Difference Problems and Differential Problems

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Abstract. We state some elementary problems concerning the relation between *difference calculus* and *differential calculus*, and we try to convince the reader that, in spite of the simplicity of the statements, a solution of these problems would be a significant contribution to the understanding of the foundations of differential and integral calculus.

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Introduction

Differential calculus can be seen as the "continuous extension of difference calculus to singular parameters".[1] This simple observation has been used in [BGN04] for developing a general approach to differential calculus in arbitrary dimension and over very general base fields and rings: let $\mathbb{K}$ be a commutative ring with unit 1 (for a first reading, think of $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) und V, W be $\mathbb{K}$-modules. For any subset $U \subset V$ and any map $f : U \to W$ we define the *first order difference quotient* by

$$f^{]1[}(x, v, t) := \frac{f(x + tv) - f(x)}{t}, \qquad (0.1)$$

wherever this makes sense, i.e., for (x, v, t) belonging to the set

$$U^{]1[} = \{(x, v, t) \in U \times V \times \mathbb{K}^{\times} \,|\, x + tv \in U\}. \qquad (0.2)$$

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[1] I thank Wilhelm Kaup for pointing out to me that this observation is folklore.

The map $f^{]1[} : U^{]1[} \to W$ thus defined is called the *first order difference quotient map*. If we want to stress the dependence on the ring $\mathbb{K}$, we write also $f_{\mathbb{K}}^{]1[}$ and $U_{\mathbb{K}}^{]1[}$. In order to keep track of all three variables, it is also useful to introduce the *extended difference quotient map*

$$\Delta^{]1[}f : U^{]1[} \to W^{]1[}, \quad (x, v, t) \mapsto \big(f(x), f^{]1[}(x, v, t), t\big) \tag{0.3}$$

which satisfies, as is immediately checked, the functorial relations

$$\Delta^{]1[}(g \circ f) = \Delta^{]1[}g \circ \Delta^{]1[}f, \qquad \Delta^{]1[}(\mathrm{id}_U) = \mathrm{id}_{U^{]1[}}. \tag{0.4}$$

Differential calculus is the continuous extension to the singular parameter value $t = 0$. More precisely, let

$$U^{[1]} = \{(x, v, t) \in U \times V \times \mathbb{K} \,|\, x + tv \in U\}; \tag{0.5}$$

this set contains $U \times V \times \{0\}$. Assume that $\mathbb{K}$ is a topological ring such that the unit group $\mathbb{K}^{\times}$ is open dense in $\mathbb{K}$; assume moreover that V and W are topological $\mathbb{K}$-modules and that U is open V; then $U^{]1[}$ is dense in $U^{[1]}$, and both are open in $U \times V \times \mathbb{K}$. We say that f is *of class $\mathcal{C}^1$* if $f^{]1[}$ admits an extension to a *continuous* map $f^{[1]} : U^{[1]} \to W$ (equivalently, if $\Delta^{]1[}$ admits an extension to a continuous map $\Delta^{[1]}f : U^{[1]} \to W^{[1]}$). By density of $\mathbb{K}^{\times}$ in $\mathbb{K}$, this extension is unique, if it exists, and hence the *differential* and the *tangent map* are well-defined by

$$df(x)v := f^{[1]}(x, v, 0), \qquad Tf(x)v := (f(x), df(x)v). \tag{0.6}$$

Again by density, Relation (0.4) implies that $T(g \circ f) = Tg \circ Tf$, i.e., the chain rule. In a similar way, all elementary rules of differential calculus (for instance, the linearity of $df(x) : V \to W$) follow "by density" from simple difference calculus. In [BGN04], it has been shown that, in all relevant cases, these definitions are equivalent to more classical ones; in particular, mappings between finite dimensional real vector spaces are $\mathcal{C}^1$ in our sense iff they are $\mathcal{C}^1$ in the usual sense, and this can even be used to simplify several arguments of usual multivariable analysis – see [Be08] for an elementary account. On the other hand, this approach can be further generalized beyond the context of differential calculus over topological fields and rings (cf. [BGN04]).

All this may look fairly trivial, but that impression changes completely if we now turn to *higher order difference and differential calculus*. In fact, definitions are made such that they can easily be iterated; for instance, f is called *of class $\mathcal{C}^2$* if it is $\mathcal{C}^1$ and if $f^{[1]}$ (or, equivalently, $\Delta^{[1]}f$) is again $\mathcal{C}^1$, giving rise to $f^{[2]} := (f^{[1]})^{[1]}$ (resp. $\Delta^{[2]}f := \Delta^{[1]}(\Delta^{[1]}f)$), and so on. Already the formula for $f^{]2[}$ is quite unpleasant (see the formula given below), and we are far from understanding the general maps $f^{[k]}$ and $\Delta^{[k]}f$ (which have $2^{k+1} - 1$ arguments!). The reader will certainly agree that differential calculus is much simpler than difference calculus because, in the limit, it only retains the "invariant" and most important features of the wild structure of the latter. However, it remains important to study *how* this limit is attained, especially if we have in mind the problem of *integration* or *anti-derivative*, i.e., of going back from differential data to local or even global ones. As we will see, the problem of understanding the precise way how differential calculus in imbedded into difference calculus is highly non-trivial.

In this note, I shall present some problems related to higher order difference and differential calculus, most of them completely open and, I hope, challenging specialists as well as non-specialists. My impression is that some of these problems are of a foundational nature, tieing together algebra, combinatorics, linear and affine geometry with analysis, and that it would be very useful to better understand what is going on here.

I have presented some aspects of these problems at the Eighth International Workshop on Differential Geometry and its Applications, held in Cluj-Napoca in August 2007, and I would like to thank the organizers for their great hospitality.

1 Difference problems

Problem 1: Explicit formulae. *Find an explicit formula for the higher order difference quotient map $f^{]k[}$, resp. for its extended version $\Delta^{]k[}f$.*

Comments. For $k = 2$, we have the explicit formula

$$
\begin{aligned}
f^{]2[}&((x_1, v_1, t_1), (x_2, v_2, t_2), t_3) \\
&= \frac{f^{]1[}((x_1, v_1, t_1) + t_3(x_2, v_2, t_2)) - f^{]1[}(x_1, v_1, t_1)}{t_3} \\
&= \frac{f\big(x_1 + t_3 x_2 + (t_1 + t_2 t_3)(v_1 + t_3 v_2)\big) - f(x_1 + t_3 x_2)}{t_3(t_1 + t_2 t_3)} - \frac{f(x_1 + t_1 v_1) - f(x_1)}{t_1 t_3},
\end{aligned}
$$

and the seven components of $\Delta^{]2[}f((x_1, v_1, t_1), (x_2, v_2, t_2), t_3)$ are

$$
\Big(\big(f(x_1), f^{]1[}(x_1, v_1, t_1), t_1 \big), \big(f^{]1[}(x_1, x_2, t_3), f^{]2[}((x_1, v_1, t_1), (x_2, v_2, t_2), t_3), t_2 \big), t_3 \Big).
$$

Clearly, it would be hopeless to try write out in this way the formulae for general k. Thus we ask:

Problem 2: The structure. *Understand the structure of the higher order difference quotient maps $f^{]k[}$, resp. of its extended version $\Delta^{]k[}f$. Extend this structure (in the above mentioned setting of differential calculus) to all singular parameter values.*

Comments. Both problems are related to each other, but are not equivalent. The explicit formulae above don't really give an idea on what goes on on the "singular set" (the set where some of the t_i's are not invertible). Indeed, there exist non-trivial relations, such as certain "homogenity relations" used in [Gl04] (Appendix B) or [Gl05] (Lemma 6.8), or the following surprising relation from [BGN04], Chapter 5:

$$
f^{[2]}\big((x, v, 0), (v, 0, 0), 0\big) = 2 f^{[2]}\big((x, v, 0), (0, 0, 1), 0\big).
$$

The explicit formula given above does not help at all to prove or guess this relation. Yet, an explicit formula may turn out to be useful for attacking Problem 2 – for instance, in the above formula for $f^{]2[}$ we see that for $t_2 = 0$ we get a rather familiar and more symmetric expression.

A first step towards the solution of Problems 1 and 2 consists in finding more appropriate notation. The maps $f^{[k]}$ and $\Delta^{[k]}f$ have $2^{k+1} - 1$ arguments: there are 2^k "space variables" and $2^k - 1$ "scalar" or "(multi-) time variables". For better book-keeping, we may write $u_0 = (x_0, x_1, t_1)$ instead of (x, v, t), and use double indices on the next level; for instance

$$(u_0, u_1, t_1) = ((x_{00}, x_{01}, t_{01}), (x_{10}, x_{11}, t_{11}), t_{10}),$$

so that the formula for $f^{]2[}$ given above reads now

$$f^{]2[}\big((x_{00}, x_{01}, t_{01}), (x_{10}, x_{11}, t_{11}), t_{10}\big) =$$

$$\frac{f\big(x_{00} + t_{10}x_{10} + (t_{01} + t_{11}t_{10})(x_{01} + t_{10}x_{11})\big) - f(x_{00} + t_{10}x_{10})}{t_{10}(t_{01} + t_{11}t_{10})} - \frac{f(x_{00} + t_{01}x_{01}) - f(x_{00})}{t_{01}t_{10}}$$

and the seven components of $\Delta^{[2]}f$ are (where we now put the multi-time variable $\mathbf{t} = (t_{01}, t_{10}, t_{11})$ at the end)

$$\Big(f(x_{00}), f^{]1[}(x_{00}, x_{01}, t_{01}), f^{]1[}(x_{00}, x_{10}, t_{10}), f^{]2[}((x_{00}, x_{01}, t_{01}), (x_{10}, x_{11}, t_{11}), t_{10}), \mathbf{t}\Big).$$

For general k, as in [Be05], we will use multi-indices $\alpha \in I_k := \{0, 1\}^k$, namely $f^{[k]}$ depends on 2^k "space variables" x_α with $\alpha \in I_k$ and on $2^k - 1$ "multitime variables" t_α with $\alpha \in I_k^* := I_k \setminus \{0\}$. We may thus denote the values of the difference quotient functions by

$$f^{[k]}(\mathbf{x}, \mathbf{t}), \qquad \Delta^{[k]}(\mathbf{x}, \mathbf{t}), \qquad \mathbf{x} \in V^{2^k}, \ \mathbf{t} \in \mathbb{K}^{2^k - 1}. \tag{1.1}$$

It is then not too difficult to find the formula expressing $\Delta^{[k]}f$ in terms of all $f^{[j]}$, $j = 0, 1, \ldots, k$, namely, for $\alpha \in I_k$, the α-component of $\Delta^{[k]}f(\mathbf{x}, \mathbf{t})$ is simply t_α for the "multi-time variable", and for the α-component of the "space variable" we get

$$f^{[\ell]}\big((x_\beta)_{\beta \subseteq \alpha}, (t_\beta)_{\substack{\beta \subseteq \alpha \\ \beta \neq 0}}\big)$$

where $\ell = |\alpha| = \sum_i \alpha_i$ is the "depth" of α and $\beta \subseteq \alpha$ means that $\beta_i \leq \alpha_i$ for all i (this gives us 2^ℓ choices for β, which is exactly the number of arguments needed). In particular, the component with maximal depth k equals $f^{[k]}(\mathbf{x}, \mathbf{t})$. We leave the proof as an exercise to the reader. However, the "fine tuning" of notation and formulae (our notation so far does not impose a "canonical order" for the arguments!) is likely to depend on the solution of the problems related to *scalar extensions*, see Problem 5 below.

Back to first order difference calculus ($k = 1$): the reader may have already remarked that, as long as an invertible scalar t is fixed, difference calculus with respect to t is rather trivial: via the (linear) change of variables $(y, w) := (x, x + tv)$, i.e., via conjugating with the matrix $\begin{pmatrix} 1 & 0 \\ 1 & t \end{pmatrix}$, the difference quotient map $(x, v) \mapsto (f(x), \frac{f(x + tv) - f(x)}{t})$ is simply conjugate to $f \times f$, and iterating this procedure for invertible values of t essentially gives iterated direct products of f with itself. However, this change of variables becomes singular as soon as t becomes non-invertible, and hence this trivial interpretation of difference calculus does *not* extend to differential calculus. In other words, this kind of "structure" of

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difference calculus does not (or at least not directly) solve our Problem 2, and it is important to keep track of the dependence on t and to pay attention whether this dependence extends to singular values. In order make clear that we want to include such singular parameter values in our considerations, we will continue our list of problems below under the headline "differential problems", and we will use systematically the superscripts $[k]$ instead of $]k[$.

Another comment may be helpful here: in the case of functions of *one* variable, i.e. of *curves* $f : I \to W$, where $I \subset \mathbb{K}$ is open, one may define inductively another kind of difference quotients, called *divided differences*, by putting

$$f^{>1<}(r, s) = \frac{f(r) - f(s)}{r - s}$$

and

$$f^{>k+1<}(t_1, \ldots, t_{k+1}) := \frac{f^{>k<}(t_1, \ldots, t_k) - f^{>k<}(t_2, \ldots, t_{k+1})}{t_1 - t_{k+1}}.$$

Then one easily sees that this expression is symmetric in its $k+1$ arguments, and one can prove the explicit formula (see Chapter 7 in [BGN04], or [Sch84], [Ro00] or [Enc]):

$$f^{>k<}(t_1, \ldots, t_{k+1}) = \sum_{j=1}^{k+1} \frac{f(t_j)}{\prod_{i \neq j}(t_j - t_i)}$$

If f is $\mathcal{C}^k$ in the sense defined above, then the function $f^{>k<}$ admits a continuous extension to a continuous map $f^{<k>}$ defined on I^{k+1} (including the "singular set", where some of the t_i's coincide), and the k-th derivative $f^{(k)}(t)$ satisfies

$$k! f^{(k)}(t) = f^{<k>}(t, \ldots, t). \tag{1.2}$$

In fact, the map $f^{<k>}$ can be seen as the restriction of $f^{[k]}$ to a suitable subset of $I^{[k]}$. The converse of this statement is considerably more difficult: if $\mathbb{K}$ a field and not merely a ring, and if $f^{>k<}$ admits a continuous extension onto I^{k+1}, then f is $\mathcal{C}^k$ (see [BGN04] or [Be08], Exercices B.6).

Problem 3: q-difference calculus. *Already on a formal level, it would be much more satisfying if the number of arguments for $f^{[k]}$ and for $\Delta^{[k]}f$ were a power of 2, and not the odd number $2^{k+1}-1$. In other words, it seems as if one "time-like" parameter were missing. But what should be its meaning? We ask for a "correct" definition of the difference quotient maps, depending on an additional variable t_0 or $q \in \mathbb{K}$.*

More specifically, we have the impression that this additional variable somehow should play a similar rôle as the "quantum" parameter q in the notion of q-derivatives (cf. [KC02]). In particular, for $q = 1$ one should find difference and differential calculus as introduced above, and for $q = -1$ one would like to find some sort of super-difference and super-differential calculus, see Problem 9 below.

Comments. Here is a tentative definition, which is likely to be not yet the right guess: recall from above the "depth" $\ell = |\alpha|$ of a variable x_α, resp. t_α. Starting from the definition

of $f^{[k]}$ and $\Delta^{[k]} f$ as above, we may define functions $\tilde{f}^{[k]}$, resp. $\tilde{\Delta}^{[k]} f$ by adding t_0 as new variable (of depth 0), and by replacing each argument x_α by $t_0^{|\alpha|} x_\alpha$, resp. t_α by $t_0^{|\alpha|} t_\alpha$, and finally re-dividing the resulting expression again by t_0^j, where j is the depth of the component in question. For $t_0 = 1$, we get the old expressions. For $t_0 = -1$, this definition means that we let act $\mathbb{K}^\times$ as if V were replaced by its dual space V^*: the natural action of the dilations on V^* is the "contragredient representation"

$$\mathbb{K}^\times \times V^* \to V^*, \quad (t, \phi) \mapsto \phi \circ (t \, \mathrm{id})^{-1} = t^{-1} \phi.$$

My impression is that, on a foundational level, both actions of $\mathbb{K}^\times$ on V should have "equal rights", and the parameter t_0 should take care of this. It is obvious that the parity of depth induces a $\mathbb{Z}/2\mathbb{Z}$-grading on the space of our variables, and hence we are automatically in a "super-space" setting – compare with Problem 9 below.

Problem 4: Barycentric difference calculus. *Instead of increasing the already incredibly big number of variables, it might seem more reasonable to try to reduce this number as much as possible. Indeed, one has the impression that much of the information encoded in the maps $f^{[k]}$ is redundant, and that it should be possible to concentrate this information in a map having less variables. We have remarked above that for* curves *(maps of one scalar variable) this is indeed possible, by looking at the "divided differences" $f^{<k>}$, which are maps of $k + 1$ variables. Is it possible to do something similar in the general case?*

Comments. My impression is "yes, this cane be done". It should be achieved by some more subtle version of the trivial change of variables $(x, v) \leftrightarrow (y, w) = (x, x+tv)$ mentioned above – at first order, this change becomes singular for non-invertible t, but at higher order only the invertibility of t_α of depth $|\alpha| = 1$ really matters; it seems that there ought to be a good change of variables involving all t_α with depth $|\alpha| > 1$, remaining continuous when these t_α tend to zero, and such that the structure after change of variables becomes similar to the one for the divided differences of a curve.

Geometrically, this change of variables should correspond to changing from the "vector space point of view" (the special rôle of x_0 corresponding to the rôle of the origin in a tangent space) to the "affine space point of view": we have to rewrite difference quotients by using barycentric calculus. For instance, one could define inductively some kind of divided differences by

$$f^{>1<}(x, y; s, t) := \frac{f\big(sx + (1 - s)y\big) - f\big(tx + (1 - t)y\big)}{s - t} \qquad (x, y \in V, s, t \in \mathbb{K}),$$

whenever this makes sense; this map clearly contains the same information as the old $f^{]1[}$. Then one would like to define inductively $f^{>k<}$ by a recursion formula as a map of $2k + 2$ variables (half of them space and half of them time variables), similar to the recursion formula for the case of curves. The "explicit formula" should be some weighted sum of values of f, evaluated at barycenters defined in terms of the $k + 1$ space variables and the $k + 1$ time variables. As in the case of curves, the higher order differentials will then appear in their "divided form" (that is, if we multiply by the factor $k!$, we get the usual differentials, see the formula for curves above). Geometrically, this corresponds to the ratio of the volume of cubes (usual vector calculus) and simplices (barycentric calculus).

Both points of view (vectorial and affine) are important; in usual multivariable calculus the vectorial point of view dominates, but it might turn out that many of the more subtle problems could be better discussed via the affine (barycentric) approach (see, for instance, Problem 7 below). Summing up, it seems as if in this issue something very fundamental were going on.

2 Differential problems

As explained above, it is important to understand the dependence of difference quotients on the "multi-time" parameter $\mathbf{t}$. On the other hand, we also have to study the behavior of these expression for fixed $\mathbf{t}$ (in such a way that the results remain valid for singular $\mathbf{t}$). It seems to me that, for fixed $\mathbf{t}$, the point of view of *scalar extensions* is most suitable, see the following

Exercise. *Fix a value* $\mathbf{t} := (t_\alpha)_{\alpha \in I_k^*}$ *for the "multi-time parameter" and let us consider the partial maps of* $\Delta^{[k]}f$ *for fixed* $\mathbf{t}$, *i.e., define*

$$\Delta_{\mathbf{t}}^{[k]}U := \{\mathbf{x} \in V^{2^k} \,|\, (\mathbf{x}, \mathbf{t}) \in \Delta^{[k]}U\},$$

$$\Delta_{\mathbf{t}}^{[k]}f : \Delta_{\mathbf{t}}^{[k]}U \to W^{2^k}, \quad \mathbf{x} \mapsto \mathrm{pr}_1(\Delta^{[k]}(\mathbf{x}, \mathbf{t})),$$

where $\mathrm{pr}_1(\mathbf{x}, \mathbf{t}) = \mathbf{x}$ *is the projection onto the spacial part of the variable* $(\mathbf{x}, \mathbf{t})$. *Show that* $\Delta_{\mathbf{t}}^{[k]}$ *is a covariant functor commuting with direct products, and that this functor can be interpreted as the functor of scalar extension from the ring* $\mathbb{K}$ *to a (commutative unital) ring* $\Delta_{\mathbf{t}}^{[k]}\mathbb{K}$. *(In particular, if* f *is a polynomial map, then* $\Delta_{\mathbf{t}}^{[k]}f$ *is its scalar extension in the algebraic sense.)*

Hints. Let us start with $k = 1$, so we look at

$$\Delta_t^{[1]}f(x, v) = \big(f(x), f^{[1]}(x, v, t)\big) = \Big(f(x), \frac{f(x + tv) - f(v)}{t}\Big),$$

the latter provided $t \in \mathbb{K}^\times$. For such t, the functoriality is easily checked (see Equation (0.4) above), and so is the property $\Delta_t^{]1[}(f \times g) = \Delta_t^{]1[}f \times \Delta_t^{]1[}g$ (under obvious identifications of sets). Again by density, the corresponding statements remain true for non-invertible scalars t; in particular, for $t = 0$ we get again the chain rule for the *tangent functor* $T = \Delta_0^{[1]}$, as defined by Equation (0.6).

Now, for any $t \in \mathbb{K}$, the functor $\Delta_t^{[1]}$, being covariant and commuting with direct products, may be applied to the ring $\mathbb{K}$ and its structure maps $\mathbb{K} \times \mathbb{K} \to \mathbb{K}$, and then gives a new ring, which has dimension 2 over $\mathbb{K}$. For invertible t, a straightforward calculation gives the ring multiplication

$$
\begin{aligned}
(x_0, x_1) \cdot (y_0, y_1) &= \Big(x_0 y_0 \,,\, \frac{(x_0 + t x_1)(y_0 + t y_1) - x_0 y_0}{t}\Big) \\
&= \big(x_0 y_0 \,,\, x_0 y_1 + x_1 y_0 + t x_1 y_1\big).
\end{aligned}
$$

In a similar way, we see that the sum in this ring is just the usual sum in $\mathbb{K}^2$. Hence as a ring, we get $\mathbb{K} \oplus \omega\mathbb{K}$ with relation $\omega^2 = t\omega$. It can also be described as the truncated polynomial ring $\mathbb{K}[X]/(X^2 - tX)$. Again by density, these statements remain true for non-invertible scalars t, and in particular for $t = 0$ we obtain the *tangent ring* $T\mathbb{K}$, which is nothing but the ring of *dual numbers over* $\mathbb{K}$, $\mathbb{K}[X]/(X^2) = \mathbb{K} \oplus \varepsilon\mathbb{K}$ with $\varepsilon^2 = 0$.

Next, we claim that, if $f : U \to W$ is $\mathcal{C}^\infty$ over $\mathbb{K}$, then $\Delta_t^{[1]}f : \Delta_t^{[1]}U \to \Delta_t^{[1]}W$ is $\mathcal{C}^\infty$ over the ring $\Delta_t^{[1]}\mathbb{K}$. In fact, here the proof of the special case $t = 0$ from [Be05], Theorem 6.2, can be applied word by word; it uses only the fact that $\Delta_t^{[1]}$ is a covariant functor commuting with direct products and with diagonal mappings. What comes out is this:

$$\Delta_{t_{01}, t_{10}, t_{11}}^{[2]} f = \Delta_{t_{01} + \omega t_{11}}^{[1]}(\Delta_{t_{10}}^{[1]} f) \tag{2.1}$$

where ω satisfies $\omega^2 = t_{10}\omega$. Thus $\Delta_{\mathbf{t}}^{[2]}$ is a composition of two scalar extension functors. By induction, it follows now that $\Delta_{\mathbf{t}}^{[k]}$ is a composition of k scalar extension functors, and hence is itself a functor of scalar extension from $\mathbb{K}$ to a ring $\Delta_{\mathbf{t}}^{[k]}\mathbb{K}$.

Problem 5: Scalar extensions. *Describe the structure of the ring* $\Delta_{\mathbf{t}}^{[k]}\mathbb{K}$ *in such a way that its dependence on* $\mathbf{t}$ *becomes understandable.*

Comments. For $k = 1$, as seen above, $\Delta_t^{[1]}\mathbb{K} \cong \mathbb{K}[X]/(X^2 - tX)$. For $k = 2$, by (2.1), this ring is isomorphic to

$$\left(\mathbb{K}[X_1]/(X_1^2 - t_{10}X_1)\right)[X_2]/(X_2^2 - (t_{01} + t_{11}X_1)X_2)$$

$$\cong \mathbb{K}[X_1, X_2]/\left((X_1^2 - t_{10}X_1), (X_2^2 - (t_{01} + t_{11}X_1)X_2)\right).$$

We can also describe this ring as

$$R = \mathbb{K} \oplus \omega_{10}\mathbb{K} \oplus \omega_{01}\mathbb{K} \oplus \omega_{11}\mathbb{K}$$

with relations

$$\omega_{10}^2 = t_{10}\omega_{10}, \quad \omega_{11} = \omega_{01}\omega_{10} = \omega_{10}\omega_{01}, \quad \omega_{01}^2 = (t_{01} + t_{11}\omega_{10})\omega_{01} = t_{01}\omega_{01} + t_{11}\omega_{11}.$$

It follows that

$$\omega_{10}\omega_{11} = t_{10}\omega_{11}, \quad \omega_{01}\omega_{11} = (t_{01} + t_{10}t_{11})\omega_{11},$$

$$\omega_{11}^2 = \omega_{10}^2\omega_{01}^2 = t_{10}\omega_{10}(t_{01} + t_{11}\omega_{10})\omega_{01} = t_{01}(t_{10} + t_{01}t_{11})\omega_{11}.$$

Some features of the structure of this ring are similar to the ones of the second order tangent ring $TT\mathbb{K}$ (cf. [Be05]). For general k, we may write

$$R = \bigoplus_{\alpha \in I_k} \omega_\alpha \mathbb{K} \qquad (\omega_0 = 1),$$

with relations

$$\omega_\alpha \omega_\beta = \sum_\gamma c_{\alpha,\beta}^\gamma \omega_\gamma,$$

where the coefficients $c^{\gamma}_{\alpha,\beta}$ depend on the multi-time parameter $\mathbf{t}$. It seems that this system of coefficients is "triangular" in a certain sense, and there should be some general pattern permitting to understand it.

Problem 6: Difference version of the de Rham complex. *We ask for a (first) necessary condition for solving difference equations: is there an operator D whose square is zero and such that $DF = 0$ is a necessary condition for $F : U^{]k[} \to W$ to be essentially of the form $f^{]1[}$ for some $f : U^{]k-1[} \to W$?*

Comments. The classical de Rham complex concerns the case $\mathbf{t} = 0$, and its construction relies entirely on the fact that the symmetric group Σ_k acts by automorphisms of the ring $T^k \mathbb{K}$ (or, equivalently, of the functor $\Delta_{\mathbf{0}}^{[k]}$), cf. [Be05], Chapter 22. Therefore also for general $\mathbf{t}$ one has to study the "Galois group" $\mathrm{Aut}_{\mathbb{K}}(\Delta_{\mathbf{t}}^{k}\mathbb{K})$. It should be some kind of deformation of the automorphism group $\mathrm{Aut}_{\mathbb{K}}(T^k \mathbb{K})$, and we expect that its quotient with respect to its "identity component" is still isomorphic to Σ_k. Therefore we should indeed have necessary conditions in terms of this group for a map $U^{[k]} \to W$ to be some $f^{[1]}$, and these conditions should look similar to closedness of k-forms.

Problem 7: The case of positive characteristic. *Differential calculus and differential geometry work perfectly well over base fields of arbitrary, possibly positive, characteristic (see [BGN04], [Be05]). However, in case of characteristic $p > 0$, it is less obvious how the "differential information" is best exploited. The approach from [Be05], based on the higher order tangent functor T^k and on its invariance under the symmetric group (see the preceding problem), still leads to some loss of information. How can we avoid this loss of information?*

Comments. It is known that, in case of characteristic $p > 0$, the usual definition of a *Lie algebra* is too weak, and that one should use stronger structures such as the *restricted Lie algebras* or *Lie p-algebras* introduced by N. Jacobson [Jac41] (and, later, for related algebras, such as *Leibniz algebras*, cf. [DL06]). Roughly, one has to add to the usual Lie algebra structure a compatible structure of a "p-th power map" $x \mapsto x^{[p]}$.

It seems as if for differential geometry in general (which comprises Lie theory) the situation were very similar: when the approach is based on usual higher differentials $d^k f$ (or, equivalently, on the higher order tangent functors $T^k = \Delta_{\mathbf{t}}^{[k]}$ for $\mathbf{t} = 0$), we inevitably lose information; but this loss of information can be avoided if we use the information encoded in the higher order slopes $\Delta_{\mathbf{t}}^{[k]}$ where *some, but not all* t_{α} are zero. In fact, the general Taylor formula, valid in any characteristic, expresses the Taylor coefficients a_j in the expansion

$$f(x + th) = f(x) + \sum_{j=1}^{k} t^j a_j(x, h) + t^k R(x, h, t)$$

in terms of the slopes (see [BGN04], [Be05] or [Be08]):

$$a_1(x, h) = f^{[1]}(x, h, 0), \quad a_2(x, h) = f^{[2]}((x, h, 0), (0, 0, 1), 0), \dots$$

In case of a curve $f : I \to W$, this can also be expressed in the context of divided differences (cf. Problem 2):

$$a_1(x, h) = f^{<1>}(x)\, h, \quad a_2(x, h) = f^{<2>}(x, x)\, h^2, \ldots, a_k(x, h) = f^{<k>}(x, \ldots, x)\, h^k.$$

For instance, in characteristic 2, the second differential of $f(x) = x^2$ at the origin is zero, but nevertheless we get the correct Taylor expansion: we have $f^{<1>}(t, s) = \frac{s^2 - t^2}{s - t} = s + t$ and $f^{<2>}(r, t, s) = \frac{(r+t) - (r+s)}{t - s} = 1$, whence the Taylor expansion $h^2 = f^{<2>}(0, 0, 0)h^2$. In fact, in the general case, it is easily checked that $f^{[2]}((x, h, t), (0, 0, 1), s))$ is exactly the divided difference $\gamma^{<2>}(0, t, s)$ for the curve $\gamma(t) = f(x + th)$, thus relating the Taylor expansion for the curve γ at 0 and the map f at x. Put differently, the (hypothetic) "barycentric" or "simplicial differential calculus" from Problem 4 should furnish the information that is lost in characteristic $p > 0$ if we use the classical "vectorial" or "rectangular" differential calculus".

It is all the more surprising that, even in positive characteristic, one can *reconstruct* some sort of approximation of a Lie group by starting from mere Lie- or Leibniz algebras (see [Di07]); it looks as if one just had to add some structure like the p-th order power map in order to fill in the missing information.

Problem 8: Non-commutative base rings. *In principle, the definition of the maps $f_{\mathbb{K}}^{[k]}$ and $\Delta_{\mathbb{K}}^{[k]} f$ makes sense over non-commutative base fields or rings $\mathbb{K}$. However, already the multiplication map $\mathbb{K} \times \mathbb{K} \to \mathbb{K}$ and the squaring map $\mathbb{K} \to \mathbb{K}$ will then no longer be of class $\mathcal{C}^1$, and hence all arguments relying on these facts break down. How can definitions be modified in order to take account of these problems?*

Comments. If $m : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ is the product in the ring $\mathbb{K}$, then

$$m^{]1[}((x, v), (y, w), t) = t^{-1}\big((x + tv)(y + tw) - xy\big) = vtw + vy + t^{-1}xtw.$$

Unfortunately, the last term does in general not admit a continuous extension to $t = 0$ if $\mathbb{K}$ is not commutative. Thus m will not be of class $\mathcal{C}^1$ over $\mathbb{K}$, and similarly, the power maps $\mathbb{K} \to \mathbb{K}$, $x \mapsto x^k$ won't be $\mathcal{C}^1$ over $\mathbb{K}$ either. Therefore we do not get a useful differential calculus over non-commutative fields or rings. (Of course, one could define the difference quotient map by putting t^{-1} to the right of the bracket, but this does not help!)

There is no simple way to avoid this problem – as is well-known, no quaternionic calculus has ever been found that in any respect shared the power of real or complex calculus. However, it is not excluded to generalize some more specific aspects of calculus to the non-commutative case. To do this, one certainly needs to impose some suitable conditions on $\mathbb{K}$. For instance, one might rather consider *Hermitian rings*, that is a pair $(\mathbb{K}, \tau)$ formed by a ring together with an involution (anti-automorphism of order 2). One may then try to use the involution for an appropriate definition of difference quotients. This might be interesting in relation with exceptional Lie groups and their geometries. (It is known that all compact forms of exceptional Lie groups arise as automorphism groups of *quaternionic symmetric spaces*, the so-called Wolf spaces. These spaces are not really manifolds over the quaternions, but it looks as if they belonged to some sort of generalized

projective geometries (in the sense of [Be02]) that should be defined in some suitable way over the quaternions.)

Problem 9: Super differential calculus. *Is it possible to see* super differential calculus *as a continuous extension of some "super difference calculus" to singular values?*

Comments. In the context of Problem 8, another suitable condition to impose on our ring $\mathbb{K}$ would be *super-commutativity*, i.e., $\mathbb{K} = \mathbb{K}_1 \oplus \mathbb{K}_{-1}$ is $\mathbb{Z}/2\mathbb{Z}$-graded and the product satisfies the sign rule $xy = (-1)^{d(x) \cdot d(y)} yx$. If it is possible to define some reasonable difference calculus in this case, one should expect it to have some link with *super differential calculus* (cf. [Dew84], [Va04]).

Another aspect of this problem is given by the scalar extension point of view: in presence of a "quantum" parameter q (see Problem 3), the arguments used above (hints to the exercise) suggest that for $q = -1$ we should obtain some calculus that can be interpreted as a scalar extension by the Grassmann algebra.

Problem 10: Pansu calculus. *Another approach to non-commutativity comes from sub-Riemannian geometry: instead of making $\mathbb{K}$ non-commutative we can "make the vector space V non-commutative" (the co-called* Pansu calculus*)[2]. Is there a corresponding "Pansu difference calculus", having the Pansu differential calculus as limit, and if so, what are its properties?*

Comments. The non-commutative version of a vector space is a *Carnot group*: let $V = \mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n$ be a graded nilpotent Lie algebra (i.e., $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$; wo do not require here equality as in [Bu06]) and define a (polynomial) group law $v * w = v + w + \frac{1}{2}[v, w] + \ldots$ on V by the Campbell-Hausdorff formula. If $\mathfrak{g}$ is abelian, then this is just usual vector addition, and hence $(V, *)$ can be seen as a non commutative analog of $(V, +)$. Now we define the analog of multiplication by scalars: for $t \in \mathbb{K}$ define the *dilation*

$$\delta_t : \mathfrak{g} \to \mathfrak{g}, \quad \sum_i x_i \mapsto \sum_i t^i x_i.$$

From the grading condition it follows immediately that δ_t is a Lie algebra-endomorphism of $\mathfrak{g}$; if t is invertible, it is an automorphism. If $\mathfrak{g} = \mathfrak{g}_1$, we get the usual multiplication by scalars. Now let $f : U \to W$ be a map between Carnot groups and define the *first difference quotient* by

$$f^{]1[}(x, v, t) := \delta_t^{-1}\Big(f(x)^{-1} * f(x * (\delta_t v))\Big).$$

(Of course, since $(V, *)$ is no longer commutative, we had to make a choice of order: all other choices have equal rights!) Now assume that our Carnot groups are topological groups, that the dilation map $\mathbb{K} \times V \to V$ is continuous and that U is open in V. Then we can define the class $\mathcal{C}^1$ as before by requiring that $f^{]1[}$ extends continuously to the suitably defined set $U^{[1]}$. To some extent, the theory goes through: for instance, one proves as before that the "differential" $df(x) : V \to W$ is a "Carnot linear map". Things

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[2]I thank Marius Buliga for pointing out to me this approach, cf. [Bu06].

start to become more involved when looking at higher order different quotients and trying to define the class $\mathcal{C}^2$: before iterating our procedure, we have to fix a Carnot structure on direct products; the direct product being $\mathbb{Z} \times \mathbb{Z}$-graded, there are several possible choices of $\mathbb{Z}$-gradings. It seems reasonable to choose the "diagonal grading"; but even then we do not get a well-defined notion of "bilinear maps", and hence we have to be very careful when talking about higher order differentials. Nevertheless, this sort of calculus might be interesting for geometries such as certain flag varieties that are indeed "Carnot groups patched together" and which could be smooth manifolds in the sense of such a calculus.

Problem 11: Pointwise concepts. *Can one define the notion of a map f being "of class $\mathcal{C}^k$ at one point"? Is it then true that f is of class $\mathcal{C}^k$ on U if f is $\mathcal{C}^k$ at all points $a \in U$?*

Comments. This should indeed be possible, but as far as I know it has not yet been really worked out. For $k = 1$ and $k = 2$, I have outlined the definition of the "classes $\mathcal{C}_a^k$" in [Be08], Section 14.3: for $k = 1$, we require that f is continuous at a and that the difference quotient $f^{]1[}$ admits an extension, denoted by $f_a^{[1]}$, to

$$U_a^{[1]} := U^{]1[} \cup V_a, \quad \text{where} \quad V_a := \{(a, v, 0) \mid v \in V\},$$

such that $f_a^{[1]}$ is continous on V_a. This notion has good properties, and f is indeed $\mathcal{C}^1$ on U iff it is $\mathcal{C}_a^1$ for all $a \in U$. In finite dimension over $\mathbb{K} = \mathbb{R}$, this property is equivalent to the *strict differentiability of f at a* (see loc cit). It is known that strict differentiability is in many respects the "good pointwise concept" (better than the weaker Fréchet differentiability); in particular, f is $\mathcal{C}^1$ on U iff it is strictly differentiable at all points of U. The definition of the class $\mathcal{C}_a^k$ is similar, though slightly more subtle: roughly, we require the existence of an extension of $f^{]k[}$ onto a set $U_a^{[k]}$ defined similarly as above, and requiring continuity on the set of all multi-space variables x_α of depth $|\alpha| \geq 1$ and all multi-time variables t_α of depth $|\alpha| \geq 2$. This still gives a "pointwise" (as opposed to "local") concept having properties strong enough to prove all basic results of differential calculus.

Problem 12: Integral formulae in the real case. *In the case $\mathbb{K} = \mathbb{R}$, when dimensions are finite or in presence of a locally convex topological vector space structure, the fundamental theorem of calculus permits to express first order difference quotients by integrals (cf. [BGN04] for the general locally convex case):*

$$f^{[1]}(x, v, t) = \int_0^1 df(x + stv)v \, ds.$$

Find an analoguous expression of $f^{[k]}(\mathbf{x}, \mathbf{t})$ by an integral formula !

Comments. In principle, we just have to iterate the preceding formula:

$$f^{[2]}((x, v, t), (x', v', t'), s) = \int_0^1 d(f^{[1]})((x, v, t) + s'(x', v', s'))(x', v', t') \, ds',$$

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then again express $f^{[1]}$ by an integral and permute the differential d with integration. We will end up with a certain integral of $d^2 f$ over a plane region, and so on. Via Problem 4, this question is related to the corresponding question for divided differences of curves. In this case, on the following integral formula holds

$$f^{<k>}(x_0, \ldots, x_k) = \int_{\Delta_k} f^{(k)}\big(x_0 + \sum_{i=1}^{k} \lambda_i (x_i - x_0)\big) \, d\lambda_1 \cdots d\lambda_k$$

(see, for instance, [Be08], Exericise B.6.7), where $f^{(k)}$ is the usual k-th order derivative of a curve and

$$\Delta_k = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k \,|\, \lambda_i \geq 0, \sum_i \lambda_i \leq 1\}$$

is the standard simplex in $\mathbb{R}^k$. (Recall that its volume is $\frac{1}{k!}$). By averaging further, one deduces that there exists a function $M_{k+1}(x_0, \ldots, x_k; t)$ (called the *Peano kernel*), such that, for all curves of class $\mathcal{C}^k$, $f : I \to \mathbb{R}^n$,

$$f^{<k>}(x_0, \ldots, x_k) = \frac{1}{k!} \int_{\mathbb{R}} f^{(k)}(t) \cdot M_{k+1}(x_0, \ldots, x_k; t) \, dt.$$

One can show that $M_{k+1}(x_0, \ldots, x_k; \cdot)$ is a function of class $\mathcal{C}^{k-2}$ such that its restriction to the interval $]x_j, x_{j+1}[$ is a polynomial of degree $\leq k - 2$, and zero outside the interval $]x_0, x_k[$.

Final remarks. In some sense, all the problems mentioned so far are only preliminaries for the Big Problem which is the *integration problem*: *is there a difference theory of an anti-derivative?* I am inclined to believe that such a theory exists, but it is likely to be fairly sophisticated, rather giving an infinite sequence of necessary conditions than a plain "existence and uniqueness theorem". In any case, before understanding anti-derivatives and "anti-differences", we have to understand derivatives and differences.

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