^{-1}az + rz], \\ \mu'_r([a], [x], [b]) &= [(1-r)bx(ax)^{-1}a + rb]. \end{aligned} \quad (2.12)$$

The transpose of $g = s \text{id}_W + (1-s)x(ax)^{-1}a$ is given by

$$\begin{aligned} g^t[b] &= [b \circ g] = [sb + (1-s)(bx)(ax)^{-1}a] \\ &= [s(bx)^{-1}b + (1-s)(ax)^{-1}a] = \mu_s([a], [x], [b]) = s_{[a],[x]}([b]), \end{aligned}$$

and finally we have obtained the relation

$$(s_{[x],[a]})^t = s_{[a],[x]}. \quad (2.13)$$

Taking $g = s_{[x],[a]}$ in Equation (2.10), we therefore get the following identity (PG1), where for simplicity we omit brackets in the notation of elements of X and X' :

$$s_{x,a}(\mu_r(y, s_{a,x}b, z)) = \mu_r(s_{x,a}(y), b, s_{x,a}(z)),$$

which can also be written

$$\mu_s(x, a, \mu_r(y, \mu_s(a, x, b), z)) = \mu_r(\mu_s(x, a, y), b, \mu_s(x, a, z)). \quad (\text{PG1})$$

Above we have assumed that s is invertible in \mathbb{K} ; however, no inverses appear in the identity (PG1), and one can check that it remains valid for any $s \in \mathbb{K}$, provided all expressions are defined – we leave the details to the reader. A dual identity holds with the rôles of X and X' reversed.

2.7. For any ternary map μ one gets, by fixing two elements, operators of *right*, *middle* and *left multiplication*, defined by $R_{b,c} = \mu(\cdot, b, c)$, $M_{a,b} = \mu(a, \cdot, b)$, $L_{a,b} = \mu(a, b, \cdot)$. The identity (PG1) describes the functorial properties of left multiplications for μ_r : for $r \in \mathbb{K}^*$, the pair $(r_{x,a}, r_{a,x}^{-1})$ is an *automorphism of* (X, X') . Right multiplications can be transformed into left multiplications because of the identity (Af3): $\mu_r(a, b, \cdot) = \mu_{1-r}(\cdot, b, a)$. The middle multiplications

$$M_{x,y} := M_{x,y}^{(r)} : X' \supset (V'_x \cap V'_y) \rightarrow X, \quad a \mapsto \mu_r(x, a, y) \quad (2.14)$$

cannot be reduced to right or left multiplications since they *exchange* the partners X and X' . We claim that the map (2.14) is \mathbb{K} -affine when we affinize X with respect to some fixed base point o' and X' with respect to the point $o := M_{y,x}^{(r)}(o')$, where we assume that $(y, o', x) \in D$. We will

check this explicitly by rewriting Formula (2.7) in an affine way, making use of the decomposition $W = L \oplus H$ corresponding to our fixed base point (o, o') : we normalize $x : L \rightarrow W$, $a : W \rightarrow L$ etc. with respect to the decomposition $W = L \oplus H$ such that

$$x = \begin{pmatrix} \text{id}_L \\ X \end{pmatrix}, \quad y = \begin{pmatrix} \text{id}_L \\ Y \end{pmatrix}, \quad a = (\text{id}_L \quad A).$$

Then our assumption that $M_{y,x}^{(r)}(o') = o$ means that $rX + (1-r)Y = 0$. Thus

$$r ax + (1-r) ay = r(\text{id}_L + AX) + (1-r)(\text{id}_L + AY) = \text{id}_L.$$

Using this, we get (for the second equality, multiplying with $(ax)(ay)$ from the right in the bracket)

$$\begin{aligned} \mu_r([x], [a], [y]) &= [(1-r)x(ax)^{-1} + ry(ay)^{-1}] \\ &= [(1-r)x(ay) + y(ay)^{-1}r(ax)(ay)] \\ &= [(1-r)x(ay) + y(ay)^{-1}(\text{id}_L - (1-r)(ay))(ay)] \\ &= [(1-r)x(ay) + ry(ax)] \\ &= [(1-r) \begin{pmatrix} \text{id}_L + AY \\ X(\text{id}_L + AY) \end{pmatrix} + r \begin{pmatrix} \text{id}_L + AX \\ Y(\text{id}_L + AX) \end{pmatrix}] \\ &= \left[\begin{pmatrix} \text{id}_L \\ (1-r)X + rY + (1-r)XAY + rYAX \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} \text{id}_L \\ X + Y - (rXAX + (1-r)YAY) \end{pmatrix} \right]. \end{aligned}$$

This is clearly affine in A , as we wanted to show. The preceding calculation has shown more than announced, namely that the term in question is furthermore essentially quadratic in X ; for $r = \frac{1}{2}$ we have $X = -Y$ and

$$\mu_{\frac{1}{2}}([x], [a], [y]) = \left[\begin{pmatrix} \text{id}_L \\ XAX \end{pmatrix} \right]. \quad (2.15)$$

Having proved that the map given by Equation (2.14) is \mathbb{K} -affine, let us rewrite this condition as an identity for the multiplication maps: we get for all $s \in \mathbb{K}$,

$$M_{x,y}^{(r)} \mu'_s(a, M_{y,x}^{(r)} b, c) = \mu_s(M_{x,y}^{(r)} a, b, M_{x,y}^{(r)} c)$$

which can also be written

$$\mu_r(x, \mu'_s(a, \mu_r(y, b, x), c), y) = \mu_s(\mu_r(x, a, y), b, \mu_r(x, c, y)). \quad (\text{PG2})$$

As for (PG1), a dual identity of (PG2) holds.

2.8. If 2 is not invertible in \mathbb{K} , the maps μ_r, μ'_r are, similarly as in the preceding chapter, not sufficient for an algebraic description of Grassmannian geometries. We have to add a new structural feature by incorporating all the maps ρ (see item (c) after Theorem 1.1) belonging to various affinizations into a single map which now depends on *four* arguments: for a linearization $([x], [a]) \in M$ and $[y] \in V_{[a]}$ let $\tau_{[y]} := \tau_{[y]}^{[x],[a]}$ be the translation by $[y]$ in the \mathbb{K} -module corresponding to the given linearization. This map is induced by an invertible operator on W and therefore acts on X and on X' by bijections; we denote the dual action on X' by

$$\tilde{\tau}_{[y]} := (\tau_{[y]})' = ((\tau_{[y]})^t)^{-1} : X' \rightarrow X'. \quad (2.16)$$

Taking the transpose inverse of the identity $\tau_{[u]}\tau_{[v]} = \tau_{[u]+[v]}$ (sum in the \mathbb{K} -module $V_{[a],[x]}$), we get

$$\tilde{\tau}_{[u]}\tilde{\tau}_{[v]} = \tilde{\tau}_{[u]+[v]} \quad (\text{T})$$

which just reflects the fact that we have an action of translation groups on the dual space. As in 2.4 one can give explicit formulae for these actions:

$$\begin{aligned} \tau_{[x]}^{[y],[a]}([z]) &= [x(ax)^{-1} - y(ay)^{-1} + z(az)^{-1}], \\ \tau_{[x]}^{[y],[a]} &= \text{P}(x(ax)^{-1}a - y(ay)^{-1}a + \text{id}), \\ (\tilde{\tau}_{[x]}^{[y],[a]})^{-1}([b]) &= [(ax)^{-1}a - (bx)^{-1}(by)(ay)^{-1}a + (bx)^{-1}b]. \end{aligned} \quad (2.17)$$

The first formula is proved in the same way as (2.7) (if we affinize with respect to a , then only the term $x - y + x$ remains which is in keeping with (0.2)); the second follows by eliminating the variable z and the third follows by transposing. An affine formula for the last expression, when $([y], [a]) = ([o], [o'])$ is a fixed base point, is given by a similar calculation as in 2.7: with the notation used there one gets the expression $(\text{id}_L + BX)^{-1}B$ which in Jordan theory is known as the *quasi-inverse*, see [Lo77]. – Summing up our discussion of Grassmannian geometry, we have

Theorem 2.9. *The multiplication maps μ_r, μ'_r of a Grassmannian geometry $(\text{Gras}_L(W, H), \text{Gras}_H(W, L))$ over \mathbb{K} satisfy the following identities: the partial maps obtained by fixing the middle element satisfy the identities (Af1) - (Af5), the left, resp. middle multiplications satisfy the relations (PG1), resp. (PG2), and the translation property (T) holds. ■*

2.10. It is not possible to reconstruct Grassmannian or projective geometries from the properties of the preceding theorem – the reason is that these properties hold for the much bigger class of *generalized projective geometries*, see [Be02]. One can show that the projective spaces are characterized among the generalized projective geometries by the property that *lines are independent of the affinization* and thus have an invariant geometric meaning; but this is not a property which can be expressed by algebraic identities. It is an interesting and to our knowledge open problem how one can intrinsically characterize geometries of Grassmannian type among the generalized projective geometries. In view of the results of [Be02], the algebraic counterpart of this problem is the intrinsic characterization of *special* Jordan pairs among general Jordan pairs (which is possible by some special identities such as e.g. Glennie's identity, cf. the appendix of [Lo77]).

2.11. (M as a symmetric space.) The action of the automorphism group $G := \text{PGL}(W)$ is transitive on the space M of vectorizations – in fact, if $W = L \oplus H = E \oplus F$ correspond to two points of M , then there are \mathbb{K} -module isomorphisms $\alpha : L \rightarrow E$ and $\beta : H \rightarrow F$ which give rise to an automorphism g of W transforming one decomposition into the other. The stabilizer group of the first decomposition is isomorphic to $\text{P}(\text{GL}(L) \times \text{GL}(H))$, and thus

$$M \cong \text{P}(\text{GL}(W))/\text{P}(\text{GL}(L) \times \text{GL}(H))$$

as a homogeneous space. If W is a finite dimensional vector space over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , then $M = \text{GL}(p+q, \mathbb{K})/(\text{GL}(p, \mathbb{K}) \times \text{GL}(q, \mathbb{K}))$ is well-known to be a *symmetric space* in the differential geometric or group theoretic sense.

2.12. (Extension of base ring.) If R is an extension of \mathbb{K} , then the data $(W \otimes_{\mathbb{K}} R, L \otimes_{\mathbb{K}} R, H \otimes_{\mathbb{K}} R)$ define a Grassmannian geometry (X_R, X'_R) over R which comes together with a map of (X, X') into this space. For instance, real Grassmannians are naturally imbedded into complex ones. Of course, it does not make sense to say that scalar restriction turns the complex Grassmannians into real ones; but the collection of multiplication maps $\mu_r, r \in \mathbb{R}$ associated to the complex

Grassmannians still satisfies all algebraic properties we have in the real case, and therefore it will make perfectly sense to consider the complex Grassmannians as a new “generalized projective geometry” over $\mathbb{K} = \mathbb{R}$. This remark also applies to *quaternionic* Grassmannians: for r belonging to the center \mathbb{R} of \mathbb{H} , the structure maps μ_r of a quaternionic Grassmannian are defined as above; they have again the properties from Theorem 2.9. More generally, if R is any \mathbb{K} -algebra, then complemented Grassmannian geometries $(X, X') = (\text{Gras}_E(W, F, R), \text{Gras}_F(W, E, R))$ can be defined as in Section 2.1, and (X, X') is then a generalized projective geometry over \mathbb{K} . For instance, if we take $E = F = R$, then (X, X') is the *projective line over R* as defined by A. Herzer in his article “Chain geometries” in [Bu95]. Algebraic properties of the algebra R (such as the property of having “stable rank 2”) can now be translated into geometric properties of the associated generalized projective geometry (see [Be02]).

3. Projective algebra + Polarity = Polar algebra

3.1. If (X, X') is a Grassmannian geometry over \mathbb{K} as in the preceding chapter, then clearly (X', X) is again a Grassmannian geometry, called the *dual geometry*. A corresponding *polar geometry* arises by a suitable identification of the partners X and X' : an *antiautomorphism* of (X, X') is a pair of bijections $(p : X \rightarrow X', p' : X' \rightarrow X)$ compatible with the multiplication maps in the sense that the identities

$$p\mu_r(x, a, y) = \mu'_r(px, p'a, py), \quad p'\mu'_r(a, x, b) = \mu_r(p'a, px, p'b) \quad (3.1)$$

hold (the first equation means that the restriction $p : V_a \rightarrow V'_{p'a}$ is \mathbb{K} -affine, and dually for the second equation). A *correlation* is an antiautomorphism “of order 2” which means that $p' = p^{-1}$. A point $p \in X$ is called *isotropic* if $(x, p(x)) \notin M$ and *non-isotropic* else. A *polarity* is a correlation having some non-isotropic points, and a *null-system* is a correlation for which all points are isotropic. (Our terminology differs from the one used e.g. in [D63] where null-systems are called “null-polarities” and no special term is introduced for what we call a polarity here.) Thus by definition, the set

$$M^{(p)} = \{x \in X \mid (x, p(x)) \in M\} \quad (3.2)$$

associated to a polarity p , is non-empty. Its complement, the *associated quadric* of a polarity p

$$Q^{(p)} = \{x \in X \mid (x, p(x)) \notin M\}, \quad (3.3)$$

may very well be empty; then the polarity is called *elliptic*.

3.2. For the general theory of polar geometries see Chapters 3 and 4 of [Be02] – there it is shown that the complement $M^{(p)}$ of the quadric carries a natural structure of a *symmetric space over \mathbb{K}* ([Be02, Theorem 4.2]), given by the multiplication map (in the sense of O. Loos [Lo69]) $\mu(x, y) := \mu_{-1}(x, p(x), y)$. If we are in the finite dimensional case over $\mathbb{K} = \mathbb{C}$ or \mathbb{R} , then the general theory of symmetric spaces ([Lo69]) implies that the topological connected components are *homogeneous symmetric spaces*, that is, they are of the form G/H , G a Lie group and H an open subgroup of the centralizer of a non-trivial involution of G . For general base fields or rings our symmetric spaces will no longer be homogeneous.

3.3. In this chapter we will describe the most important examples of polar geometries and symmetric spaces associated to Grassmannians. In general, however, a Grassmannian geometry will not admit any polarity, and therefore we will have to impose various restrictions in order to ensure the existence of polarities.

3.4. Assume that $W = L \oplus L$ is the direct sum of two copies of L and take $H = L$ for the definition of X and X' . Then $X = \text{Gras}_L(W, L)$ and $X' = \text{Gras}_L(W, L)$ are really the same sets, and $\text{id} : X \rightarrow X'$ clearly is an antiautomorphism; it is a null-system and not a polarity since every element of X is incident with itself. Note that this null-system is canonical in the sense that it commutes with the whole automorphism group (see [Be03] for an axiomatic approach to canonical null-systems for generalized projective geometries). We can easily define polarities: for instance, we may take

$$g_1 = \begin{pmatrix} 0 & \text{id}_L \\ \text{id}_L & 0 \end{pmatrix} \quad \text{or} \quad g_2 = \begin{pmatrix} 0 & \text{id}_L \\ -\text{id}_L & 0 \end{pmatrix}$$

(matrix with respect to the decomposition $W = L \oplus L$); then the square of g_i in $\text{PGL}(W)$ is the identity, and $g_i : X \rightarrow X'$, $g_i^{-1} : X' \rightarrow X$ defines a polarity (the first copy of L is mapped onto the second one). In order to describe the quadric and its complement associated to g_1 , we look at the polarity given by $p = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}$ which is conjugate to g_1 by the element $R := \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ (the *Cayley transform*) of $\text{PGL}(W)$. Then $M^{(p)}$ is given by all subspaces $F \subset W$ isomorphic to L and such that $W = F \oplus p(F)$. This implies that the intersection of F with both factors L is trivial, and the two projections $\text{pr}_i : F \rightarrow L$, $i = 1, 2$, to the first, resp. to the second factor are injective. Therefore we can write $F = \{(x, \alpha(x)) \mid x \in \text{pr}_1(F)\}$ with some linear map $\alpha : L \supset \text{pr}_1(F) \rightarrow L$. The condition $W = F \oplus p(F)$ then implies that α is in fact a bijection from L onto L . Summing up, F is the *graph* of a uniquely determined element $\alpha \in \text{GL}_{\mathbb{K}}(L)$:

$$F = \Gamma_{\alpha} := \{(x, \alpha(x)) \mid x \in L\} \subset L \oplus L$$

The “graph map” $\alpha \mapsto \Gamma_{\alpha}$ is a bijection of $\text{GL}_{\mathbb{K}}(L)$ onto $M^{(p)}$. In fact, it is also an isomorphism with respect to the symmetric space structures on both sets: the subgroup of $\text{GL}_{\mathbb{K}}(W)$ fixed under conjugation by p is $\text{GL}_{\mathbb{K}}(L) \times \text{GL}_{\mathbb{K}}(L)$, and $M^{(p)}$ is homogeneous under the action of this group and can be written as the homogeneous space $(\text{GL}_{\mathbb{K}}(L) \times \text{GL}_{\mathbb{K}}(L))/\text{GL}_{\mathbb{K}}(L)$, the stabilizer being diagonally imbedded. In the finite dimensional case over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , this is precisely the Lie group $\text{GL}(n, \mathbb{K})$, seen as a symmetric space. We observe that also for general base rings the “symmetric space over \mathbb{K} ” $\text{GL}_{\mathbb{K}}(L)$ is always homogeneous.

The matrix g_1 leads to another realization of $\text{GL}_{\mathbb{K}}(L)$ in the Grassmannian, related to the one just described via the Cayley transform R . The matrix g_2 leads to a different symmetric space. In fact, one can apply arguments known from the real finite dimensional case showing that the associated symmetric space is *c-dual* to the preceding one (see [Be00, Prop. X.1.3]). For $\mathbb{K} = \mathbb{R}$, the *c-dual* of $\text{GL}(n, \mathbb{R})$ is $\text{GL}(n, \mathbb{C})/\text{GL}(n, \mathbb{R})$; in general, the *c-dual* of $\text{GL}_{\mathbb{K}}(L)$ is isomorphic to $\text{GL}_{\mathbb{K}}(L)$ itself if \mathbb{K} contains a square root i of -1 , and to

$$\text{GL}_{\mathbb{K}(i)}(L \otimes_{\mathbb{K}} \mathbb{K}(i))/\text{GL}_{\mathbb{K}}(L)$$

else.

3.5. The preceding example was somewhat special: in ordinary projective geometry, it corresponds to the projective line $\mathbb{K}\mathbb{P}^1 = \text{P}(\mathbb{K} \oplus \mathbb{K})$ which is the only projective space admitting a canonical null-system. In general, one will look for polarities defined by *non-degenerate bilinear forms*. Let us assume that $b_1 : L \times L \rightarrow \mathbb{K}$ and $b_2 : H \times H \rightarrow \mathbb{K}$ are non-degenerate bilinear forms and define a non-degenerate bilinear form on W by $b := b_1 \oplus b_2$. For a subspace $E \subset W$ we let $p(E) = \{w \in W \mid b(E, w) = 0\}$ and $p'(E) = \{w \in W \mid b(w, E) = 0\}$. We are interested in the following two properties:

- (1) $E \in X \Leftrightarrow p(E) \in X'$, $F \in X' \Leftrightarrow p'(F) \in X$,
- (2) for $E \in X$, $(E, p(E)) \in M$ iff $b|_{E \times E}$ is non-degenerate.

In general, neither (1) nor (2) will hold; thus we have to make suitable assumptions which imply these properties. In this section we will assume that \mathbb{K} is a field and that W is finite-dimensional over \mathbb{K} (in other words, $(X, X') = (\text{Gras}_{p,q}(\mathbb{K}), \text{Gras}_{q,p}(\mathbb{K}))$); another possible assumption would be that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and L, H are Hilbert spaces over \mathbb{K} . In both cases, $p : X \rightarrow X'$ and $p' : X' \rightarrow X$ are then well-defined bijections, and the pair (p, p') is an anti-automorphism in the sense defined in Section 3.1: in fact, if $F \in X$ is the image of the injection $x : L \rightarrow W$, then $p(F)$ is the kernel of the (right) adjoint map $x^* : W \rightarrow L$, and if $E \in X'$ is the kernel of the surjection $a : W \rightarrow L$, then $p'(E)$ is the image of the (left) adjoint map $a^* : L \rightarrow W$, and (3.1) holds since

$$\begin{aligned} p(\mu_r([x], [a], [y])) &= [((1-r)a(xa)^{-1} + ra(ya)^{-1})^*] \\ &= [((1-r)a^*(x^*a^*)^{-1} + ra^*(y^*a^*)^{-1})] = \mu_r(p[x], p'[a], p[y]). \end{aligned}$$

If the form b is symmetric or skew-symmetric, then we have $p^{-1} = p'$, i.e. (p, p') is a correlation. From Condition (2) we see that the space of non-isotropic points for this correlation is

$$M^{(p)} = \{E \in X \mid b|_{E \times E} \text{ non-degenerate}\}.$$

By assumption, b is non-degenerate on L and on H , and $p(L) = H$; thus $M^{(p)}$ is not empty, and p is a polarity. The orthogonal group $O(b)$ of the form b acts as a group of automorphisms of the polar geometry (X, X', p, p') ; in particular, it preserves $M^{(p)}$. Now we have to distinguish two cases:

- (i) The forms b_1, b_2, b are symmetric. Then in general the action of $O(b)$ on $M^{(p)}$ is not transitive. It is so if $\mathbb{K} = \mathbb{C}$; in this case $M^{(p)}$ is the symmetric space $O(n, \mathbb{C}) / (O(p, \mathbb{C}) \times O(q, \mathbb{C}))$. If $\mathbb{K} = \mathbb{R}$ and if b_1 and b_2 are positive definite, then the polarity is elliptic, and

$$M^{(p)} = X = O(n) / (O(p) \times O(q)) \quad (3.5)$$

is the Grassmannian, considered as a (compact) symmetric space. The c -dual case arises if b_1 is negative and b_2 positive definite; then the topological connected component containing the base point is

$$M_o^{(p)} \cong O(p, q) / (O(p) \times O(q)); \quad (3.6)$$

this is a *real bounded symmetric domain* in the sense of [Lo77], realized inside its compact dual via a real version of the Borel-embedding. In the general choice, if $B_1 = I_{r_1, s_1}$ is the matrix of b_1 and $B_2 = I_{r_2, s_2}$ the matrix of b_2 (where $I_{k, l} = \begin{pmatrix} 1_k & 0 \\ 0 & -1_l \end{pmatrix}$), we get a symmetric space of the type

$$O(r, s) / (O(r_1, s_1) \times O(r_2, s_2)) \quad (3.7)$$

with $r_1 + r_2 = r$, $s_1 + s_2 = s$. If $\mathbb{K} = \mathbb{Q}$, then for $B_1 = \mathbf{1}_p$, $B_2 = \mathbf{1}_q$, we still have an elliptic polarity, whence $M^{(p)} = X$, but the action of $O(n, \mathbb{Q})$ is far from being transitive on X . These remarks show that the orbit structure in the general case is fairly complicated. As mentioned above, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, we could also work in a Hilbert space context; then we get in a similar way as in (3.5) and (3.6) some of the infinite dimensional real bounded symmetric domains introduced by W. Kaup (see [Kau97]) as well as their ‘‘compact-like’’ duals.

- (ii) The forms b_1, b_2, b are symplectic. We may assume that $p = 2r$, $q = 2s$, $B_1 = J_r$, $B_2 = J_s$ with $J_l = \begin{pmatrix} 0 & 1_l \\ -1_l & 0 \end{pmatrix}$. In this case the action of the group $O(b)$ (i.e. of the symplectic group) is transitive on $M^{(p)}$, and we can write

$$M^{(p)} \cong \text{Sp}(p+q, \mathbb{K}) / (\text{Sp}(p, \mathbb{K}) \times \text{Sp}(q, \mathbb{K})). \quad (3.8)$$

If we use the preceding construction with a symplectic form b , but p and q are odd, then the complement map p is a null-system and not a polarity.

3.6. Instead of bilinear forms we may also take *sesquilinear* forms associated to some involution of a (skew-)field \mathbb{F} having \mathbb{K} (or a homomorphic image of \mathbb{K}) as a subfield fixed under the involution; again we can define polarities in this way. Thus we get, similar as in (3.5), an elliptic polarity of complex Grassmannians, turning them into the symmetric space

$$X = \mathbb{C}P^n = \mathrm{U}(n)/(\mathrm{U}(p) \times \mathrm{U}(q)),$$

and for $B_1 = -\mathbf{1}_p$, $B_2 = \mathbf{1}_q$ we get as the zero-component the bounded symmetric domains $\mathrm{U}(p, q)/(\mathrm{U}(p) \times \mathrm{U}(q))$ (including, for $p = 1$, the *complex hyperbolic spaces*) and, in general, symmetric spaces of a form similar to (3.7). Taking $\mathbb{F} = \mathbb{H}$ and the elliptic polarity, we get the quaternionic Grassmannians as symmetric spaces,

$$X = \mathbb{H}P^n = \mathrm{Sp}(n)/(\mathrm{Sp}(p) \times \mathrm{Sp}(q)),$$

as well as their non-compact duals and other related symmetric spaces – cf. [Be00, Ch. IV and Ch. XII] for a fairly exhaustive list of the classical real and complex spaces.

4. Projective algebra: the Lagrangian case

4.1. *Lagrangian geometries* are generalized projective geometries of the form (X, X') , where $X = X'$ is a space of maximal isotropic subspaces of a symplectic or quadratic neutral form. When speaking about bilinear forms, we need the properties (1) and (2) from Section 3.5, and for simplicity, let us make the same assumptions as there: we assume that \mathbb{K} is a field, that $L \cong \mathbb{K}^n$ is finite-dimensional and that β is a non-degenerate symmetric or skew-symmetric form on $W := L \oplus L$. (It is possible to relax these assumptions or to work in a Hilbert space context.) Then we let

$$X := X' := \{E \subset W \mid E = E^\perp\},$$

where $E^\perp = p(E)$ is the orthogonal complement, as in Section 3.5. We assume that X is non-empty. Then $E \in X$ implies that $E \cong L$, i.e. $\dim E = n$. The pair (X, X') can be seen as a subgeometry of the Grassmannian geometry $(Y, Y') := (Y, Y) := (\mathrm{Gras}_L(W, L), \mathrm{Gras}_L(W, L))$: in fact, as we have seen in Section 3.5, the orthocomplement map (p, p) is compatible with all structure maps (μ_r, μ_r) of (Y, Y) , and this implies without any further calculation that the set of all Lagrangians that are complementary to a given Lagrangian is an affine space over \mathbb{K} . Therefore we can restrict the structure maps (μ_r, μ_r') to (X, X') , and all formal properties from Theorem 2.9 now carry over from (Y, Y') to (X, X') . The space $M \subset (X \times X')$ is now the space of complementary Lagrangian subspaces of W ; from Witt's theorem it follows that M is a homogeneous space,

$$M \cong \mathrm{O}(\beta)/\mathrm{GL}_{\mathbb{K}}(L) \tag{4.1}$$

with $\mathrm{GL}_{\mathbb{K}}(L)$ realized as $\left\{ \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \mid g \in \mathrm{GL}_{\mathbb{K}}(L) \right\}$.

4.2. If β is a symplectic form, (X, X') is called a *symplectic Lagrangian geometry*, and if β is a (neutral) quadratic form, (X, X') is called an *orthogonal Lagrangian geometry*.

4.3. *Polarities* and *null-systems* of (X, X') are defined in the same way as in the preceding chapter; as mentioned there, any polarity gives rise to a symmetric space over \mathbb{K} . Note that the identification $n : X \cong X'$ is a *canonical null-system* (it commutes with the whole automorphism

group $\text{Aut}(X, X')$ which is the orthogonal group of the form). Composing with n , antiautomorphisms are identified with automorphisms; correlations thus correspond to automorphisms g of order 2. Combining choices of β and g we obtain several interesting polar geometries. Let us mention here some of them:

(1) Assume $g = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}$. Then the arguments from 3.4 show that $F \cap g(F) = 0$ iff $F = \Gamma_\alpha$ for a unique $\alpha \in \text{GL}(L)$. Next we fix β . Let A be a non-degenerate quadratic or symplectic form on L (which we may identify with a symmetric or skew-symmetric matrix if we write $L \cong \mathbb{K}^n$).

(1.1) $\beta = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}$. Then the defining relation of the adjoint operator with respect to A ,

$$0 = A(\alpha v, w) - A(v, \alpha^* w) = \beta((v, \alpha v), (w, \alpha^* w)), \quad (4.2)$$

implies that Γ_α is Lagrangian for β iff α is self-adjoint for A . Therefore $M^{(p)}$ is the space of graphs of non-degenerate A -self-adjoint operators. Again we distinguish two cases:

- A is symplectic. (Thus β is symmetric and of signature $(2r, 2r)$.) Then $M^{(p)}$ is identified with the *space of symplectic forms on L* . Any two symplectic forms are conjugate under the general linear group, and thus

$$M^{(p)} \cong \text{GL}(2r, \mathbb{K}) / \text{Sp}(r, \mathbb{K}) \quad (4.3)$$

as a homogeneous space and also as a “symmetric space over \mathbb{K} ”.

- A is symmetric. (Thus β is symplectic.) Then $M^{(p)}$ is identified with the *space of non-degenerate quadratic forms on L* . The action of the general linear group is in general far from being transitive on $M^{(p)}$. This is so if \mathbb{K} is algebraically closed; then

$$M^{(p)} \cong \text{GL}(n, \mathbb{K}) / \text{O}(n, \mathbb{K})$$

as a “symmetric space over \mathbb{K} ”. If $\mathbb{K} = \mathbb{R}$, then the topological connected components are symmetric spaces in the ordinary sense, isomorphic to the homogeneous spaces $\text{GL}(n, \mathbb{R}) / \text{O}(n-l, l)$ parametrized by the *signature* of the form.

(1.2) $\beta = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$. Then the defining relation of the adjoint operator with respect to A implies in a similar way as above that Γ_α is Lagrangian for β iff $\alpha^* = \alpha^{-1}$ for the form A on L . Therefore $M^{(p)}$ is the orthogonal group $\text{O}(A, \mathbb{K})$ of A . This is a “symmetric space over \mathbb{K} of group type” and therefore always homogeneous under the group $\text{O}(A, \mathbb{K}) \times \text{O}(A, \mathbb{K})$ acting from the left and from the right. For $A = J_r$, we get for $M^{(p)}$ the *symplectic groups* $\text{Sp}(r, \mathbb{K})$, and for A a diagonal matrix with entries a_1, \dots, a_p we get the *orthogonal groups* $\text{O}(a_1, \dots, a_p; \mathbb{K})$. In particular, the complex and real cases lead to the orthogonal Lie groups $\text{O}(p, \mathbb{C})$ and $\text{O}(r, s)$, $r + s = p$.

(2) *Elliptic polarities* exist for instance if $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$, given by the orthogonal complement with respect to a scalar product. Then, for $\mathbb{K} = \mathbb{R}$, if β is symplectic, $X \cong \text{U}(n) / \text{O}(n)$, and if β is symmetric neutral, $X \cong \text{O}(n)$ as a symmetric space.

4.3. As in the Grassmannian case, polarities may also be defined via sesquilinear forms, and also starting with skew-fields (see 3.6). The number of symmetric spaces thus obtained increases considerably – see [Be00, Sections I.6 and XII] for a classification in the real case, including among others the group cases $\text{U}(r, s)$ and $\text{Sp}(r, s)$ and spaces of non-degenerate Hermitian forms.

5. Affine metric algebra

5.1. It is known that a vector space V with a non-degenerate quadratic form can be “conformally” imbedded into a quadric – the best known case is the one-point-compactification of the Euclidean vector space $V = \mathbb{R}^n$ given by the sphere S^n , where the imbedding is defined by stereographic projection. More generally, the “conformal compactification” of the space \mathbb{R}^n with a form of signature (p, q) is the projective quadric defined by a form of signature $(p+1, q+1)$, which leads to the action of the “conformal group” $\text{SO}(p+1, q+1)$ on \mathbb{R}^n . This “conformal compactification” can be found, e.g., in the literature related to the classification of Cartan-domains (where it corresponds to the type IV-domains), see e.g. [Sa80], but its construction usually is not intrinsic: one uses external direct sums in terms of which a new quadratic form is defined. In order to give an intrinsic version of this construction, we return in this chapter to the context of affine spaces (Chapter 1) to which we add a new structure given by a “field of quadratic forms”. We assume throughout that \mathbb{K} is a field of characteristic different from 2; with some minor modifications, our construction also works for commutative rings with $2 \in \mathbb{K}^\times$ (see Section 5.8).

5.2. We define a *quadratic affine space* to be an affine space V over \mathbb{K} together with a binary map $q : V \times V \rightarrow \mathbb{K}$, such that:

- (q1) For all $x \in V$, $q_x := q(x, \cdot) : V \rightarrow \mathbb{K}$ is a non-zero quadratic form (in the usual sense) on the vector space (V, x) with zero vector x .
- (q2) The “field of quadratic forms” q_x , $x \in V$, is translation invariant in the sense that $q \circ (\tau \times \tau) = q$ for all translations τ ; in other terms, having fixed a base point o , we have

$$q(x+v, y+v) = q(x, y).$$

The space is called *non-degenerate* if the symmetric bilinear form

$$g_p(x, y) = q_p(x+y) - q_p(x) - q_p(y)$$

associated to one (and hence to all) of the q_p 's is non-degenerate. An *isometry* (resp. *similarity*) between quadratic affine spaces is an affine map α such that, for all $x, y \in V$, $q'(\alpha x, \alpha y) = q(x, y)$ (resp. $q(\alpha x, \alpha y) = \lambda_\alpha q(x, y)$ for a scalar λ_α depending only on α).

5.3. If (V, q) is a quadratic affine space, then $q(x, y) = q(y, x)$ for all $x, y \in V$: in fact, from the translation invariance (q2) we get, with respect to a base point o ,

$$q(y, x) = q_y(x) = q_o(x-y) = q_o(y-x) = q(x, y).$$

5.4. For a fixed origin $o \in V$ and $p \in V$, the function $q_p : V \rightarrow \mathbb{K}$ is a (in general non-homogeneous) quadratic function. In fact,

$$q_p(x) = q_o(x-p) = q_o(x) - g_o(x, p) + q_o(p), \tag{5.1}$$

which can also be written

$$q_p = q_o - p^* + q_o(p), \tag{5.2}$$

where $p^* = g_o(\cdot, p)$. Hence the set

$$Q := \{q_p \mid p \in V\}$$

and its \mathbb{K} -span $\langle Q \rangle$ are contained in the space of all quadratic functions on V . We claim that, if $f \in \langle Q \rangle$, then also the homogeneous components of f (with respect to the fixed base point o)

are contained in $\langle Q \rangle$: in fact, from $q_x - q_y = -(x-y)^* + q_o(x-y)$ we see that all affine functions of the form $h_v = v^* + q_o(v)$ with $v \in V$ belong to $\langle Q \rangle$; then $h_{v+w} - h_v - h_w$ also belongs to $\langle Q \rangle$, which implies that all constants of the form $q_v(w)$ belong to $\langle Q \rangle$. Since q_v is non-zero, the constant 1 and hence all constants belong to $\langle Q \rangle$; we deduce that all linear functions p^* belong to $\langle Q \rangle$. Summing up,

$$\langle Q \rangle = \{p^* \mid p \in V\} \oplus \mathbb{K}1 \oplus \mathbb{K}q_o \quad (5.3)$$

as a vector space.

Theorem 5.5. *Let (V, q) be a non-degenerate quadratic affine space over \mathbb{K} . Then there is a unique symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\langle Q \rangle$ such that for all $x, y \in V$ we have*

$$\langle q_x, q_y \rangle = q(x, y).$$

If a base point $o \in V$ is fixed, then $\langle Q \rangle$ has a direct sum decomposition $\langle Q \rangle \cong V \oplus \mathbb{K} \oplus \mathbb{K}$ into linear, constant and homogeneous quadratic terms, and the bilinear form $\langle \cdot, \cdot \rangle$ is given with respect to this decomposition by the matrix

$$\begin{pmatrix} -g_o & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}.$$

Moreover, the quadratic map

$$V \rightarrow \langle Q \rangle, \quad y \mapsto q_y$$

is an imbedding of V into the quadric associated to $\langle \cdot, \cdot \rangle$; its image is naturally identified with a paraboloid in a hyperplane of $\langle Q \rangle$.

Proof. Uniqueness of the bilinear form is clear since Q generates $\langle Q \rangle$. In order to prove existence, we fix a base point $o \in V$ and define a bilinear form with respect to the decomposition (5.3) by the matrix given in the theorem, i.e. if $f_i(x) = p_i^*(x) + a_i + b_i p_o(x)$, $i = 1, 2$, then

$$\langle f_1, f_2 \rangle := -g_o(p_1, p_2) + a_1 b_2 + a_2 b_1.$$

Thus

$$\begin{aligned} \langle q_x, q_y \rangle &= \langle q_o - x^* + q_o(x), q_o - y^* + q_o(y) \rangle \\ &= q_o(x) - g_o(x, y) + q_o(y) = q(x, y), \end{aligned}$$

proving existence of the desired bilinear form. Moreover, it follows that $\langle q_x, q_x \rangle = q(x, x) = q_x(x) = 0$, i.e. all q_x are isotropic vectors. It remains to show that the image of the imbedding is naturally identified with a paraboloid: according to (5.2), the homogeneous quadratic part of q_p is independent of p , i.e., Q is contained in the affine hyperplane E of $\langle Q \rangle$ having constant last coordinate, equal to q_o . We claim that Q is a paraboloid in E . In fact, identifying E with $V \oplus \mathbb{K}1$, Q is equal to the set

$$\{(-p^*, q_o(p)) \mid p \in V\}.$$

This set can be identified with the graph of the quadratic map $V \rightarrow \mathbb{K}$, $p \mapsto -q_o(p)$. But the graph of a quadratic form $V \rightarrow \mathbb{K}$ is (almost by definition) a paraboloid in $V \oplus \mathbb{K}$. ■

Proposition 5.6. *The preceding construction is functorial: an isometry $f : (V, q) \rightarrow (V', q')$ (resp. a similarity f with $\lambda_f \in \mathbb{K}^*$) induces an imbedding $Q \rightarrow Q'$ which extends to an isometry (resp. similarity) $\tilde{f} : \langle Q \rangle \rightarrow \langle Q' \rangle$. In particular, the group of bijective similarities of (V, q) acts by automorphisms of the quadric in $P(\langle Q \rangle)$.*

Proof. One checks that the extension given by $\tilde{f}(q_x) = q'_{fx}$ is well-defined and has the desired properties. ■

5.7. The bottom right form in the matrix from the theorem is diagonalized by the “real Cayley transform”, yielding 1 and -1 on the diagonal. For instance, if $g_o(x, x) = x^t I_{p,q} x$ is the form on \mathbb{R}^n with signature (p, q) , then the new form $\langle \cdot, \cdot \rangle$ has signature $(p+1, q+1)$. We will call \widehat{Q} the “conformal completion” of the affine space (V, g) . Note that every projective quadric can affinely be realized as a paraboloid, and thus every projective quadric arises in the way just described.

5.8. It is possible to prove an analog of Theorem 5.5 under weaker assumptions: if the form g_o is possibly degenerate, then the space $\{p^* | p \in V\}$ is isomorphic to $V/\ker(g_o)$, with a well-defined form given by $\tilde{g}_o(x^*, y^*) := g_o(x, y)$. If \mathbb{K} is merely a ring with $2 \in \mathbb{K}^\times$, then we have to assume that q_o admits at least one value in \mathbb{K}^\times in order to ensure that the constants appear in the decomposition (5.3).

6. Projective algebra: the conformal case

6.1. In this chapter we will re-consider affine metric spaces from the point of view of projective quadrics, i.e. without a preferred affine realization. We will see that projective quadrics also have the properties from Theorem 2.9 characterizing “generalized projective geometries”. For simplicity, we assume that W is a vector space over a field \mathbb{K} of characteristic different from 2, equipped with a non-degenerate symmetric bilinear form $b : W \times W \rightarrow \mathbb{K}$. We consider the associated projective quadric

$$X = \{[x] \in \mathbb{P}(W) | b(x, x) = 0\}$$

together with its dual projective quadric which is, by definition, the space

$$X' = \{[x^*] | [x] \in X\} \subset \mathbb{P}(W)'$$

of tangent hyperplanes of X (as usual, we write $x^* = b(x, \cdot)$ and identify the class of a linear form with its kernel). We say that elements $[x] \in X$, $[y^*] \in X'$ are *remote* if they are remote in $(\mathbb{P}(W), \mathbb{P}(W)')$, i.e. if $b(x, y) \neq 0$, and define $M \subset (X \times X')$ to be the space of remote pairs. Note that $[x] \in X$ and $[x^*] \in X'$ are never remote (they are “as incident as possible”), and thus the canonical bijection

$$n : X \rightarrow X', \quad [x] \mapsto [x^*]$$

turns out to be a null-system (in the sense of 3.1) and not a polarity; it is canonical in the sense that it commutes with the natural automorphism group $O(b)$. For instance, if X is a sphere in $\mathbb{R}(\mathbb{P}^{n+2})$, then any tangent space intersects X in only one point, and therefore M is essentially the complement of the diagonal in $X \times X$.

6.2. We will show that every element $[y^*] \in X'$ defines an affinzation of $V_{[y^*]} = \{[x] \in X | b(x, y) \in \mathbb{K}^*\}$: denote by $E_{[y^*]} := \{[x] \in \mathbb{P}(W) | b(x, y) \in \mathbb{K}^*\}$ the affinzation of $\mathbb{P}(W)$ having $[y^*]$ as “hyperplane at infinity”. Since $[y]$ lies at infinity, it corresponds to a family of parallel lines in $E_{[y^*]}$ whose projective completions all intersect in $[y]$. Let $\langle y \rangle$ be the corresponding group of translations in the direction of these lines and denote by

$$F_{[y^*]} := E_{[y^*]} / \langle y \rangle$$

the corresponding quotient space; it is, in a natural way, an affine space over \mathbb{K} . Now, $F_{[y^*]}$ is in bijection with $V_{[y^*]}$: indeed, the projective completion of any line l in direction $\langle y \rangle$ intersects the quadric X transversally at one point (namely at $[y]$) and therefore has to have exactly one other intersection point $l_X \in X$ which is remote from $[y^*]$ since l is by definition remote from

$[y^*]$. Thus the map $l \mapsto l_X$ is a bijection $F_{[y^*]} \rightarrow V_{[y^*]}$ which we use to define on $V_{[y^*]}$ the structure of an affine space. The same construction can be done dually, with the rôles of X and X' exchanged. Note that, in case X is a sphere, our construction is really nothing but ordinary stereographic projection, where one usually identifies $F_{[y^*]}$ with the tangent space at some “antipode” of $[y]$ which, in fact, is not canonical from a projective point of view.

6.3. We can recover the structure of a quadratic affine space (in the sense of 5.2) on $V_{[y^*]}$: it is given, for $[x], [z] \in V_{[y^*]}$, by the formula

$$q([x], [z]) = b(x, z).$$

6.4. The multiplication maps μ_r, μ'_r can now be defined in the same way as in the preceding chapters. We claim that μ_r is given by the explicit formula

$$\mu_r([x], [y^*], [z]) = [(1-r)b(z, y)x + rb(x, y)z - r(1-r)b(x, z)y]. \quad (6.1)$$

If $[x]$ and $[z]$ belong to $V_{[y^*]} \subset X$ and $[f] := [f(x, y, z)]$ denotes the right hand side of (6.1), then an easy calculation shows that $b(f, f) = 0$. Moreover, since $b(x, y), b(z, y) \in \mathbb{K}^*$, one deduces that $f \neq 0$, i.e. $[f] \in X$. Now we pass to the affine picture by taking $[y^*]$ as hyperplane at infinity: this means to normalize x and z such that $b(x, y) = b(z, y) = 1$. Then the first two terms in $f(x, y, z)$ reduce to $(1-r)x + rz$ and the last term just adds a translation into direction of $[y]$, and by definition of the affine chart this shows that $[f(x, y, z)]$ is indeed the image of $(1-r)x + rz$ under our chart.

6.5. Clearly the construction from 6.2 is invariant under the natural automorphism group $G = O(b)$, i.e. we have

$$g\mu_r([x], g^t[y^*], [z]) = \mu_r(g[x], [y^*], g[z]) \quad (6.2)$$

for all $g \in O(b)$, which can of course also be directly verified from Formula (6.1). If we identify X and X' , then μ_r is invariant under $O(b)$ in the usual sense.

6.6. Let us write left and middle multiplications corresponding to μ_r in operator form: from (6.1) we get

$$r_{[x], [y^*]} = [(1-r)y^* \otimes x + rb(x, y) \text{id} + r(r-1)x^* \otimes y], \quad (6.3)$$

$$M_{[x], [z]} = [(1-r)z^* \otimes y + rx^* \otimes y - r(1-r)b(x, z) \text{id}_W]. \quad (6.4)$$

Via b , the transpose of, say, $y^* \otimes x$ is identified with $x^* \otimes y$, and thus (6.3) and (6.4) imply

$$(r_{[x], [y^*]})^t = r_{[y], [x^*]}, \quad (M_{[x], [z]})^t = M_{[z], [x]}, \quad (6.5)$$

which combined with (6.2) yields once again the identities (PG1) and (PG2) from Chapter 2.

6.7. (Polarities and symmetric spaces.) *Correlations, null systems and polarities* of (X, X') are defined as in 3.1. As already mentioned, the canonical identification $X \cong X'$ is then a null-system. There also exist polarities: take any element $p \in O(b)$ with $p^2 = \text{id}$ and admitting a point $[x] \in X$ such that $b_p(x, x) := b(x, px) \in \mathbb{K}^*$; then $p : X \rightarrow X \cong X'$ is a polarity. As in the preceding cases, the general theory ([Be02, Theorem 4.1]) implies that $M^{(p)} = \{[x] \in X \mid b(x, px) \in \mathbb{K}^*\}$ carries a natural structure of a “symmetric space over \mathbb{K} ”. For example, in the real case the usual symmetric space structure $\text{SO}(n+1)/\text{SO}(n)$ on the sphere M arises for b of signature $(n, 1)$ and b_p positive definite (elliptic polarity). Another choice of p leads to the *real hyperbolic spaces* $M = \text{SO}(n, 1)/\text{SO}(n)$. In the complex case, an elliptic polarity is obtained if b_p is a positive definite scalar product; then $M = X$ is the compact Hermitian symmetric space $\text{SO}(n+2)/(\text{SO}(n) \times \text{SO}(2))$. For general \mathbb{K} , the corresponding symmetric spaces are no longer homogeneous.

6.8. (Remarks on the incidence structure.) Our discussion of Grassmannian, Lagrangian and conformal geometry has featured the *algebraic* aspects and not the *incidence geometry* of the spaces in question. In fact, we have seen that the algebraic description of these geometries can be done in a rather uniform way (via affinizations plus the identities (PG1) and (PG2)). The incidence structure of these spaces is fairly complicated and differs significantly from case to case (see, e.g., [Ch49], [D63], [Hua45], [F96] for important results on the incidence structure in special cases); it even depends highly on the base field and partially breaks down over rings (where incidence is not the same as non-remoteness). For these reasons it is remarkable that in fact the algebraic structure entirely determines the incidence structure (work in progress).

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