A FUNCTORIAL APPROACH TO DIFFERENTIAL CALCULUS

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Abstract. We show that differential calculus (in its usual form, or in the general form of *topological differential calculus*) can be fully imdedded into a functor category (functors from a small category of *anchord tangent algebras* to anchored sets). To prepare this approach, we define a new, symmetric, presentation of differential calculus, whose main feature is the central rôle played by the *anchor map*, which we study in detail. Our aim for developing this theory is twofold: (1) define a setting for calculus over *any* commutative ring, including finite rings; (2) define a setting that can be generalized to categories of *graded rings* (super differential calculus).

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Introduction

Differential Calculus is a central ingredient of modern mathematics. While the "working mathematician" takes this tool for granted, thinking about its conceptual foundations remains a potentially important topic. In the present work, we continue the line of research started with [BGN04, Be08, BeS14, Be17], and combine it with what Grothendieck once called the "simple idea of a good functor from rings to sets" (according to W. Lawvere, cf. n-lab)[1]. The "simple idea" mentioned by

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[1]Here the quote from the n-lab: "The 1973 Buffalo Colloquium talk by Alexander Grothendieck had as its main theme that the 1960 definition of scheme ... should be abandoned AS the FUNDAMENTAL one and replaced by the *simple idea of a good functor from rings to sets.* The needed restrictions could be more intuitively and more geometrically stated directly in terms of the topos of such functors, and of course the ingredients from the "baggage" could be extracted when needed as auxiliary explanations of already existing objects, rather than being carried always as core elements of the very definition."

Grothendieck is currently used in algebraic geometry, and in Lie Theory, where one often considers a real "space" – for instance, a Lie group $\underline{G}$ – as set of "real points" $G_{\mathbb{R}}$ of a *complex* Lie group $G_{\mathbb{C}}$. This is a kind of non-linear analog of the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of a real vector space (or of a real Lie algebra). Grothendieck's insight was that this idea of "complexification" should not be limited to *field* extensions, but enlarged to more general *ring* extensions, in order to incorporate operations belonging to *infinitesimal calculus*: a $\mathbb{K}$-Lie group G, or a general $\mathbb{K}$-smooth manifold M, should admit "scalar extensions" $M_{\mathbb{A}}$ akin to a hypothetic tensor product $M \otimes_{\mathbb{K}} \mathbb{A}$, for certain $\mathbb{K}$-algebras $\mathbb{A}$. The simplest example of such an extension is the one by *dual numbers*,

$$\mathbb{K}[\varepsilon] := \mathbb{K}[X]/(X^2) = \mathbb{K} \oplus \varepsilon\mathbb{K} \quad (\varepsilon^2 = 0), \tag{0.1}$$

where the nilpotent element ε is the class $[X]$ modulo (X^2). Grothendieck, following ideas of Weil [We53], realized that the tangent bundle TM of a "space" M, which is "defined over $\mathbb{K}$", could be understood as something like $M \otimes_{\mathbb{K}} \mathbb{K}[\varepsilon]$. This idea has been used by Demazure and Gabriel in their theory of algebraic groups [DG], in differential calculus over general base field and rings [Be08], and in the approach to natural operations in differential geometry via the so-called *Weil functors* ([KMS93], cf. also [BeS14]). The most elaborate and systematic development of these ideas leads to what is called nowadays *synthetic differential geometry* (SDG, see [MR91]). The approach to be presented here pursues the same goals as SDG, but by different means: we keep closer to the idea of generalizing the algebraic tensor product. In a very direct sense, our problem is to generalize the algebraic scalar extension $V_{\mathbb{A}} := V \otimes_{\mathbb{K}} \mathbb{A}$ of a $\mathbb{K}$-module V (basic facts are recalled in Appendix A), to more general spaces M, like, e.g., manifolds – where we face the problem that such an operation won't be possible for *all* $\mathbb{K}$-algebras $\mathbb{A}$, so we have to single out a good class (good category) of algebras for which such an extension is possible. Such a class, called *the category of (anchored) tangent algebras*, will be defined in this paper. It arises naturally, when questioning the very shape of differential calculus, instead of taking it for granted. Let us briefly explain the main ideas.

0.1. **Topological differential calculus.** In differential calculus we consider maps f whose domain U and codomain U' are *locally linear sets* – by this we mean $U \subset V$ and $U' \subset V'$ are non-empty subsets of linear (or affine, if one prefers) spaces V and V'. In this situation, we may define the *slope* or *difference quotient map*: when $t, s \in \mathbb{K}$ are such that $t - s$ is invertible, we look at the difference quotient

$$f^{[1]}(v_0, v_1; t, s) := f^{[1]}_{(t,s)}(v_0, v_1) := \frac{f(v_0 + tv_1) - f(v_0 + sv_1)}{t - s}. \tag{0.2}$$

To speak of *topological* calculus, we shall assume that V, V' are topological vector spaces or modules over topological fields or rings $\mathbb{K}$, and U, U' are open. For the moment, let's consider the "classical case" $\mathbb{K} = \mathbb{R}$ and $V = \mathbb{R}^n$, $V' = \mathbb{R}^m$. Then the following holds (cf. [BGN04, Be08]): *The map f is of class C^1 if, and only if, the difference quotient map $f^{[1]}$ extends continuously to a map defined on the set*

$$U^{[1]} := \left\{ (v_0, v_1; t, s) \in V^2 \times \mathbb{K}^2 \,\middle|\, \begin{array}{l} v_0 + tv_1 \in U \\ v_0 + sv_1 \in U \end{array} \right\}. \tag{0.3}$$

If this is the case, we denote still by $f^{[1]} : U^{[1]} \to U'$ the extended map. Then $f^{[1]}(v_0, v_1; 0, 0) = df(v_0)v_1$ gives the differential df of f. Now, these conditions make perfectly sense for any "good" topological ring $\mathbb{K}$ and for maps defined on open locally linear sets, and thus can serve as *definition* of differentiability over $\mathbb{K}$ – the "topological differential calculus" thus defined has excellent functorial properties allowing to give a "purely algebraic" presentation of certain features of usual calculus (see [BGN04, Be11]). To understand the structure of formulae like (0.2) and (0.3), the following *way of talking* turns out to be useful:

- call $\mathbf{v} = (v_0, v_1)$ "space variables", with v_0 the "foot point" and v_1 the "direction" (in which we differentiate),
- call (t, s) "time variables", and t "target time", and s "source time",
- call (t, s) "regular", or "finite", if $t - s$ is invertible in $\mathbb{K}$, and "singular" or "infinitesimal" else, with $t - s = 0$ being the "most singular value",
- call $v_0 + sv_1$ the "source", and $v_0 + tv_1$ the "target evaluation point",
- for fixed (t, s), call $\alpha\big((v_0, v_1)\big) := v_0 + sv_1$ the "source map", and define the "target map" $\beta\big((v_0, v_1)\big) := v_0 + tv_1$.

The slogan summarizing topological calculus is: *the slope extends continuously (jointly in space and time variables) from finite to singular times.* The notable difference with [BGN04, Be11] is that here we shall use a *pair* of time parameters (t, s), instead of a single parameter t as in loc. cit. Although the expression (0.2) is of course symmetric under switch of target and source time, it will be important to distinguish "target" and "source". The setting of [BGN04, Be11] is gotten by restricting to $s = 0$ (we call this "target calculus"); symmetrically, the theory could also be formulated when letting $t = 0$ ("source calculus"). But now we can take advantage to define a third calculus, the "symmetric calculus", which corresponds to the case $t = -s$: then $v_0 = \frac{v_0+sv_1+v_0+tv_1}{2}$, so the footpoint is the midpoint of target and source evaluation point.[2] This symmetric setting is best suited for a transition of our methods towards *super-calculus* (cf. Section 4).

0.2. **The underlying algebraic structure: anchor.** In the second section we shall carve out the algebraic structures underlying topological differential calculus. As in general groupoid theory, the pair (α, β) given by source and target will be called *anchor map*[3]. We use the same term when considering the pair of time variables (t, s) as a "frozen parameter" (temporarily considered to be fixed); then we write (t, s) as lower index – for instance,

$$U^{[1]}_{(t,s)} := \{(v_0, v_1) \mid (v_0, v_1; t, s) \in U^{[1]}\}. \tag{0.4}$$

For fixed (t, s), we call again *anchor* the (linear) map sending the space variables $\mathbf{v} = (v_0, v_1)$ to the pair of evaluation points:

(0.5)
$$\Upsilon_{(t,s)} : U^{[1]}_{(t,s)} \to U \times U, \quad \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \mapsto \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & s \\ 1 & t \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} v_0 + sv_1 \\ v_0 + tv_1 \end{pmatrix} = \begin{pmatrix} \alpha(\mathbf{v}) \\ \beta(\mathbf{v}) \end{pmatrix}.$$

[2] A price has to be paid: one will have to require that 2 be invertible in $\mathbb{K}$. Analysts won't bother, some algebraists might...

[3] This map is indeed the anchor map of a groupoid structure, see Lemma 4.1.

Of course, a choice is made here: the "first" component of $U \times U$ shall be associated with "source", and the "second" with "target". One of our concerns in the sequel will be to formalize the levels on which such choices are operated. Anyhow, by direct computation, the anchor is seen to be invertible if, and only if, $t-s$ is invertible, and then its inverse is given by

$$\Upsilon_{(t,s)}^{-1} : U \times U \to U_{(t,s)}^{[1]}, \quad \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto \frac{1}{t-s} \begin{pmatrix} t & -s \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{tx_0 - sx_1}{t-s} \\ \frac{x_1 - x_0}{t-s} \end{pmatrix}. \tag{0.6}$$

The first component is an affine combination $v_0 = \frac{s}{s-t} x_1 + \frac{t}{t-s} x_0$, and the second a "difference quotient". From this, comparing with (0.2), we see that $f_{(t,s)}^{[1]}$ is precisely the second component of the map $f_{(t,s)}^{\{1\}} := \Upsilon_{(t,s)}^{-1} \circ (f \times f) \circ \Upsilon_{(t,s)}$, given by

$$f_{(t,s)}^{\{1\}} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} \frac{tf(v_0+sv_1) - sf(v_0+tv_1)}{t-s} \\ \frac{f(v_0+tv_1) - f(v_0+sv_1)}{t-s} \end{pmatrix}. \tag{0.7}$$

The big advantage is that $f_{(t,s)}^{\{1\}}$ *depends functorially on* f: the "chain rule" simply reads $(g \circ f)_{(t,s)}^{\{1\}} = g_{(t,s)}^{\{1\}} \circ f_{(t,s)}^{\{1\}}$. Now we can reformulate the property of being $C^1_{\mathbb{K}}$ (Lemma 1.2): *The map* $f : U \to U'$ *is of class* $C^1_{\mathbb{K}}$ *if, and only if, for all* $(t,s) \in \mathbb{K}^2$ *there exists a continuous map* $f_{(t,s)}^{\{1\}} : U_{(t,s)}^{\{1\}} \to (U')_{(t,s)}^{\{1\}}$*, jointly continuous also in the parameter* $(t,s) \in \mathbb{K}^2$*, such that*

$$\Upsilon_{(t,s)} \circ f_{(t,s)}^{\{1\}} = (f \times f) \circ \Upsilon_{(t,s)} : \qquad \begin{array}{ccc} U_{(t,s)} & \overset{f_{(t,s)}^{\{1\}}}{\longrightarrow} & U'_{(t,s)} \\ \Upsilon \downarrow & & \downarrow \Upsilon \\ U \times U & \overset{f \times f}{\longrightarrow} & U \times U \end{array} \tag{0.8}$$

In a nutshell, this diagram contains the essential ingredients needed for our approach: our aim is to translate diagram (0.8) into a "categorical" formulation, so that it will make sense in an abstract setting, not requiring topology any more. In a first step, we generalize this diagram at higher order $n \in \mathbb{N}$ (Theorem 1.7): indeed, differentiability at order n is characterized by a diagram of the same kind, replacing $f_{(t,s)}^{\{1\}}$, etc., by higher order maps $f_{(\mathbf{t},\mathbf{s})}^{\{\mathsf{n}\}}$, etc., where $(\mathbf{t},\mathbf{s}) = (t_1, \ldots, t_n; s_1, \ldots, s_n) \in \mathbb{K}^{2n}$. Technically, we work with 2^n-fold direct products, which have to be indexed by elements A of the *n-hypercube* $\mathcal{P}(\mathsf{n})$ (power set of $\mathsf{n} = \{1, \ldots, n\}$).

0.3. **The simple idea of a good functor from rings to sets.** In order to formalize the idea that the extended domains and maps $(U_{(\mathbf{t},\mathbf{s})}^{\{\mathsf{n}\}}, f_{(\mathbf{t},\mathbf{s})}^{\{\mathsf{n}\}})$ are scalar extensions $(U \otimes_{\mathbb{K}} \mathbb{A}, f \otimes_{\mathbb{K}} \mathbb{A})$, we look at the case $U = \mathbb{K}$. From functoriality, it follows that the spaces $\mathbb{K}_{(\mathbf{t},\mathbf{s})}^{\{\mathsf{n}\}}$ are in fact $\mathbb{K}$-*algebras*, which can easily be identified,

(1) in terms of polynomial rings: they are polynomial algebras $\mathbb{K}[X_1, \ldots, X_n]$, quotiented by the relations $(X_i - t_i)(X_i - s_i) = 0$, for $i = 1, \ldots, n$,
(2) in terms of tensor products: they are n-fold tensor products of "first order algebras" $\mathbb{K}_{(t_1,s_1)} \otimes \ldots \otimes \mathbb{K}_{(t_n,s_n)}$.

Likewise, the n-fold anchor is identified as the natural *evaluation morphism* of the polynomial algebra into $\mathbb{K}^{\mathcal{P}(\mathsf{n})} \cong \mathbb{K}^{2^n}$, or as the n-fold tensor product of first order anchors. This allows us to find an explicit formula for the n-fold anchor (Theorem 2.24), and for its inverse when $(\mathbf{t},\mathbf{s})$ is *regular* (Theorem 2.25). These are the essential ingredients to define a small category of $\mathbb{K}$-algebras, the category of *n-th order anchored tangent algebras* $\mathbf{talg}_{\mathbb{K},n}$, by an explicit list of its objects and morphisms: its *objects* are precisely the anchors $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})}$ with $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$, and there is an explicit list of *morphisms*. Putting these categories together, for all $n \in \mathbb{N}$, we get the category of all *anchord tangent algebras*, $\mathbf{talg}_{\mathbb{K}}$. This setting emphasizes the central rôle of the anchor – its status has been "upgraded": it is not a *morphism* like the others, but is considered as the central *object* of our theory.

Now, the "simple idea of a good functor from rings to sets" is to view "n times $\mathbb{K}$-differentiable spaces" as functors $\underline{M}$ from the (small) category $\mathbf{talg}_{\mathbb{K},\mathsf{n}}$ to the category of *anchored sets*, assigning to an anchor $\Upsilon : \mathbb{A} \to \mathbb{A}'$ a pair of sets $(M_{\mathbb{A}}, M_{\mathbb{A}'})$ together with a map $M_\Upsilon : M_{\mathbb{A}} \to M_{\mathbb{A}'}$. The family $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ then defines a *natural transformation* of such functors, and which behaves in all respect like a family of "algebraic scalar extensions" $f \otimes_{\mathbb{K}} \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, thus achieving our goal.

In order to fully justify such a functorial approach to differential calculus, one usually requires in SDG that the model be *well-adapted*, that is, that we obtain a *full and faithful* imbedding of a "usual" category of differential calculus into the "functorial" one. We show that, for our setting, this is indeed the case (Theorem 3.5). The proof is much easier than the one of analogs in SDG, because, in essence, the whole setting is designed for such a theorem to hold: it is merely the translation of Theorem 1.7 into a more abstract language.

0.4. **Towards higher (super) algebra.** The present functorial approach is designed to fit best with further developments: on the one hand, *higher algebra* (by n-fold iteration, the functor categories are in fact *n-categories*, and they take values in *n-fold groupoids*), and *super-calculus*, on the other hand – see Section 4).

0.5. **Appendices on linear algebra and on functor categories.** In Appendix A we recall some basic facts about scalar extension $V \otimes_{\mathbb{K}} \mathbb{A}$ of $\mathbb{K}$-modules V by $\mathbb{K}$-algebras $\mathbb{A}$, in Appendix B we develop some linear algebra on "2^n-spaces", and in Appendix C we recall the definition of *functor categories* (cf. e.g., [CWM, MM92]), with particular attention to categories of functors from anchored $\mathbb{K}$-algebras to anchored sets.

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Notation. We let $\mathsf{n} = \{1, 2, \ldots, n\}$. Categories are denoted in boldface characters: small letters for small categories, such as $\mathbf{talg}_{\mathbb{K}}$, and capital letters for large categories, such as **Sets** (category of sets). The letter **Fn** stands for "functor category", so $\mathbf{Fn}(\mathbf{c}, \mathbf{Sets})$ is the category of (covariant) functors from a (small) category $\mathbf{c}$ to **Sets** (cf. Appendix C). Throughout, $\mathbb{K}$ is a commutative base ring with unit 1.

1. Topological differential calculus

In differential calculus, one usually is mostly interested in the *morphisms*, that is, in *maps of class C^n*. However, let us first say some words about the *objects*:

1.1. Locally linear sets, and the anchor. A *locally linear set* is a pair (U, V), where V is a $\mathbb{K}$-module, and $U \subset V$ a non-empty subset. Without fixing time parameters, we define the set $U^{[1]}$ by (0.3), and the *anchor* by

$$\Upsilon : U^{[1]} \to (U \times \mathbb{K})^2, \quad (v_0, v_1; t, s) \mapsto (v_0 + sv_1, s; v_0 + tv_1, t). \tag{1.1}$$

When time parameters $(t, s) \in \mathbb{K}^2$ are fixed, we define $U_{(t,s)} := U^{[1]}_{(t,s)} := U^{\{1\}}_{(t,s)}$ by (0.4), and the *anchor*

$$\Upsilon_{(t,s)} := \Upsilon^{\{1\}}_{(t,s)} : U^{[1]}_{(t,s)} \to U \times U$$

is given by restricting the map Υ defined above, i.e., it is given by (0.5). Direct computation shows that $\Upsilon_{(t,s)}$ is invertible iff $s - t$ is invertible in $\mathbb{K}$, with inverse given by (0.6). Note that $(U^{[1]}_{(t,s)}, V^2)$ is again a locally linear set, and hence the construction can be iterated, with some new parameter (t_2, s_2), and so on. Explicit formulae, describing this, will be given later (restricted iteration, Def. 1.4).

1.2. The topological setting. In the remainder of this section we assume that $\mathbb{K}$ is a *good topological ring* (a topological ring whose unit group $\mathbb{K}^\times$ is open and dense, and inversion is a continuous map), that all $\mathbb{K}$-modules are topological modules, and that all locally linear sets (U, V), (U', V'), … are *open* inclusions.

Definition 1.1. *We say that $f : U \to V'$ is* of class $C_1^{\mathbb{K}}$ *if the* slope *given by (0.2) extends to a* continuous *map $f^{[1]} : U^{[1]} \to V'$. We then define $df(x)v := \partial_v f(x) := f^{[1]}(x, v; 0, 0)$.*

Remark 1.1. Letting $s = 0$, this clearly implies that f is of class $C^1_{\mathbb{K}}$ in the sense of [BGN04] or [Be08]. Conversely, the map denoted here by $f^{[1]}$ can be expressed by the one denoted $f^{[1]}$ in loc. cit., and hence the $C^1_{\mathbb{K}}$-notions used there are equivalent to the one given above. We call the calculus obtained by restricting to $s = 0$ *target calculus*. Recall from [BGN04] that, in the real or complex finite dimensional case this definition is equivalent to all usual ones, and in the infinite dimensional locally convex case it is equivalent to Keller's definition of differentiability.

Lemma 1.2. *For a map $f : U \to U'$, the following are equivalent:*

(1) f is $C^1_{\mathbb{K}}$,

(2) for all $(t, s) \in \mathbb{K}^2$, there exists a map $f_{(t,s)} = f^{\{1\}}_{(t,s)} : U_{(t,s)} \to U'_{(t,s)}$, such that

- *(a) the map $U^{[1]} \to (U')^{[1]}$, $(x, v; t, s) \mapsto f_{(t,s)}(x, v)$ is continuous,*
- *(b) for all $(t, s) \in \mathbb{K}^2$,*

$$\Upsilon_{(t,s)} \circ f^{\{1\}}_{(t,s)} = (f \times f) \circ \Upsilon_{(t,s)} : \quad \begin{array}{ccc} U_{(t,s)} & \xrightarrow{f_{(t,s)}} & U'_{(t,s)} \\ \Upsilon \downarrow & & \downarrow \Upsilon \\ U \times U & \xrightarrow{f \times f} & U' \times U' \end{array}$$

Proof. As we have already seen, when $t-s$ is invertible in $\mathbb{K}$, then $f_{(t,s)}$ is necessarily given by (0.7). Since its second component is the slope $f^{[1]}$, existence of $f_{(t,s)}$, jointly continuous in $(x,v;t,s)$, implies existence of a continuous extension of the slope, whence (2) ⇒ (1). To prove the converse, assume (1). Then the second component of $f_{(t,s)}$ (that is, the slope) admits a continuous extension, by definition. Let us show that the first component of $f_{(t,s)}$ also admits a continuous extension. Indeed, let $x_0 := f(x+sv)$ and $x_1 := f(x+tv)$ and $v_1 = f^{[1]}(x,v;t,s)$. Then $x_0 = v_0 + sv_1$, $x_1 = v_0 + tv_1$, whence the first component of $f_{(t,s)}$ is

$$v_0 = x_1 - tv_1 = f(x+tv) - tf^{[1]}(x,v;t,s),$$

showing that this expression extends continuously for all (t,s) if so does $f^{[1]}$. □

Example 1.1. If $f : V \to V'$ is *linear* and continuous, then direct computation using (0.7) shows that $f_{(t,s)}(v_0,v_1) = (f(v_0), f(v_1))$, so f is $C^1_{\mathbb{K}}$.

Remark 1.2. Letting $v_1 = 0$ in (0.7), we always get $f_{(t,s)}(v_0,0) = (f(v_0),0)$. In diagrammatic form, the map f itself imbeds into $f_{(t,s)}$: we define the imbedding

$$\iota_{(t,s)} : U \to U_{(t,s)}, \quad v_0 \mapsto (v_0,0) \tag{1.2}$$

then the computation just mentioned shows that $f_{(t,s)} \circ \iota_{(t,s)} = \iota_{(t,s)} \circ f$:

$$\begin{array}{ccc} U_{(t,s)} & \xrightarrow{f_{(t,s)}} & U'_{(t,s)} \\ \iota \uparrow & & \uparrow \iota \\ U & \xrightarrow{f} & U \end{array} \tag{1.3}$$

Note that $\Upsilon \circ \iota$ is the diagonal imbedding $\Delta : U \to U \times U$, $x \mapsto (x,x)$.

In this setting, the usual rules of first order calculus hold (chain rule, product rule, linearity of first differential) – for a systematic exposition we refer to [BGN04, Be08, Be11]. Most important for our purposes is the Chain Rule, which we write in functorial form

$$\forall (t,s) \in \mathbb{K}^2 : \quad (g \circ f)_{(t,s)} = g_{(t,s)} \circ f_{(t,s)}. \tag{1.4}$$

This follows easily from Lemma 1.2: for invertible $t-s$, we have functoriality $(g \times g) \circ (f \times f) = (g \circ f) \times (g \circ f)$, and for general (t,s), it follows "by density".

1.3. **Full versus restricted iteration.** Higher order differentiability is defined by iterating first order differentiability. However, there are various ways of doing so, and it is important to distinguish them. In [BGN04], f is defined to be of class $C^2_{\mathbb{K}}$ if it is C^1 and if $f^{[1]}$ also is C^1, so that we can define $f^{[2]} := (f^{[1]})^{[1]}$, etc.:

Definition 1.3 (Full iteration)**.** *We say that f is* of class $C^n_{\mathbb{K}}$ *if: f is of class $C^1_{\mathbb{K}}$, and $f^{[1]}$ is of class $C^{n-1}_{\mathbb{K}}$. In this case we let $f^{[n]} := (f^{[1]})^{[n-1]}$.*

Remark 1.3. In the real or complex finite dimensional case this is equivalent to the usual definitions (see [BGN04, Be11]). However, since full iteration repeats the procedure for all variables together, the number of variables exploses, and it is hard to get control over the structure of the maps $f^{[n]}$ (see [Be15b]). To reduce the number of variables, in *restricted iteration* we consider in each step time variables to be frozen, and take difference quotients only with respect to space variables.

Notation. For each $k \in \mathbb{N}$, we denote by an upper index $\{k\}$ a copy of the procedure $\{1\}$ that has been defined above. An upper index $\{i,j\}$ $(i<j)$ indicates a double application of the procedure (first $\{i\}$, then $\{j\}$), etc. E.g., an upper index $\mathsf{n} := \{1,\ldots,n\}$ indicates that we first apply $\{1\}$, then $\{2\}$, etc., and finally $\{n\}$.

To abbreviate, in the sequel, we let $(\mathbf{t},\mathbf{s}) = (t_1,\ldots,t_n;s_1,\ldots,s_n) \in \mathbb{K}^{2n}$.

Definition 1.4 (Restricted iteration). *A map $f: U \to U'$ is called* of class $C_{\mathbb{K},n}$ *if: it is of class $C^1_{\mathbb{K}}$, and, for all $(t_1,s_1)\in\mathbb{K}^2$, the map $f^{\{1\}}_{(t_1,s_1)}$ is of class $C_{\mathbb{K},n-1}$. In this case we define*

$$f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := (f^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to (U')^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}.$$

The map $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ is defined on the restricted iterated domain

$$U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := (U^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)}.$$

We also require that $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ be jointly continuous both in space and in time variables. In other terms, the inductive definition of $U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ and of $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ amounts to

$$U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = U^{\{1,\ldots,n\}}_{(\mathbf{t},\mathbf{s})} = (\ldots(U^{\{1\}}_{t_1,s_1})^{\{2\}}_{(t_2,s_2)})\ldots)^{\{n\}}_{(t_n,s_n)},$$
$$f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = f^{\{1,\ldots,n\}}_{(\mathbf{t},\mathbf{s})} := (\ldots(f^{\{1\}}_{t_1,s_1})^{\{2\}}_{(t_2,s_2)})\ldots)^{\{n\}}_{(t_n,s_n)}.$$

Theorem 1.5. *When $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, and V is a locally convex topological vector space, then the conditions $C^n_{\mathbb{K}}$ and $C_{\mathbb{K},n}$ are both equivalent to the usual (Keller's) definition of C^n-maps.*

Proof. As already mentioned, $C^n_{\mathbb{K}}$ clearly implies $C_{\mathbb{K},n}$, and equivalence of $C^n_{\mathbb{K}}$ with Keller's definition has been proved in [BGN04]. On the other hand, $C_{\mathbb{K},n}$ obviously implies Keller's C^n-definition, which arises simply by taking $(\mathbf{t},\mathbf{s}) = (0,\ldots,0)$ in the $C_{\mathbb{K},n}$-condition. Thus all three conditions are equivalent. □

Remark 1.4. For general $\mathbb{K}$, properties $C^n_{\mathbb{K}}$ and $C_{\mathbb{K},n}$ cease te be equivalent: in *positive characteristic*, condition $C^n_{\mathbb{K}}$ appears to be strictly stronger than $C_{\mathbb{K},n}$ (cf. the proof of the general Taylor formula in [BGN04, Be11], which really uses *full* iteration; concerning this item, cf. also [Be13]). It would be interesting to have a criterion allowing to decide when $C^n_{\mathbb{K}}$ and $C_{n,\mathbb{K}}$ are equivalent.

Definition 1.6. *For all $(\mathbf{t},\mathbf{s})\in\mathbb{K}^{2n}$, the n*-th order anchor *of $U\subset V$ is defined as follows: for all locally linear sets $(U,V),(U',V')$, we consider the map*

$$(U\times U')_{(t,s)} \to U_{(t,s)} \times U'_{(t,s)}, \quad ((v_0,v_0'),(v_1,v_1')) \mapsto ((v_0,v_1),(v_0',v_1'))$$

as identification. Under such identifications, the map $\Upsilon := \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} :=$

$$(\Upsilon^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to (U^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)} \times (U^{\{1\}}_{(t_1,s_1)})^{\{2,\ldots,n\}}_{(t_2,\ldots,t_n,s_2,\ldots,s_n)}$$

inductively gives rise to a map $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to U^{2^n}$ which we call the n-fold anchor.

Remark 1.5. In order to fully formalize this definition, we need an explicit labelling of the 2^n copies of U in U^{2^n}. For the moment, this is not needed, and will be taken up later (Def. 2.23). Let us, however, give the result for $n=2$: space variables have labels $0,1,2,12$ corresponding to the subsets of $\{1,2\}$, so we write

$\mathbf{v} = (v_0, v_1, v_2, v_{12}) \in U^{\{1,2\}}_{(t_1,t_2,s_1,s_2)}$. Then iteration shows that the linear map Υ is given by the (block) matrix (Kronecker product of two first-order anchors)

$$\begin{pmatrix} 1 & s_1 \\ 1 & t_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & s_2 \\ 1 & t_2 \end{pmatrix} = \begin{pmatrix} 1 & s_1 & s_2 & s_1 s_2 \\ 1 & t_1 & s_2 & t_1 s_2 \\ 1 & s_1 & t_2 & s_1 t_2 \\ 1 & t_1 & t_2 & t_1 t_2 \end{pmatrix}, \tag{1.5}$$

so we have four "evaluation points" given by the four lines of the (block) matrix:

$$\begin{array}{rcl} \Upsilon_\emptyset(\mathbf{v}) & = & v_\emptyset + s_1 v_1 + s_2 v_2 + s_1 s_2 v_{12}, \\ \Upsilon_1(\mathbf{v}) & = & v_\emptyset + t_1 v_1 + s_2 v_2 + t_1 s_2 v_{12}, \\ \Upsilon_2(\mathbf{v}) & = & v_\emptyset + s_1 v_1 + t_2 v_2 + s_1 t_2 v_{12}, \\ \Upsilon_{12}(\mathbf{v}) & = & v_\emptyset + t_1 v_1 + t_2 v_2 + t_1 t_2 v_{12}. \end{array} \tag{1.6}$$

The inverse matrix of (1.5) is the Kronecker product of the inverses of the respective first order anchors (when these are invertible): writing $d_\Upsilon := (t_1 - s_1)(t_2 - s_2)$,

$$\frac{1}{d_\Upsilon} \begin{pmatrix} t_1 & -s_1 \\ -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} t_2 & -s_2 \\ -1 & 1 \end{pmatrix} = \frac{1}{d_\Upsilon} \begin{pmatrix} t_1 t_2 & -s_1 t_2 & -t_1 s_2 & s_1 s_2 \\ -t_2 & t_2 & s_2 & -s_2 \\ -t_1 & s_1 & t_1 & -s_1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \tag{1.7}$$

For the general case, see Theorem 2.25.

Theorem 1.7. *For a map $f : U \to U'$, the following are equivalent:*

(1) f is $C_{\mathbb{K},n}$,
(2) for all $(\mathbf{t}, \mathbf{s}) \in \mathbb{K}^{2n}$, there exists a map $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to (U')^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, such that
 (a) $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v})$ is jointly continuous in space and time variables $(\mathbf{v}; \mathbf{t}, \mathbf{s})$,
 (b) for all $(\mathbf{t}, \mathbf{s}) \in \mathbb{K}^{2n}$, $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \circ f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = f^{2^n} \circ \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$:

$$\begin{array}{ccc} U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} & \xrightarrow{f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} & (U')^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \\ \Upsilon^{\mathsf{n}}_{\mathbf{t},\mathbf{s}} \downarrow & & \downarrow \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \\ U^{2^n} & \xrightarrow{f^{2^n}} & (U')^{2^n}. \end{array}$$

The map $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ depends functorially on f: $(f \circ g)^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \circ g^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ (Chain Rule).

Proof. By induction: for $n = 1$, this is Lemma 1.2. Assume the claim holds on level $n - 1$ and apply it to f replaced by $f^{\{1\}}_{(t_1,s_1)}$. From the inductive definitions, it follows readily that the properties are again equivalent on level n. The (higher order) Chain Rule now also follows by induction. □

Example 1.2. Using Formula (1.7), let us give explicit formulae for $n = 2$:

$$f^{2}_{(t_1,t_2,s_1,s_2)}(\mathbf{v}) = \Upsilon^{-1}\big(f(\Upsilon_\emptyset(\mathbf{v})), f(\Upsilon_1(\mathbf{v})), f(\Upsilon_2(\mathbf{v})), f(\Upsilon_{12}(\mathbf{v}))\big)$$

$$= \frac{1}{d_\Upsilon} \begin{pmatrix} t_1 t_2 f(\Upsilon_\emptyset \mathbf{v}) - s_1 t_2 f(\Upsilon_1 \mathbf{v}) - t_1 s_2 f(\Upsilon_2 \mathbf{v}) + s_1 s_2 f(\Upsilon_{12} \mathbf{v}) \\ -t_2 f(\Upsilon_\emptyset \mathbf{v}) + t_2 f(\Upsilon_1 \mathbf{v}) + s_2 f(\Upsilon_2 \mathbf{v}) - s_2 f(\Upsilon_{12} \mathbf{v}) \\ -t_1 f(\Upsilon_\emptyset \mathbf{v}) + s_1 f(\Upsilon_1 \mathbf{v}) + t_1 f(\Upsilon_2 \mathbf{v}) - s_1 f(\Upsilon_{12} \mathbf{v}) \\ f(\Upsilon_\emptyset \mathbf{v}) - f(\Upsilon_1 \mathbf{v}) - f(\Upsilon_2 \mathbf{v}) + f(\Upsilon_{12} \mathbf{v}) \end{pmatrix} \tag{1.8}$$

Since $d_\Upsilon = (t_1 - s_1)(t_2 - s_2)$, the first term is in fact an affine combination of values of f at the four evaluation points, whereas the other three terms are "zero-sum combinations" of these values, and hence correspond to "true" difference quotients. In order to state results at arbitrary order, we need some notation:

1.4. **Hypercube notation, and formula for higher order slopes.**

Definition 1.8. *We call* n-hypercube *the power set* $\mathcal{P}(\mathsf{n}) = \mathcal{P}(\{1,\ldots,n\})$. *It serves as index set for space variables, which we write in the form* $\mathbf{v} = (v_A)_{A\in\mathcal{P}(\mathsf{n})}$. *Recall that* $\mathcal{P}(\mathsf{n})$ *is a semigroup for union* $\cup$ *and intersection* $\cap$, *and a group with respect to the* symmetric difference

$$A\Delta B = (A\cup B)\setminus(A\cap B) = (A\cap B^c)\cup(B\cap A^c),$$

where $A^c = \mathsf{n}\setminus A$ *is the complement of* A *in* n. *Recall also that* $A^c\Delta B^c = A\Delta B$, *and that* $A\Delta B^c = (A\Delta B)^c = A^c\Delta B$, *whence* $|A\Delta B^c| = n - |A\Delta B|$.

Definition 1.9. *For all* $\mathbf{t},\mathbf{s}\in\mathbb{K}^n$ *and* $A\in\mathcal{P}(\mathsf{n})$, *we let* $\mathbf{t}_\emptyset = 1 = \mathbf{s}_\emptyset$, *and*

$$\mathbf{t}_A = \prod_{k\in A} t_k, \qquad \mathbf{s}_A = \prod_{k\in A} s_k, \qquad (\mathbf{t}-\mathbf{s})_A = \prod_{k\in A}(t_k - s_k).$$

Call $(\mathbf{t},\mathbf{s})$ regular, *or* finite, *if,* $\forall i = 1,\ldots,n : (t_i - s_i)\in\mathbb{K}^\times$, *and* singular *if* $\forall i = 1,\ldots,n : (t_i - s_i)\notin\mathbb{K}^\times$, *and* mixed *else.*

Theorem 1.10. *Let* $f : U\to U'$ *be of class* $C_{\mathbb{K},n}$. *Then, for all regular* $(\mathbf{t},\mathbf{s})\in\mathbb{K}^{2n}$, *and all* $B\in\mathcal{P}(\mathsf{n})$, *the component* $(f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v}))_B$ *is given by*

$$(f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}(\mathbf{v}))_B = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}}\sum_{A\in\mathcal{P}(\mathsf{n})}(-1)^{|A\Delta B|}\mathbf{s}_{B^c\cap A}\mathbf{t}_{B^c\cap A^c}\, f\Big(\sum_{C\in\mathcal{P}(\mathsf{n})}\mathbf{s}_{C\cap A^c}\mathbf{t}_{C\cap A}v_C\Big).$$

The proof will be given in Subsection 2.7. For $B=\emptyset$, the component is an affine combination of values of f at the 2^n evalation points, and for all other components it is again a "zero sum combination".

1.5. **Categories of locally linear sets and $C_{\mathbb{K},n}$-maps.** To summarize, we have defined a *category of locally linear sets and their* $C_{\mathbb{K},n}$*-morphisms*:

Definition 1.11. *We denote by* $\mathbf{Llin}_{\mathbb{K},n}$ *the category whose objects are pairs* (U,V), *where* V *is a topological* $\mathbb{K}$*-module and* $U\subset V$ *a non-empty open subset, and morphisms are* $C_{\mathbb{K},n}$*-maps* $f : U\to U'$. *(For* $n=0$, *morphisms are continuous maps, and for* $n=\infty$, *these are maps that are* $C_{\mathbb{K},n}$ *for all* $n\in\mathbb{N}$.*)*

Definition 1.12. *For* $m\ge n$ *and* $(\mathbf{t},\mathbf{s})\in\mathbb{K}^{2n}$, *the* $(n;\mathbf{t},\mathbf{s})$-tangent functor *is the functor from* $\mathbf{Llin}_{\mathbb{K},m}$ *to* $\mathbf{Llin}_{\mathbb{K},m-n}$ *given by* $(U,V)\mapsto(U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})},V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})})$ *and* $f\mapsto f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$.

Remark 1.6 (Manifolds). By the usual glueing procedures, one may now define $C_{\mathbb{K},n}$*-manifolds over* $\mathbb{K}$, modelled on locally linear sets – since these methods are independent of the particular form of differential calculus, we do not wish to go here into details (see [Be16] for a formulation of such principles, adapted to most general contexts). The $(n;\mathbf{t},\mathbf{s})$-tangent functor then carries over to manifolds : for every $\mathbb{K}$-smooth manifold M we have a "generalized higher order tangent bundle" $M^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, depending functorially on M, and coming with an anchor map $M^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}\to M^{2^n}$.

2. The rings of calculus: tangent algebras

Our next aim is to understand the $(n;\mathbf{t},\mathbf{s})$-tangent functor as a *functor of scalar extension*, from $\mathbb{K}$ to a ring $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, as a generalisation of the algebraic scalar extension functor $-\otimes_{\mathbb{K}} \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$. To this end, let's first restrict attention to the case $V=U=\mathbb{K}$. Since $\mathbb{K}$ carries canonical structures, so will, by functoriality, the spaces $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$; and the anchor map will be a ring morphism, which we can compute explicitly. In this section, we continue to assume that $\mathbb{K}$ is a good topological ring.

2.1. **Bilinear maps.** We can differentiate bilinear continuous maps $\beta : V\times W \to Y$ in the usual way. Since we think of β as a "product", let us write $v \bullet w := \beta(v,w)$. Concerning Cartesian products, we use the convention from Def. 1.6, so that $\beta^{\{1\}}_{(t,s)}$ is considered as a map

$$\beta^{\{1\}}_{(t,s)} : V_{(t,s)} \times W_{(t,s)} \to Y_{(t,s)}, \quad (\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}) \mapsto \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \bullet^{\{1\}}_{(t,s)} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}$$

which by an explicit computation using Formula (0.7) is given by

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \bullet^{\{1\}}_{(t,s)} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} v_0 \bullet w_0 - st\, v_1 \bullet w_1 \\ v_0 \bullet w_1 + v_1 \bullet w_0 + (s+t) v_1 \bullet w_1 \end{pmatrix}. \tag{2.1}$$

Moreover, by Lemma 1.2, the anchor $\Upsilon_{(t,s)}$ is a "morphism" from the product $\beta^{\{1\}}_{(t,s)}$ to $\beta \times \beta$ (the "direct product algebra").

2.2. **First order tangent algebras.** Now take $V=W=\mathbb{K}$ and $\beta(x,y)=xy$.

Definition 2.1 (Normalization: source, target, anchor). *When $V=W=\mathbb{K}$, the anchor is a map $\Upsilon : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}^2 \times \mathbb{K}^2$, and when $(t,s)\in\mathbb{K}^2$ is fixed, then we have a (linear) map $\Upsilon_{(t,s)} : \mathbb{K}^2 \to \mathbb{K}^2$. The standard basis in the* domain *of $\Upsilon_{(t,s)}$ will be denoted by $e_\emptyset := (1,0)$, $e_1 := (0,1)$, and the standard basis in its* range *by $E_\emptyset := (1,0)$ and $E_1 := (0,1)$, so $\mathbf{v} = v_\emptyset e_\emptyset + v_1 e_1$. We make the* choice *to associate $E_\emptyset$ with "source" and E_1 with "target", that is, we let*

$$\begin{aligned} \alpha(\mathbf{v}) &= \alpha(v_0 e_\emptyset + v_1 e_1) = (v_0 + s v_1) E_\emptyset, \\ \beta(\mathbf{v}) &= \beta(v_0 e_\emptyset + v_1 e_1) = (v_0 + t v_1) E_1, \\ \Upsilon(\mathbf{v}) &= \Upsilon(v_0 e_\emptyset + v_1 e_1) = (v_0 + s v_1) E_\emptyset + (v_0 + t v_1) E_1. \end{aligned}$$

In other terms,

$$\Upsilon(e_\emptyset) = E_\emptyset + E_1, \qquad \Upsilon(e_1) = s E_\emptyset + t E_1.$$

With respect to ordered bases, with "usual" ordering $(e_\emptyset, e_1)$, $(E_\emptyset, E_1)$, these linear maps correspond to the matrices

$$\alpha = \begin{pmatrix} 1 & s \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & t \end{pmatrix}, \qquad \Upsilon_{(t,s)} = \begin{pmatrix} 1 & s \\ 1 & t \end{pmatrix}.$$

As usual, dual bases are denoted by $((e_\emptyset)^*, (e_1)^*)$, resp., $((E_\emptyset)^*, (E_1)^*)$. Thus

$$\begin{aligned} \alpha &= e_\emptyset^* + s \cdot e_1^*, \\ \beta &= e_\emptyset^* + t \cdot e_1^*, \\ \Upsilon_{(t,s)} &= \alpha \otimes E_\emptyset + \beta \otimes E_1 = e_\emptyset^* \otimes E_\emptyset + e_1^* \otimes E_1 + s\, e_1^* \otimes E_\emptyset + t\, e_\emptyset^* \otimes E_1. \end{aligned}$$

The *choice* appearing in this definition should be seen as an additional structure, akin to the choice of an "orientation".

Lemma 2.2. *For all* $(t,s) \in \mathbb{K}^2$*, the map induced by* $m : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$*,* $(x,y) \mapsto xy$*,*

$$m^{\{1\}}_{(t,s)} : \mathbb{K}^{\{1\}}_{(t,s)} \times \mathbb{K}^{\{1\}}_{(t,s)} \to \mathbb{K}^{\{1\}}_{(t,s)}, \quad ((v_0,v_1),(w_0,w_1)) \mapsto (v_0,v_1)\cdot(w_0,w_1)$$

turns $\mathbb{K}^{\{1\}}_{(t,s)}$ *into an associative commutative* $\mathbb{K}$*-algebra which, under identification of the* $\mathbb{K}$*-bases* $e_\emptyset = 1 = [1]$*,* $e_1 = e = [X]$*, is isomorphic to the quotient algebra*

$$\mathbb{K}^{\{1\}}_{(t,s)} \cong \mathbb{K}[X]/(X-t)(X-s).$$

Proof. The fact that $\mathbb{K}^{\{1\}}_{(t,s)}$ is an associative commutative algebra follows by functoriality from the fact that so is $\mathbb{K}$ ([Be08, BeS14]), but it can also be obtained as a consequence of the following computational argument: we compare Formula (2.1), relative to $e_\emptyset, e_1$, with the product in $\mathbb{K}[X]/(X-t)(X-s)$, relative to the basis $[1], [X]$. In both cases, $1 = e_\emptyset$ is a neutral element, and we have

$$e^2 = -st \cdot 1 + (s+t)e, \tag{2.2}$$

showing that, with respect to these bases, the formulae describing the product in $\mathbb{K}_{(t,s)}$ and in $\mathbb{K}[X]/(X-t)(X-s)$ are the same. □

Definition 2.3. *We call* $\mathbb{K}^{\{1\}}_{(t,s)}$ *the* (t,s)-tangent algebra of $\mathbb{K}$*. For* $t = s = 0$ *we get the* algebra of dual numbers*, see Eqn. (0.1).*

Lemma 2.4. *The anchor*

$$\Upsilon^{\{1\}}_{(t,s)} : \mathbb{K}_{(t,s)} \to \mathbb{K} \times \mathbb{K}, \quad (v_0,v_1) \mapsto (v_0 + sv_1, v_0 + tv_1)$$

is an algebra morphism into the direct product algebra. It is an isomorphism iff $s - t \in \mathbb{K}^\times$*. Source* α *and target* β *are characters (morphisms* $\mathbb{K}_{(t,s)} \to \mathbb{K}$*).*

Proof. Under the identification $\mathbb{K}^{\{1\}}_{(t,s)} \cong \mathbb{K}[X]/((X-s)(X-t))$, we have $\Upsilon([P]) = (P(s), P(t)))$, the evaluation morphism of the quotient algebra at (s,t). □

Theorem 2.5 (Structure of the first order tangent algebra $\mathbb{K}^{\{1\}}_{(t,s)}$)**.**

(1) *The ideals* $\ker(\alpha)$ *and* $\ker(\beta)$ *satisfy* $\ker(\alpha)\cdot\ker(\beta) = 0$.

(2) *The product of* $w, v \in \mathbb{K}^{\{1\}}_{(t,s)}$ *is given by the "fundamental relation"*

$$w \cdot v = \alpha(w)v - \alpha(w)\beta(v) + \beta(v)w.$$

(3) *The map*

$$\kappa : \mathbb{K}^{\{1\}}_{(t,s)} \to \mathbb{K}^{\{1\}}_{(t,s)}, \quad v \mapsto (\alpha+\beta)(v)\cdot 1 - v$$

is an algebra automorphism of order 2 *such that* $\alpha \circ \kappa = \beta$*. Moreover,*

$$\forall v \in \mathbb{K}^{\{1\}}_{(t,s)} : \quad v \cdot \kappa(v) = \alpha(v)\beta(v)1.$$

(4) *An element* v *is invertible in* $\mathbb{K}^{\{1\}}_{(t,s)}$ *if, and only if,* $\alpha(v)\beta(v) \in \mathbb{K}^\times$*, and then the inverse is*

$$v^{-1} = \frac{1}{\alpha(v)\beta(v)}\kappa(v) = (\frac{1}{\alpha(v)} + \frac{1}{\beta(v)})1 - \frac{v}{\alpha(v)\beta(v)}.$$

Proof. (1) $\ker(\alpha) = \mathbb{K}(e - s)$ and $\ker(\beta) = \mathbb{K}(e - t)$, and, by the defining relation of the algebra, $(e - s)(e - t) = [(X - t)(X - s)] = 0$.

(2) Since $\alpha(v - \alpha(v)1) = 0$ and $\beta(w - \beta(w)1) = 0$, the preceding item implies

$$0 = (v - \alpha(v))(w - \beta(w)) = vw - \alpha(v)w - \beta(w)v + \alpha(v)\beta(w).$$

(3) Note that $\kappa(1) = 1 + 1 - 1 = 1$ and $\kappa(e) = s + t - e$, whence $\kappa(\kappa(e)) = s + t - (s + t - e) = e$, so $\kappa^2 = \mathrm{id}$. Next,

$$\alpha(\kappa(v)) = (\alpha + \beta)(v) - \alpha(v) = \beta(v).$$

To prove that κ is an automorphism, since $\kappa(1) = 1$, it suffices to show that $\kappa(e^2) = \kappa(e)^2$. Indeed, $\kappa(e)^2 = (t + s)^2 - 2(t + s)e + e^2 = (t + s)^2 - ts - (t + s)e$ and $\kappa(e^2) = \kappa(-ts + (t + s)e) = -ts + (t + s)\kappa(e) = -ts + (t + s)^2 - (t + s)e$. Finally,

$$v \cdot \kappa(v) = \alpha(v)\kappa(v) - \alpha(v)\beta(\kappa v) + \beta(\kappa v)v = \alpha(v)\beta(v)1.$$

(4) If v is invertible, then applying the morphisms α and β, it follows that both $\alpha(v)$ and $\beta(v)$ are invertible. Conversely, the last formula from (3) shows that under this condition v has an inverse given by v^{-1} as in the claim. □

Remark 2.1. Under Υ, the corresponding "fundamental relation" in $\mathbb{K}^2$ reads

$$(x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2) = x_1(y_1, y_2) - x_1y_2(1, 1) + y_2(x_1, x_2),$$

and inversion in $\mathbb{K}^2$ can be written $(x_1, x_2)^{-1} = \frac{1}{x_1x_2}(x_2, x_1)$.

Definition 2.6. *We define the following elements in* $\mathbb{K}^{\{1\}}_{(t,s)}$ *(the latter if* $2 \in \mathbb{K}^\times$*)*

$$\begin{aligned} a &:= e - t1 = [X - t], \\ b &:= e - s1 = [X - s], \\ j &:= \frac{a + b}{2} = e - \frac{t + s}{2}. \end{aligned}$$

Remark 2.2. The pair $(1, a)$ is a basis of $\mathbb{K}^{\{1\}}_{(t,s)}$, and so is $(1, b)$. When (t, s) is regular, then (a, b) also is a basis, but when $t = s$, we have $a = b$. In the symmetric case ($t = -s$), we have $e = j$. This situation is represented by Figure 1 (where $t = 0.4 = -s$). The arrows indicate the direction of the kernel of the projection α, resp. β, cf. the following lemma.

Lemma 2.7. *The elements* a, b, j *satisfy the relations*

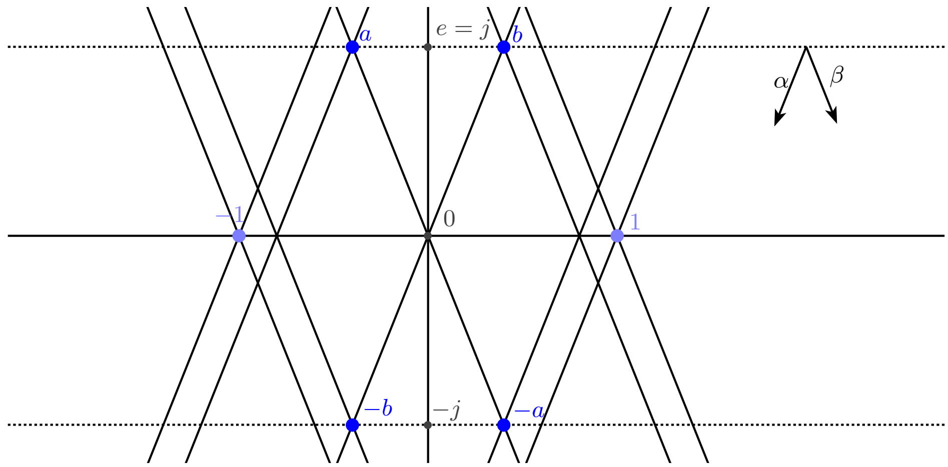
(1) $ab = 0, \qquad a^2 = (s - t)a, \qquad b^2 = (t - s)b, \qquad j^2 = \frac{(t-s)^2}{4}1$
(2) $\alpha(a) = s - t = -\beta(b), \qquad \alpha(b) = 0 = -\beta(a),$
(3) $\kappa(a) = -b, \qquad \kappa(j) = -j$
(4) $\Upsilon(a) = (s - t)E_\emptyset, \qquad \Upsilon(b) = (t - s)E_1, \qquad \Upsilon(j) = \frac{s-t}{2}(E_\emptyset - E_1).$

Proof. Since $\alpha(b) = \alpha(e) - s = s - s = 0$, and $\beta(a) = \beta(e) - t = t - t = 0$, we get from Theorem 2.5, for all $v \in V$,

$$v \cdot a = \alpha(v)a, \quad v \cdot b = \beta(v)b,$$

and everything follows more or less directly from this. □

Figure 1. The plan $\mathbb{K}_{(t,s)}$ and its distinguished elements.



Definition 2.8. *Assume (t,s) is regular, that is, $t-s$ is invertible. Then we define the* idempotent basis $(\tilde{E}_\emptyset, \tilde{E}_1)$ *in* $\mathbb{K}_{(t,s)}$ *by*

$$\tilde{E}_\emptyset := \Upsilon^{-1}(E_0) = \frac{a}{s-t} = \frac{e}{s-t} - \frac{t}{s-t},$$
$$\tilde{E}_1 := \Upsilon^{-1}(E_1) = \frac{b}{t-s} = \frac{e}{t-s} - \frac{s}{t-s}.$$

By this definition, the elements $\tilde{E}_0, \tilde{E}_1$ are orthogonal idempotents, and the matrix of Υ with respect to the $\tilde{E}$-basis in its codomain and the E-basis in its domain is the identity matrix. The base change matrix from the e-basis to the the $\tilde{E}$-basis is the "usual" matrix of Υ: indeed,

$$e_\emptyset = 1 = \tilde{E}_\emptyset + \tilde{E}_1, \qquad e_1 = e = s\tilde{E}_\emptyset + t\tilde{E}.$$

2.3. **The category of first order anchored tangent algebras.** Our aim is to define a (small) *category* of tangent algebras: so, we have to say, what are the *objects*, and what are the *morphisms* ?

Definition 2.9. *Objects of the* category of anchored first order $\mathbb{K}$-tangent algebras *are all anchor maps*

$$\Upsilon_{(t,s)} = \Upsilon^{\{1\}}_{(t,s)} : \mathbb{K}_{(t,s)} = \mathbb{K}^{\{1\}}_{(t,s)} \to \mathbb{K}^2$$

for $(t,s) \in \mathbb{K}^2$, as well as the "trivial" anchor id : $\mathbb{K} \to \mathbb{K}$. *Thus, objects can be considered as triples $(\mathbb{K}_{(t,s)}, \mathbb{K}^2, \Upsilon_{(t,s)})$, resp. $(\mathbb{K}, \mathbb{K}, \mathrm{id})$ (domain and codomain of the anchor, and its "formula"). For every anchored tangent algebra Υ, there is an* opposite anchored tangent algebra $\Upsilon' := \tau \circ \Upsilon$, *where $\tau(x,y) = (y,x)$ is the* exchange automorphism *of* $\mathbb{K}^2$.

Definition 2.10. *A pair of algebra morphisms*

$$\Phi : \mathbb{K}_{(t',s')} \to \mathbb{K}_{(t,s)}, \qquad \Phi' : \mathbb{K}^2 \to \mathbb{K}^2,$$

is called anchor-compatible *if*

$$\Upsilon_{(t,s)} \circ \Phi = \Phi' \circ \Upsilon_{(t',s')} : \mathbb{K}_{(t',s')} \to \mathbb{K}^2.$$

We say that Φ *is* direct *if the pair* $(\Phi, \mathrm{id}_{\mathbb{K}^2})$ *is anchor-compatible, and* indirect *if the pair* (Φ, τ) *is anchor-compatible. Equivalently,* Φ *is indirect iff* $\alpha \circ \Phi = \beta$, $\beta \circ \Phi = \alpha$, *and direct if* $\alpha \circ \Phi = \alpha$, $\beta \circ \Phi = \beta$.

We shall require that morphisms are anchor-compatible, either direct or indirect. When (t, s) is regular and Φ is direct, then necessarily $\Phi = (\Upsilon_{(t,s)})^{-1} \circ \Upsilon_{(t',s')}$, so there is exactly one direct morphism from $\mathbb{K}_{(t',s')}$ to $\mathbb{K}_{(t,s)}$. However, when (t, s) is singular, then we cannot always "classify" direct morphisms, and therefore we will proceed to give an explicit list of morphisms that are admitted in our small category. The case of indirect morphisms is reduced to the case of direct ones:

Lemma 2.11. *For every* $(t, s) \in \mathbb{K}^2$, *the automorphism* $\kappa : \mathbb{K}_{(t,s)} \mapsto \mathbb{K}_{(t,s)}$ *defined in Theorem 2.5 is indirect. There is a bijection between the sets of direct and indirect morphisms.*

Proof. We have seen in Theorem 2.5 that $\alpha \circ \kappa = \beta$ and $\beta \circ \kappa = \alpha$, so J is indirect. Clearly, Φ is direct iff $\Phi \circ \kappa$ is indirect, whence the claimed bijection. □

Remark 2.3. The map κ can be interpreted as the *inversion map of the groupoid* $U_{(t,s)}$, see Lemma 4.1. As such, it exchanges source and target of the groupoid.

Lemma 2.12. *Let* $(t, s), (t', s') \in \mathbb{K}^2$. *Any* $\mathbb{K}$*-affine map* $\phi : \mathbb{K} \to \mathbb{K}$ *such that* $\phi(t') = t$, $\phi(s') = s$, *induces a direct algebra morphism*

$$\Phi := \phi^* : \mathbb{K}_{(t,s)} = \mathbb{K}[X]/(X - t)(X - s) \to \mathbb{K}_{(t',s')}, \quad [P] \mapsto [P \circ \phi].$$

Proof. Any affine map ϕ induces an algebra-morphism $\mathbb{K}[X] \to \mathbb{K}[X]$, $P \mapsto P \circ \phi$. By our assumption on ϕ, this map passes to the quotient and thus defines a morphism of quotient algebras. It is direct,

$$\Upsilon(\phi^*([P])) = \Upsilon([P \circ \phi]) = (P(\phi(t)), P(\phi(s))) = (P(t'), P(s')) = \Upsilon([P]). \qquad \square$$

Definition 2.13. *A morphism defined by the preceding lemma is called* of affine type. *We distinguish the following cases:*

(1) *morphism of* translation type*:* $(t', s') = (t + \mu, s + \mu)$, $\phi(x) = x - \mu$,
(2) *morphism of* scaling type*:* $(t, s) = (\lambda t', \lambda s')$, $\phi(x) = \lambda x$,

Remark 2.4. The affine map $\phi(x) = s - x + t$ induces $\phi^* : \mathbb{K}_{(t',s')} \to \mathbb{K}_{(t,s)}$ with $s' = t, t' = s$. As algebra morphism, it is the same as κ; but κ acts from the label (t, s) to itself, and thus is considered as indirect, whereas ϕ^* acts from the label to the reversed label, and thus is considered as direct.

Remark 2.5. By the lemma, each tangent algebra $\mathbb{K}_{(t,s)}$ is isomorphic to an algebra $\mathbb{K}_{(r,0)}$ with r belonging to a system of representatives of the $\mathbb{K}^\times$-orbits in $\mathbb{K}$. Thus, when $\mathbb{K}$ is a field, there are only two isomorphism classes, corresponding to $r = 1$ (direct product algebra $\mathbb{K}^2$) and $r = 0$ (dual numbers $T\mathbb{K}$).

Definition 2.14. Morphisms of the category of anchored first order $\mathbb{K}$-tangent algebras *are all morphisms of affine type (direct), as well as their compositions with* κ *(indirect), and the canonical injection of the trivial anchor morphism, given by*

$$\Phi : \mathbb{K} \to \mathbb{K}_{(t,s)},\ x \mapsto x1, \qquad \Phi' : \mathbb{K} \to \mathbb{K}^2,\ x \mapsto x(1, 1).$$

Remark 2.6. It may turn out to be useful, or necessary, to add more morphisms to this list when developing the theory further (for instance, for the moment we refrain from adding α and β to our list, although they give rise to anchor-compatible morphisms). In any case, the list should be explicit, in order to control the behaviour of calculus with respect to morphisms, see the following proposition. Note also that the anchor "morphism" does not appear in our list of morphisms, since it is considered as "object"!

Proposition 2.15. *Let $U \subset V$ be open, $(t,s) \in \mathbb{K}^2$, $\lambda, \mu \in \mathbb{K}$. Then the maps*

$$\begin{aligned} H_\lambda : U_{(\lambda t, \lambda s)} \to U_{(t,s)}, &\quad (v_0, v_1) \mapsto (v_0, \lambda v_1), \\ T_\mu : U_{(t+\mu, s+\mu)} \to U_{(t,s)}, &\quad (v_0, v_1) \mapsto (v_0 + \mu v_1, v_1), \\ J : U_{(t,s)} \to U_{(t,s)}, &\quad (v_0, v_1) \mapsto ((s+t)v_1 + v_0, -v_1) \end{aligned}$$

are well-defined, and, for every map $f : U \to U'$ of class $C^1_{\mathbb{K}}$, they commute with tangent maps, in the sense that

$$H_\lambda \circ f_{(\lambda t, \lambda s)} = f_{(t,s)} \circ H_\lambda, \qquad T_\mu \circ f_{(t+\mu, s+\mu)} = f_{(t,s)} \circ T_\mu, \qquad \kappa \circ f_{(t,s)} = f_{(t,s)} \circ \kappa.$$

Proof. Direct computation using (0.3) and (0.7). □

2.4. **Higher order tangent algebras.** Now we apply restricted iteration to the construction of the tangent algebra. Recall that $\mathbb{K}^{\{k\}}_{(t_k,s_k)}$ is simply a copy of $\mathbb{K}^{\{1\}}_{(t_1,s_1)}$ which we call "of k-th generation". Now, each iteration step doubles the $\mathbb{K}$-dimension of our $\mathbb{K}$-algebra, so that in degree n we get an algebra of dimension 2^n. We will index basis elements by the hypercube $\mathcal{P}(\mathsf{n})$ of $\mathsf{n} = \{1, \ldots, n\}$, which is in keeping with notation for restricted iteration introduced above.

Definition 2.16. *For any $\mathbb{K}$-module V, the canonical isomorphism*

$$V \oplus V = (V \otimes_{\mathbb{K}} \mathbb{K}) \oplus (V \otimes_{\mathbb{K}} \mathbb{K}) = V \otimes_{\mathbb{K}} \mathbb{K}^2$$

gives an identification $V^{\{1\}}_{(t,s)} = V \times V = V \otimes_{\mathbb{K}} \mathbb{K}^{\{1\}}_{(t,s)}$. By restricted iteration, let

$$\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = \mathbb{K}^{\{1\}}_{(t_1,s_1)} \otimes \ldots \otimes \mathbb{K}^{\{n\}}_{(t_n,s_n)}.$$

The canonical basis $e_\emptyset, e_k$ in each factor $\mathbb{K}^{\{k\}}_{(t_k,s_k)}$ gives rise to a canonical basis in the $\mathbb{K}$-module $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, indexed by elements $A \in \mathcal{P}(\mathsf{n})$, and given by

$$e_A = \otimes_{k=1}^n e_{\{k\} \cap A}.$$

Lemma 2.17. *Applying restricted iteration n-times, with parameter $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$, to the product map of $\mathbb{K}$, we obtain an algebra structure on $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, isomorphic to*

(1) the n-fold tensor product of algebras $\mathbb{K}^{\{1\}}_{(t_1,s_1)} \otimes \ldots \otimes \mathbb{K}^{\{n\}}_{(t_n,s_n)}$,
(2) equivalently, to the quotient algebra

$$\mathbb{K}[X_1, \ldots, X_n]/((X_i - t_i)(X_i - s_i), i = 1, \ldots, n).$$

The canonical basis $(e_A)_{A \in \mathcal{P}(\mathsf{n})}$ of the tensor product corresponds to the canonical $\mathbb{K}$-basis of the polynomial algebra,

$$e_A = [\prod_{k \in A} X_k].$$

If V is a (topological) $\mathbb{K}$-module, then applying n-fold restricted iteration to the scalar multiplication map $\mathbb{K} \times V \to V$ gives the scalar action of the ring $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ on the algebraic scalar extension $V \otimes_{\mathbb{K}} \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$.

Proof. Formula (2.1) shows that, if $\mu : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$, $(x,y) \mapsto x \bullet y$ is any (finite-dimensional) algebra (associative or not), then $\mu^{\{k\}}_{(t_k,s_k)}$ is, as algebra, isomorphic to the tensor product of algebras $\mathbb{A} \otimes \mathbb{K}^{\{k\}}_{(t_k,s_k)}$, that is, the algebraic scalar extension of the algebra $\mathbb{A}$ by the ring $\mathbb{K}^{\{k\}}_{(t_k,s_k)}$. If $\mathbb{A}$ is an associative commutative algebra, then this algebraic scalar extension is also isomorphic to $\mathbb{A}[X_k]/((X_k - t_k)(X_k - s_k))$. Applying this remark n times, starting with $\mathbb{A} = \mathbb{K}$, the lemma follows. Similarly for the bilinear salar multiplication map $\mathbb{K} \times V \to V$. □

Note that, since $e_\emptyset$ is the unit element in $\mathbb{K}_{(t,s)}$, we have, e.g.,

$$e_{\{1,2\}} = e_1 \otimes e_2 = (e_1 \otimes e_\emptyset) \cdot (e_\emptyset \otimes e_2) = e_1 \cdot e_2,$$

by identifying $x \otimes 1$ with x and $1 \otimes y$ with y. And so on: we may write

$$e_A = \prod_{k \in A} e_k.$$

Definition 2.18. *Elements $\mathbf{v} \in \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, or of $V \otimes_{\mathbb{K}} \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, are written*

$$\mathbf{v} = \sum_{A \in \mathcal{P}(\mathsf{n})} v_A e_A, \quad v_A \in V.$$

*We also define the a-*basis*, resp. b-*basis*, of $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ by $a_\emptyset = 1 = b_\emptyset$,*

$$a_A := \otimes_{k=1}^n a_{\{k\} \cap A} = \prod_{i \in A} a_i, \qquad a_i := [X_i - t_i] = e_i - t_i, \qquad b_A = \prod_{i \in I} b_i,$$

*again indexed by $A \in \mathcal{P}(\mathsf{n})$. If 2 is invertible in $\mathbb{K}$, we define the j-*basis* by*

$$j_i := \frac{a_i + b_i}{2} = e_i - \frac{s_i + t_i}{2}, \qquad j_A := \prod_{i \in A} j_i.$$

Lemma 2.19. *We have the product rules, for all $A, B \in \mathcal{P}(\mathsf{n})$,*

$$\begin{aligned} a_A a_B &= (\mathbf{s} - \mathbf{t})_{A \cap B}\, a_{A \cup B}, \\ b_A b_B &= (\mathbf{t} - \mathbf{s})_{A \cap B}\, b_{A \cup B}, \\ j_A j_B &= \frac{1}{4^{|A \cap B|}} (\mathbf{t} - \mathbf{s})^2_{A \cap B}\, j_{A \Delta B}. \end{aligned}$$

Proof. The first product rule comes from $a_k^2 = (s_k - t_k) a_k$ and $a_i a_j = a_{\{i,j\}}$ when $i \neq j$, and similarly for the other two, with $j_k^2 = \frac{1}{4}(t_k - s_k)^2 = \frac{1}{4}(t_k - s_k)^2 j_\emptyset$. □

Corollary 2.20. *When $\mathbf{t} = 0$ and $\mathbf{s} = (1, \ldots, 1)$, then $\mathbb{K}^{\mathsf{n}}_{(0,1)}$ with a-basis is the semigroup algebra of $(\mathcal{P}(\mathsf{n}), \cup)$, and when $\mathbf{s} = -\mathbf{t} = (1, \ldots, 1)$, then $\mathbb{K}^{\mathsf{n}}_{(1,-1)}$ with j-basis is the group algebra of $(\mathcal{P}(\mathsf{n}), \Delta)$.*

Proof. In the first case, $a_A a_B = a_{A \cup B}$, which are the defining relations for the semigroup algebra $\mathbb{K}[\mathcal{P}(\mathsf{n})]$ (with $\cup$), and in the second, $j_A j_B = j_{A \Delta B}$, which defines the group algebra of $\mathbb{K}[\mathcal{P}(\mathsf{n})]$ with Δ. □

2.5. **The n-fold anchor and its inverse.** By restricted iteration, the n-fold anchor, taken for $U = V = \mathbb{K}$, has been defined, in Definition 1.6, to be a (linear) map $\Upsilon^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})} : \mathbb{K}^{\mathbf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}^{2^n}$. By induction, it will be an algebra morphism, but in order to put hands on it, we now have to fix a basis in $\mathbb{K}^{2^n}$, in such a way that it is compatible with iteration procedures. Forgetting the algebra structure, we get a linear space, called *hypercubic space*, see Appendix B.

Definition 2.21. *For any subset $N \subset \mathbb{N}$ of finite cardinal n, the* 2^n-fold direct product algebra of $\mathbb{K}$, in standard order, *is the algebra*

$$\mathbb{K}^{2^n} := \mathbb{K}^{\mathcal{P}(N)} := \bigoplus_{A \in \mathcal{P}(N)} \mathbb{K} E^N_A$$

with $\mathbb{K}$-basis $(E^N_A)_{A \in \mathcal{P}(N)}$, and product defined on basis elements by

$$E_A \cdot E_A = E_A, \quad \forall A \neq B : E_A \cdot E_B = 0.$$

When N is fixed, the notation E_A suffices. In particular, this notation is in keeping with our preceding one for the standard basis in $\mathbb{K}^2$: $E_\emptyset = E^{\{1\}}_\emptyset$, $E_1 = E^{\{1\}}_{\{1\}}$. We simplify notation by writing, e.g., k instead of $\{k\}$, and ij instead of $\{i,j\}$, etc.

Lemma 2.22. *The algebra $\mathbb{K}^{\mathcal{P}(N)}$ is an associative algebra canonically isomorphic to the algebra of functions from $\mathcal{P}(N)$ to $\mathbb{K}$ with pointwise product, with E^N_A corresponding to the function having value 1 at A and 0 else. For two disjoint finite subsets $N_1, N_2 \subset \mathbb{N}$, we have an isomorphism of algebras*

$$\mathbb{K}^{\mathcal{P}(N_1)} \otimes \mathbb{K}^{\mathcal{P}(N_2)} \to \mathbb{K}^{\mathcal{P}(N_1 \sqcup N_2)}, \quad E^{N_1}_A \otimes E^{N_2}_B \mapsto E^{N_1 \sqcup N_2}_{A \sqcup B}.$$

Proof. For the first statement, just recall that the algebra $\mathbb{K}^S$ of $\mathbb{K}$-valued functions on a finite set S has canonical basis $(1_x)_{x \in S}$, where $1_x(y) = \delta_{x,y}$, so E_A corresponds to the element 1_A. For the second statement, recall that for finite sets S, T,

$$\mathbb{K}^S \otimes \mathbb{K}^T \to \mathbb{K}^{S \times T}, \quad f \otimes g \mapsto ((x,y) \mapsto f(x)g(y))$$

defines an isomorphism of algebras sending $1_x \otimes 1_y$ to $1_{(x,y)}$, and combine this with the canonical bijection

$$\mathcal{P}(N_1) \times \mathcal{P}(N_2) \to \mathcal{P}(N_1 \sqcup N_2), \quad (A,B) \mapsto A \cup B$$

having inverse $C \mapsto (C \cap N_1, C \cap N_2)$. □

E.g., using the lemma, for $n = 2$, the algebra $\mathbb{K}^{\mathcal{P}(\{1,2\})}$ has $\mathbb{K}$-basis

$$\begin{array}{ll} E^{12}_\emptyset = E^1_\emptyset \otimes E^2_\emptyset, & E^{12}_1 = E^1_1 \otimes E^1_\emptyset, \\ E^{12}_2 = E^1_\emptyset \otimes E^2_2, & E^{12}_{12} = E^1_1 \otimes E^2_2, \end{array}$$

In general, the neutral element of $\mathbb{K}^{\mathcal{P}(N)}$ is the function that is 1 everywhere, that is, the sum of all basis elements:

$$1 = \sum_{A \in \mathcal{P}(N)} E^N_A.$$

Definition 2.23. *The* n-fold anchor *is the tensor product of* n *copies of the first order anchor, with respect to bases defined as above: it is the algebra morphism*

$$\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} := \otimes_{i=1}^{n} \Upsilon^{\{i\}}_{(t_i,s_i)} : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})},$$

where for each $k \in \mathbb{N}$, $\Upsilon^{\{k\}}_{(t_k,s_k)} : \mathbb{K}^{\{k\}}_{(t_k,s_k)} \to \mathbb{K}^{\mathcal{P}(\{k\})}$ *is a copy of the first order anchor. Thus, by definition,*

$$\Upsilon^{\{k\}}_{(t_k,s_k)}(e_\emptyset) = E^k_\emptyset + E^k_k, \qquad \Upsilon^{\{k\}}_{(t_k,s_k)}(e_k) = s_k E^k_\emptyset + t_k E^k_k.$$

Recall Formula (1.5) for the matrix of the second order anchor. Note that, when $s_1 = 1 = s_2$, then this matrix is a *symmetric matrix*, whereas for $t_1 = 1 = t_2$, this is not the case. Using notation introduced above, we generalize:

Theorem 2.24. *Fix* $n \in \mathbb{N}$, *and* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$. *With respect to the bases* $(e_A)_{A\in\mathcal{P}(\mathsf{n})}$ *in its domain and* $(E_A)_{A\in\mathcal{P}(\mathsf{n})}$ *in its range, the* n*-fold anchor is given by*

$$\Upsilon = \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = \sum_{(A,B)\in\mathcal{P}(\mathsf{n})^2} \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c}\; e_A^* \otimes E_B.$$

In other terms, it is characterized by the following equivalent conditions:

(1) $\Upsilon(e_A) = \sum_{B\in\mathcal{P}(\mathsf{n})} \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c} E_B$,
(2) $\Upsilon(\sum_{A\in\mathcal{P}(\mathsf{n})} v_A e_A) = \sum_{B\in\mathcal{P}(\mathsf{n})} \big(\sum_{A\in\mathcal{P}(\mathsf{n})} \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c} v_A\big) E_B$,
(3) *the matrix of* Υ *with respect to these bases has coefficients*

$$\Upsilon_{(B,A)} := E_B^*(\Upsilon(e_A)) = \mathbf{t}_{A\cap B}\mathbf{s}_{A\cap B^c}, \qquad (A,B) \in \mathcal{P}(\mathsf{n})^2.$$

In particular, in the symmetric case $\mathbf{s} = -\mathbf{t}$, *we have* $\Upsilon_{(B,A)} = (-1)^{|A\cap B|}\mathbf{s}_A$*:*

$$\Upsilon = \Upsilon^{\mathsf{n}}_{(-\mathbf{s},\mathbf{s})} = \sum_{A\in\mathcal{P}(\mathsf{n})} \mathbf{s}_A \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\cap B|} e_A^* \otimes E_B.$$

Proof. This is the special case of Theorem B.3 for $\mathbf{a} = 1 = \mathbf{c}$, $\mathbf{b} = \mathbf{s}$, $\mathbf{d} = \mathbf{t}$. □

Next, to compute the inverse of the anchor, in the regular case, recall Formula (1.7) concerning the case $n = 2$. This generalizes as follows:

Theorem 2.25. *Fix* $n \in \mathbb{N}$ *and* $(\mathbf{t},\mathbf{s}) \in \mathbb{K}^{2n}$. *Recall the notation* $(\mathbf{t}-\mathbf{s})_{\mathsf{n}} = \prod_{k=1}^{n}(t_k - s_k)$. *The anchor map* $\Upsilon = \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ *is invertible if, and only if,* $t_k - s_k$ *is invertible for all* $k = 1,\ldots,n$, *and then its inverse map is given by the formula*

$$\Upsilon^{-1} = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{(A,B)\in\mathcal{P}(\mathsf{n})^2} (-1)^{|A\Delta B|}\mathbf{s}_{A^c\cap B}\mathbf{t}_{B^c\cap A^c}\; E_A^* \otimes e_B.$$

Equivalently,

(1) $\Upsilon^{-1}(E_A) = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|}\mathbf{s}_{A^c\cap B}\mathbf{t}_{B^c\cap A^c}\; e_B$,
(2) $\Upsilon^{-1}(\sum_{A\in\mathcal{P}(\mathsf{n})} y_A E_A) = \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\Delta B|} y_A \mathbf{s}_{A^c\cap B}\mathbf{t}_{B^c\cap A^c}\; e_B$.

In particular, in case $\mathbf{s} = -\mathbf{t}$, *we get (using* $(A\Delta B) \sqcup (A^c \cap B^c) = (A\cap B)^c$*)*

$$\Upsilon^{-1}(E_A) = \frac{1}{(-2)^n \mathbf{s}_{\mathsf{n}}} \sum_{B\in\mathcal{P}(\mathsf{n})} (-1)^{|A\cap B|}\mathbf{s}_{B^c}\, e_B.$$

Proof. This is a special case of Theorem B.4. □

2.6. **The idempotent basis.** Just like in case $n=1$, we define, for regular $(\mathbf{t},\mathbf{s})$ the basis $(\tilde{E}_A)_{A\in\mathcal{P}(\mathsf{n})}$ of $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ by

$$\tilde{E}_A := \Upsilon^{-1}(E_A).$$

Thus, by definition, the matrix $M^E_{\tilde{E}}(\Upsilon)$ of Υ for these bases is the identity matrix, whereas $\underline{\Upsilon} := M^E_e(\Upsilon)$ is its "usual" matrix, computed in Theorem 2.24. It follows that the base change matrix from $\tilde{E}$ to e is given by

$$M^e_{\tilde{E}}(\mathrm{id}) = M^e_E(\Upsilon)\, M^E_{\tilde{E}}(\Upsilon^{-1}) = \underline{\Upsilon}\cdot 1 = \underline{\Upsilon}.$$

Thus the idempotent basis is given by base change with the coeffiencts of the matrix Υ^{-1} computed in Theorem 2.25.

Remark 2.7. For $A\in\mathcal{P}(\mathsf{n})$, the linear form $E^*_A : \mathbb{K}^{\mathcal{P}(\mathsf{n})}\to\mathbb{K}$ is the A-projection, which is a *character*, i.e., an algebra morphism into the base ring. Thus

$$\Upsilon_A := \tilde{E}^*_A = E^*_A\circ\Upsilon : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}\to\mathbb{K}$$

also is a character (generalising α,β from $n=1$). Seen this way, the anchor Υ can be considered as an analog of the Fourier transform for the algebra $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$.

2.7. **The n-th order restriced slope map.** Having established the explicit formulae for Υ and Υ^{-1}, we can prove the already anounced formula from Theorem 1.10 for $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = \Upsilon^{-1}\circ f^{\mathcal{P}(\mathsf{n})}\circ\Upsilon$ when $(\mathbf{t},\mathbf{s})$ is regular. We decompose $\mathbf{v}\in V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} = V\otimes_{\mathbb{K}}\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ in the form $\mathbf{v}=\sum_{A\in\mathcal{P}(\mathsf{n})} v_A e_A$, and $\Upsilon(\mathbf{v}) = \sum_{A\in\mathcal{P}(\mathsf{n})}\Upsilon_A(\mathbf{v})E_A$, with the 2^n *evaluation points* given by

$$\Upsilon_A(\mathbf{v}) = \sum_{C\in\mathcal{P}(\mathsf{n})} \mathbf{s}_{C\cap A^c}\mathbf{t}_{C\cap A} v_C.$$

Then

$$\begin{aligned}
f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}\Big(\sum_{A\in\mathcal{P}(\mathsf{n})} v_A e_A\Big) &= \Upsilon^{-1}\Big(\sum_{A\in\mathcal{P}(\mathsf{n})} f\big(\Upsilon_A(\mathbf{v})\big)\Big)\\
&= \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}}\sum_{B\in\mathcal{P}(\mathsf{n})} e_B\Big(\sum_{A\in\mathcal{P}(\mathsf{n})}(-1)^{|A\Delta B|}\mathbf{t}_{A^c\cap B^c}\mathbf{s}_{B^c\cap A} f\big(\Upsilon_A(\mathbf{v})\big)\Big)\\
&= \frac{1}{(\mathbf{t}-\mathbf{s})_{\mathsf{n}}}\sum_{B\in\mathcal{P}(\mathsf{n})} e_B\Big(\sum_{A\in\mathcal{P}(\mathsf{n})}(-1)^{|A\Delta B|}\mathbf{t}_{A^c\cap B^c}\mathbf{s}_{B^c\cap A} f\big(\sum_{C\in\mathcal{P}(\mathsf{n})}\mathbf{t}_{C\cap A}\mathbf{s}_{C\cap A^c} v_C\big)\Big).
\end{aligned}$$

2.8. **The category of anchored n-th order tangent algebras.** We generalize the concepts defined in 2.3 to the case of any order $n\in\mathbb{N}$:

Definition 2.26. *Objects of the category* $\mathbf{talg}_{\mathbb{K},n}$ *of* anchored n-th order $\mathbb{K}$-tangent algebras *are all anchor morphisms*

$$\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}\to\mathbb{K}^{\mathcal{P}(\mathsf{n})}$$

for $(\mathbf{t},\mathbf{s})\in\mathbb{K}^{2n}$*, along with the trivial anchor morphism* $\mathrm{id}:\mathbb{K}\to\mathbb{K}$.

Definition 2.27. *A pair of algebra morphisms*

$$\Phi : \mathbb{K}^{\mathsf{n}'}_{(\mathbf{t}',\mathbf{s}')} \to \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}, \qquad \Psi : \mathbb{K}^{\mathcal{P}(\mathsf{n}')} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})}$$

is called anchor-compatible *if*

$$\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \circ \Phi = \Psi \circ \Upsilon^{\mathsf{n}'}_{(\mathbf{t}',\mathbf{s}')} : \qquad \begin{array}{ccc} \mathbb{K}^{\mathsf{n}'}_{(\mathbf{t}',\mathbf{s}')} & \to & \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \\ \downarrow & & \downarrow \\ \mathbb{K}^{\mathcal{P}(\mathsf{n}')} & \to & \mathbb{K}^{\mathcal{P}(\mathsf{n})}. \end{array}$$

It is called anchor preserving *if* $n = n'$ *and* $\Psi = \mathrm{id}$.

Note that, when $(\mathbf{t}', \mathbf{s}')$ is regular, then this condition is automatically satisfied, by taking $\Psi = \Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \circ \Phi \circ (\Upsilon^{\mathsf{n}'}_{(\mathbf{t}',\mathbf{s}')})^{-1}$. From Lemma 2.12 we directly get, by taking tensor products, the following anchor preserving morphisms:

Definition 2.28. *Let* $\phi_i : \mathbb{K} \to \mathbb{K}$ *affine maps,* $i = 1, \ldots, n$, *and* $\phi = \phi_1 \times \ldots \times \phi_n : \mathbb{K}^n \to \mathbb{K}^n$. *Then* ϕ *induces an anchor-preserving morphism, called* of affine type*,* $\phi^* : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t}',\mathbf{s}')} \to \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$. *In particular, we say that* ϕ^* *is*

(1) of translation type *if* ϕ^* *is the tensor product of morphisms of translation type, parametrized by* $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{K}^n$, $t'_i = t_i + \mu$, $s'_i = s_i + \mu$,
(2) of scaling type *if* ϕ^* *is the tensor product of morphisms of scaling type, parametrized by* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n$, $t'_i = \lambda_i t_t$, $s'_i = \lambda_i s_i$.

Remark 2.8. When defined, the composition of morphisms of affine type is again of affine type. In more elaborated language, we get a small category which is the *action category associated to the affine monoid*.

On the most basic level of the theory, it would be enough to consider only anchor-preserving morphisms of affine type. However, certain other anchor-compatible morphisms will soon become relevant, which are the higher order analogs of the indirect morphisms defined for $n = 1$.

Definition 2.29. *We denote by* $\mathfrak{B}_n$ *the* hyperoctahedral group*, which is defined as follows: it is the group acting on* $\mathcal{P}(\mathsf{n})$ *generated by*

(1) the symmetric group $\mathfrak{S}_n$ *acting on* $\mathcal{P}(\mathsf{n})$ *in the natural way by* $\sigma.A = \sigma(A)$,
(2) the action of the abelian group $(\mathcal{P}(\mathsf{n}), \Delta) \cong (\mathbb{Z}/(2))^n$ *by translations on itself:* $B.A = A \Delta B$ *(symmetric difference).*

Remark 2.9. The group $\mathfrak{B}_n$ is a semidirect product of $\mathfrak{S}_n$ with the normal subgroup $(\mathcal{P}(\mathsf{n}), \Delta)$, whence is of cardinal $n!2^n$. It is a Coxeter group of type B_n. It has a non-trivial center: the element n acts by the *complement map* $A \mapsto \mathsf{n}\Delta A = A^c$, which belongs to the center of $\mathfrak{B}_n$.

Definition 2.30. *We call* admissible automorphism *of* $\mathbb{K}^{\mathcal{P}(\mathsf{n})}$ *the algebra automorphisms induced by elements of* $\tau \in \mathfrak{B}_n$, *via* $\Psi(E_A) = E_{\tau(A)}$. *In particular,*

(1) Ψ *is of* σ-permutation type *if* $\tau = \sigma$ *belongs to the subgroup* $\mathfrak{S}_n$ *of* $\mathfrak{B}_n$ *(in particular, we say* Ψ *is a* flip *if* τ *is a transposition),*
(2) Ψ *is of* B-inversion type *if* $\tau(A) = B \Delta A$ *belongs to the subgroup* $(\mathcal{P}(\mathsf{n}, \Delta))$ *of* $\mathfrak{B}^n$. *In particular, we say that* Ψ *is an* elementary inversion *if* B *is a singleton,* $B = \{i\}$, *and* Ψ *is the* central inversion*, if* $B = \mathsf{n}$.

We use the same terminology for the corresponding pairs of anchor-compatible morphisms, having the corresponding type as base-map Ψ*.*

Lemma 2.31. *The action of the hyperoctahedral group on* $\mathbb{K}^{\mathcal{P}(\mathsf{n})}$ *lifts to an action by anchor compatible isomorphisms on the objects of the category of anchored* n*-th order tangent algebras. More precisely, for every* $B \in \mathcal{P}(\mathsf{n})$*, there is an automorphism of* B*-inversion type* $\Phi : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$*, namely*

$$\kappa_B := \otimes_{k=1}^{n} \kappa_k^{|B \cap \{k\}|}$$

(tensor product of inversions for each $k \in B$*), and for every* $\sigma \in \mathfrak{S}_n$*, there is a morphism of* σ*-permutation type (with* $\mathbf{t}' = \sigma(\mathbf{t})$*,* $\mathbf{s}' = \sigma(\mathbf{s})$*), namely*

$$P_\sigma : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t}',\mathbf{s}')} \to \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}, \quad e_A \mapsto e_{\sigma(A)}.$$

Proof. Concerning inversions, this follows by taking tensor products of the first order inversions, as in the definition of κ_B.

The symmetric group $\mathfrak{S}_n$ acts canonically on $\mathbb{K}[X_1, \ldots, X_n]$. This action passes to the quotient and defines morphisms

$$\mathbb{K}[X_1, \ldots, X_n]/(X_i - t_{\sigma(i)})(X_i - s_{\sigma(i)}) \to \mathbb{K}[X_1, \ldots, X_n]/(X_i - t_i)(X_i - s_i). \qquad \square$$

Definition 2.32. Morphisms of the category of anchored n-th order $\mathbb{K}$-tangent algebras *are all morphisms of affine type (direct), as well as their compositions with all morphisms coming from the action of the hyperoctahedral group, along with the canonical injection of the trivial anchor morphism, given by the canonical maps* $\Phi : \mathbb{K} \to \mathbb{K}_{(t,s)}$*,* $x \mapsto x1$*,* $\Phi' : \mathbb{K} \to \mathbb{K}^2$*,* $x \mapsto x1$*. In particular, the* central inversion *belongs to the central element of* $\mathfrak{B}_n$ *(exchanging* a_i *and* b_i *for all* i*, and acting by the complement map on* $\mathbb{K}^{\mathcal{P}(\mathsf{n})}$*).*

The higher order analog of Proposition 2.15 holds: morphisms of scaling and translation type induce maps that are compatible with tangent maps $f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$:

Proposition 2.33. *Let* V *be a* $\mathbb{K}$*-module and* $\phi : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t}',\mathbf{s}')} \to \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ *be a morphism of affine type, or permutation type, or of inversion type. Then, for all open subsets* $U \subset V$*, the (linear) map* $\Phi = \mathrm{id} \otimes \phi : U_{(\mathbf{t}',\mathbf{s}')} \to U_{(\mathbf{t},\mathbf{s})}$ *is well-defined, and if* $f : U \to U'$ *is of class* $C_{\mathbb{K},n}$*, then*

$$\Phi \circ f^{\mathsf{n}}_{(\mathbf{t}',\mathbf{s}')} = f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \circ \Phi.$$

Proof. Concerning morphisms of affine type, and of inversion type, this follows by iteration from Proposition 2.15. Concerning morphisms of permutation type, for $n = 2$ and ϕ the "flip" map corresponding to the transposition (12), the claim amounts to "Schwarz's Theorem", which holds in topological calculus (see [BGN04, Be11]). By induction, the claim then follows for all morphisms of permutation type. $\square$

Remark 2.10. Let us call *diagonal case* the case where $t_i = t$ and $s_i = s$ for all $i = 1, \ldots, n$. In this case, the permutation group $\mathfrak{S}_n$ acts by automorphisms on $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$. This is in particular the case for $t = 0 = s$, giving the canonical action of $\mathfrak{S}_n$ on higher order tangent bundles, extensively used in [Be08].

3. Good functor categories

In this section, $\mathbb{K}$ is an abstract (i.e., without topology) unital commutative ring.

3.1. The category $\mathbf{talg}_{\mathbb{K}}$ of anchored $\mathbb{K}$-tangent algebras.

So far, we have defined for each $n \in \mathbb{N}$ a category of *n-th order anchored tangent algebras.* Now, we shall put them together to define a category comprising all of them. The following diagram may help to "visualize" the objects of the category:

$$\begin{array}{ccc} & 0 & \\ \swarrow & & \searrow \\ \mathbf{talg}_{\mathbb{K}}^{Ana} & \longrightarrow & \mathbf{talg}_{\mathbb{K}}^{Syn} \end{array}$$

On the left, we have the "analytic leg" (algebras of type $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$), on the right, the "synthetic leg" (algebras of type $\mathbb{K}^{\mathcal{P}(\mathsf{n})}$), and the arrow from left to right represents the anchor. The left hand side takes account of the "analytic" (local, chart dependent) aspects of calculus, and the right of "synthetic" aspects (independent of the language of charts and local affinizations).

Definition 3.1. *Objects of the category* $\mathbf{talg}_{\mathbb{K}}$ *are all objects of* $\mathbf{talg}_{\mathbb{K},n}$*, as n ranges over* $\mathbb{N}$*. Morphisms of* $\mathbf{talg}_{\mathbb{K}}$ *are:*

(1) *for fixed n, all morphisms of* $\mathbf{talg}_{\mathbb{K},n}$*,*
(2) *for two objects* $\Upsilon_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}'$, $\Upsilon_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}'$*, the natural morphism to* $\Upsilon_{\mathbb{A}} \otimes \Upsilon_{\mathbb{B}}$*,*

$$\mathbb{A} \to \mathbb{A} \otimes \mathbb{B},\ x \mapsto x \otimes 1, \quad \mathbb{A}' \to \mathbb{A}' \otimes \mathbb{B}',\ z \mapsto z \otimes 1.$$

Remark 3.1. The category $\mathbf{talg}_{\mathbb{K}}$ is stable under taking tensor products: if $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ and $\Upsilon^{\mathsf{k}}_{(\mathbf{t}',\mathbf{s}')}$ are objects, then $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \otimes \Upsilon^{\mathsf{k}}_{(\mathbf{t}',\mathbf{s}')}$ is naturally identified with the anchor at order $n+k$ given by parameters $(\mathbf{t},\mathbf{t}';\mathbf{s},\mathbf{s}') \in \mathbb{K}^{2(n+k)}$.

3.2. The good functor category.

Recall from Appendix C definition and some basic facts on functor categories.

Definition 3.2. *The functor category* $\mathbf{Fn}(\mathbf{talg}_{\mathbb{K},n}, \mathbf{Sets^2})$ *is called* the category of $(\mathbb{K}, n)$-space laws*: its objects are called* $(\mathbb{K}, n)$-space laws $\underline{M}$*, and its morphisms* $\underline{f}$ *are called* $(\mathbb{K}, n)$-mapping laws*. A* $(\mathbb{K}, \infty)$-space law *(or just:* $\mathbb{K}$*-space law) is a functor in* $\mathbf{Fn}(\mathbf{talg}_{\mathbb{K}}, \mathbf{Sets}^2)$*, and a* $(\mathbb{K}, \infty)$-mapping law *is a morphism in this category.*

We demand explicitly that the category of locally linear sets together with their (restrictions) of linear maps be considered as subcategory of this functor category in the way given by the following Example 3.1.

Example 3.1 (Locally linear sets). Let (U, V) be a locally linear set, and $\Upsilon : \mathbb{A} \to \mathbb{A}'$ an object of $\mathbf{talg}_{\mathbb{K}}$. We define $V_{\mathbb{A}} := V \otimes_{\mathbb{K}} \mathbb{A}$ (algebraic scalar extension), and

(1) for $\mathbb{A} = \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$, the set $U_{\mathbb{A}} := U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ is defined as in Definition 1.4,
(2) for $\mathbb{A}' = \mathbb{K}^{\mathcal{P}(\mathsf{n})}$, we let $U_{\mathbb{K}^{\mathcal{P}(\mathsf{n})}} := \times_{A \in \mathcal{P}(\mathsf{n})} U \subset V_{\mathbb{K}^{\mathcal{P}(\mathsf{n})}}$,
(3) the anchor $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to U^{\mathbb{K}^{\mathcal{P}(\mathsf{n})}}$ then is defined by the algebraic formula from Theorem 2.24.

We have seen in Subsection 2.8 that morphisms of tangent algebras induce maps on the level of sets. For a $\mathbb{K}$-linear map $f : V \to V'$ we define $f_{\mathbb{A}} = f \otimes \mathrm{id}_{\mathbb{A}}$ to be its algebraic scalar extension, restricted to $U_{\mathbb{A}}$. In this way, in virtue of Proposition 2.15, we get a functor $\underline{U}$ from $\mathbf{talg}_{\mathbb{K}}$ to $\mathbf{Sets^2}$, and a natural transformation $\underline{f}$ from $\underline{U}$ to $\underline{U}'$.

Remark 3.2. Note that the preceding example cannot be generalized to "arbitrary" functor categories from $\mathbb{K}$-algebras to sets, since for general algebras $\mathbb{A}$ there is no good definition of domain $U_{\mathbb{A}}$. Already for $\mathbb{K} = \mathbb{R}$ and $\mathbb{A} = \mathbb{C}$ this fails, in general (the "complexification" of a real set cannot be defined in a purely algebraic way).

Remark 3.3 (Category of extensions of a given domain). For any locally linear set (U, V), the collection

$$\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} : U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to U_{\mathbb{K}^{\mathcal{P}(\mathsf{n})}}$$

of all scalar extensions (for $n \in \mathbb{N}, (\mathbf{t}, \mathbf{s}) \in \mathbb{K}^{2n}$), is a category (image of the functor $\underline{U}$), with morphisms $\mathrm{id}_U \otimes \Psi$ where Ψ is a morphism of $\mathbf{talg}_{\mathbb{K}}$. Call it the *category of tangent extensions of* U.

Example 3.2 (The scaloid). In the preceding remark, take $(U, V) = (0, 0) =: 0$, the zero set in the zero-module. The category of tangent extensions of 0 is called the *scaloid* and denoted by $\mathbf{scal}_{\mathbb{K}}$: since $0 \otimes_{\mathbb{K}} \Upsilon = 0$, the underlying map of each object is a trivial map on a trivial $\mathbb{K}$-module 0; however, each of these maps and sets 0 still carries a label – for instance, $\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \otimes 0 = 0$ as set, but still keeping its label $(n; \mathbf{t}, \mathbf{s})$. Therefore the scaloid $\mathbf{scal}_{\mathbb{K}}$ is the small category of labels belonging to objects and morphisms of $\mathbf{talg}_{\mathbb{K}}$, and the functor $\underline{0}$ can be seen as the forgetful functor

$$* : \mathbf{talg}_{\mathbb{K}} \to \mathbf{scal}_{\mathbb{K}}$$

associating to each object $\Upsilon^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ from $\mathbf{talg}_{\mathbb{K}}$ its label $(\mathbf{t}, \mathbf{s}; n)$, and to each morphism a label characterizing it. Note also that $\underline{0}$ is a terminal object in the category of $C_{\mathbb{K},\infty}$-laws: every law $\underline{M}$ admits a unique morphism $\underline{f} : \underline{M} \to \underline{0}$, attaching to $M_{\mathbb{A}}$ the label of $\mathbb{A}$.

Example 3.3 (The line). Next, consider the case $U = \mathbb{K}$ (the line). Then, since $\mathbb{A} \otimes \mathbb{K} = \mathbb{A}$, $\mathrm{id} \otimes \Upsilon = \Upsilon$, the category of tangent extensions of $\mathbb{K}$ gives us back the category $\mathbf{talg}_{\mathbb{K}}$. Since $\mathbb{K}$ carries an associative bilinear product, so do the objects of $\mathbf{talg}_{\mathbb{K}}$.

Example 3.4 (Scalar function laws). A *scalar law* is a natural transformation $\underline{f} : \underline{M} \to \underline{\mathbb{K}}$. Such laws can be turned into a ring $\mathcal{O}_{\underline{M}}$ by letting

$$(\underline{f} + \underline{g})_{\mathbb{A}} := f_{\mathbb{A}} + g_{\mathbb{A}}, \qquad (\underline{f} \cdot \underline{g})_{\mathbb{A}} := f_{\mathbb{A}} \cdot g_{\mathbb{A}}.$$

Thus $\underline{M}$ gives rise to a functor $\underline{\mathcal{O}_M} : \mathbb{A} \mapsto (\mathcal{O}_{\underline{M}})_{\mathbb{A}}$, which is *contravariant*, that is, a functor from $(\mathbf{talg}_{\mathbb{K}})^{\mathrm{opp}}$ to $\mathbf{Sets^2}$ (and to $\mathbf{Ring^2}$, the category of anchored rings).

3.3. **Full imbedding of topological calculus into functor categories.** Let's return to the topological case, and recall from Definition 1.11 the category $\mathbf{Llin}_{\mathbb{K},n}$ of locally linear sets with $C_{\mathbb{K},n}$-maps as morphisms.

Theorem 3.3. *Let $\mathbb{K}$ be a good topological ring. There is an imbedding of the category $\mathbf{Llin}_{\mathbb{K},n}$ into the functor categories $\mathbf{Fn}(\mathbf{talg}_{\mathbb{K},n}, \mathbf{Sets^2})$, resp. $\mathbf{Fn}(\mathbf{talg}_{\mathbb{K}}, \mathbf{Sets^2})$, given by associating to an object (U,V) the functor $\underline{U}$ defined in Example 3.1, and to a $C_{\mathbb{K},n}$-map $f: U \to U'$ the natural transformation*

$$\underline{f}: \qquad f_{\mathbb{K}} = f, \quad f_{\mathbb{K}^{\mathcal{P}(\mathsf{n})}} = f^{\mathcal{P}(\mathsf{n})}, \quad f_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} = f^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}.$$

This imbedding has the following properties:

(1) functors $\underline{U}$ respect direct products,
(2) for locally linear spaces and linear maps, functors are given by algebraic scalar extension, as in Example 3.1,
(3) functors take values in topological spaces, morphisms are continuous and jointly continuous with respect to parameters.

Proof. First of all, $\underline{U}$ is indeed a functor: all morphisms between tangent algebras belonging to our explicit list do indeed induce set-maps, in a functorial way. Next, $C_{\mathbb{K},n}$-maps indeed induce natural transformations (for this issue, it was necessary to limit morphisms between algebras to an explicit list: there is no general theorem ensuring that these natural transformations commute with maps induced by "arbitrary" algebra morphisms). Finally, the underlying functor gives us back the original objects and morphisms, $U_{\mathbb{K}} = U, f_{\mathbb{K}} = f$, so we have indeed an imbedding. Properties (1), (2), (3), which we re-state formally in the following definition, are clearly satisfied by the constructions that have been described in the preceding sections. □

Definition 3.4. *Let $\mathbb{K}$ be a good topological ring. A functor category $\mathbf{Fn}(\mathbf{c}_{\mathbb{K}}, \mathbf{Sets^2})$ may have (or not) the following properties:*

(1) it respects direct products (cf. Definition C.3),
(2) it satisfies the axiom of algebraic scalar extension *stated in (2) above,*
(3) it is a continuous functor category*: the scaloid $\mathbf{scal}_{\mathbb{K}}$ (cf. example 3.2) is a* topological space*, functors $\underline{M}$ take values in* topological spaces*, morphisms $\underline{f}$ take values in* continuous *set-maps $f_{\mathbb{A}}$ that are* jointly continuous in the scaloid*, i.e.: for all locally linear sets (U,V) and morphisms $\underline{f}$, the following map is continuous (where $V^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \cong V^{2^n}$ via the e-basis, and likewise for W^{2^n}):*

$$\mathbb{K}^{2n} \times V^{2^n} \supset \{(\mathbf{t},\mathbf{s};\mathbf{v}) \mid \mathbf{v} \in U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}\} \to W^{2^n}, \quad (\mathbf{t},\mathbf{s};\mathbf{v}) \mapsto f_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}}(\mathbf{v}).$$

We denote by a superscript "top" the functor categories satisfying (3).

Theorem 3.5. *The imbedding from Theorem 3.3 defines a full imbedding of the category $\mathbf{Llin}_{\mathbb{K},n}$ into $\mathbf{Fn}^{top}(\mathbf{talg}_{\mathbb{K},n}, \mathbf{Sets^2})$ and a full imbedding of the category $\mathbf{Llin}_{\mathbb{K},\infty}$ into $\mathbf{Fn}^{top}(\mathbf{talg}_{\mathbb{K}}, \mathbf{Sets^2})$.*

Proof. Let $\underline{f}: \underline{U} \to \underline{U}'$ be a continuous morphism of laws. We have to show that $\underline{f}$ is induced by a map of class $C_{\mathbb{K},n}$; more precisely, we show that the underlying map $f = f_{\mathbb{K}} : U_{\mathbb{K}} = U \to U' = (U')_{\mathbb{K}}$ is of class $C_{\mathbb{K},n}$, and that it induces $\underline{f}$. By definition of the functor category, the anchor $\Upsilon : \mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})}$ induces a map $\Upsilon_U : U_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} \to U_{\mathbb{K}^{\mathcal{P}(\mathsf{n})}}$. This anchor map is indeed the same as the one considered in the preceding chapters, since Υ_U is the restriction to $U_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}}$ of $\mathrm{id}_V \otimes \Upsilon_{\mathbb{K}}$. Since $\underline{f}$ is a natural transformation, it commutes with the anchor in the sense that

$$\Upsilon \circ f_{\mathbb{K}^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}} = f_{\mathbb{K}^{\mathcal{P}(\mathsf{n})}} \circ \Upsilon .$$

By the continuity property (3) from Definition 3.4, these maps are continuous and jointly continuous also in $(\mathbf{t}, \mathbf{s})$, whence satisfy the condition from Theorem 1.7, showing that the base map $f = f_{\mathbb{K}}$ is of class $C_{\mathbb{K},n}$, with the components of $\underline{f}$ given the construction from topological differential calculus; thus $f_{\mathbb{K}}$ induces the natural transformation $\underline{f}$. □

Remark 3.4 (Manifold laws). As already mentioned (Remark 1.6), one can define $C_{\mathbb{K},n}$*-manifolds over a good topological ring* $\mathbb{K}$ ($n \in \mathbb{N} \cup \{\infty\}$). Then every such manifold defines a $(\mathbb{K}, n)$-space law $\underline{M}$, and every $C_{\mathbb{K},n}$-map $f : M \to M'$ a $(\mathbb{K}, n)$-law $\underline{f}$, such that properties (1), (2), (3) from Definition 3.4 are satisfied (where (3) applies, by definition, only to locally linear sets, that is, to chart domains of the manifold). Then the preceding arguments apply on each chart domain, showing that morphisms between manifold laws are induced by smooth maps, and hence we get a full imbedding of the category of $C_{\mathbb{K},n}$-manifolds into $\mathbf{Fn}^{top}(\mathbf{talg}_{\mathbb{K},n}, \mathbf{Sets^2})$.

Remark 3.5 (Infinitesimal vs. local and global). A remark on comparison with the case of *Weil laws* as defined in [Be14] is in order here. Taking for $\mathbf{c}_{\mathbb{K}}$ the category of *Weil algebras*, we get a formally quite similar theory, leading to an imbedding of topological differential calculus into the *category of Weil spaces and their morphisms*, which, essentially, is the so-called the Dubuc topos, [Du79]. As shown by Dubuc, loc. cit., the Dubuc topos contains morphisms that are *not* induced by smooth maps. Roughly speaking, the reason for this is that Weil algebras are by nature *infinitesimal objects* (because of the nilpotency condition), and the link with the local and global theory is not encoded on the algebra side, whereas in our approach it is. To ensure fullness, this link has to be encoded in some way – either by using more sophisticated topoi (see [MR91]), or (as done here) by taking account of anchors and allowing regular parameters $(\mathbf{t}, \mathbf{s})$, thus taking account of non-infinitesimal mathematics.

4. Towards higher (super) algebra

With Theorem 3.5, we have shown that the functor category $\mathbf{Fn}(\mathbf{talg}_{\mathbb{K}}, \mathbf{Sets^2})$ can be considered as a "well adapted model" for general differential calculus. In subsequent work, we will develop the theory further: on the one hand, comparing with SDG, we will investigate categorical questions, on the other hand, by enriching the structure of our category of algebras, the theory naturally offers links with *higher algebra* and with *super-calculus*. Let us briefly describe some basic ingredients.

4.1. **Groupoids, and higher algebra.** In topological calculus, the extended domains $U^{\mathsf{n}}_{(\mathbf{t},\mathbf{s})}$ carry a natural structure of *n-fold groupoid* (see [Be15a, Be15b, Be17], for the case of target calculus). Indeed, this follows by iteration from

Lemma 4.1. *Let (U,V) be a linear set and $(t,s) \in \mathbb{K}^2$. Then $U^{\{1\}}_{(t,s)}$ is a* groupoid, *with target and source projections*

$$\alpha : U^{\{1\}}_{(t,s)} \to U, \quad (v_0, v_1) \mapsto v_0 + sv_1,$$
$$\beta : U^{\{1\}}_{(t,s)} \to U, \quad (v_0, v_1) \mapsto v_0 + tv_1,$$

unit section $1(v_0) = (v_0, 0) = v_0 1$, inversion κ, and groupoid multiplication

$$\forall (u,w) \in U^{\{1\}}_{(t,s)} \times_{\alpha,\beta} U^{\{1\}}_{(t,s)} : \qquad u * w = u - \alpha(u)1 + w.$$

Proof. The defining properties of a groupoid are checked by direct computation (cf. loc. cit.). □

In order to implement this aspect in functorial calculus, we observe that the structure maps of this groupoid are induced by algebra morphisms:

Theorem 4.2. $(\mathbb{K}_{(t,s)}, \alpha, \beta, \mathbb{K}, *)$ *is a* groupoid object in the category of algebras, *i.e., the groupoid structure from the preceding lemma is given by algebras and algebra morphisms: source and target projections and unit section are algebra morphisms, the subset*

$$\mathbb{K}_{(t,s)} \times_{\alpha,\beta} \mathbb{K}_{(t,s)} = \{(u,v) \in \mathbb{K}_{(t,s)} \times \mathbb{K}_{(t,s)} \mid \alpha(u) = \beta(v)\}$$

is a subalgebra of the direct product algebra $\mathbb{K}_{(t,s)} \times \mathbb{K}_{(t,s)}$, the groupoid law

$$* : \mathbb{K}_{(t,s)} \times_{\alpha,\beta} \mathbb{K}_{(t,s)} \to \mathbb{K}_{(t,s)}, \quad (u,v) \mapsto u * v = u - \alpha(u) + v$$

is a morphism of algebras, and inversion J is an algebra automorphism.

Proof. Source α and target β are algebra morphisms, hence the subset $\mathbb{K}_{(t,s)} \times_{\alpha,\beta} \mathbb{K}_{(t,s)}$ of $\mathbb{K}_{(t,s)} \times \mathbb{K}_{(t,s)}$ is stable under addition $+$ and product $\cdot$, and it contains the unit $(1,1)$, so it is a (unital) subalgebra. To see that $*$ is a morphism, we use the "fundamental relation": for all $v, w \in \mathbb{K}_{(t,s)}$,

$$0 = (v - \alpha(v))(w - \beta(w)) = vw - \alpha(v)w - \beta(w)v + \alpha(v)\beta(w).$$

Thus if $\alpha(x) = \beta(z)$ and $\alpha(x') = \beta(z')$,

$$\begin{aligned}
(x * z) \cdot (x' * z') &= (x - \alpha(x) + z) \cdot (x' - \alpha(x') + z') \\
&= ((x - \alpha(x)) + z) \cdot (x' + (z' - \beta(z'))) \\
&= xx' - \alpha(x)x' + zx' + zz' - z\beta(z') \\
&= xx' - \alpha(x)\alpha(x') + zz' + (zx' - \beta(z)x' + \alpha(x)\alpha(x') - z\alpha(x')) \\
&= (xx') * (zz') + (z - \beta(z))x' + (\beta(z) - z)\alpha(x') \\
&= (xx') * (zz') + (z - \beta(z)) \cdot (x' - \alpha(x')) \\
&= (xx') * (zz').
\end{aligned}$$

Finally, we have already seen that κ is an algebra automorphism (Theorem 2.5). □

We will discuss in subsequent work how to implement this groupoid aspect into a functorial calculus. In this context, it will be important to have an analog of the sequence $TTM \to TM \times_M TM \to TM$ playing an important rôle for second and higher order tangent bundles (see [Be08, Be14]). This analog reads as follows:

Theorem 4.3. *The product map*

$$\mu : \mathbb{K}_{(t,s)} \otimes \mathbb{K}_{(t,s)} \to \mathbb{K}_{((t,s)}, \quad u \otimes v \mapsto uv$$

is an algebra morphism, which factorizes over the groupoid law from the preceding theorem via $\mu = * \circ \psi$, where ψ is the morphism

$$\psi = \mathrm{id} \otimes \beta + \alpha \otimes \mathrm{id} : \mathbb{K}_{(t,s)} \otimes \mathbb{K}_{(t,s)} \to \mathbb{K}_{(t,s)} \oplus_{\alpha,\beta} \mathbb{K}_{(t,s)}, \\ u \otimes v \mapsto (\beta(v)u, \alpha(u)v).$$

Proof. For any commutative algebra $\mathbb{A}$, the product map $\mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ is a morphism. Since $\alpha(u \cdot \beta(v)) = \alpha(u)\beta(v) = \beta(v \cdot \alpha(u))$, the morphism $\mathrm{id} \otimes \beta + \alpha \otimes \mathrm{id}$ takes indeed values in $\mathbb{K}_{(t,s)} \oplus_{\alpha,\beta} \mathbb{K}_{(t,s)}$. Let $u, v \in \mathbb{K}_{(t,s)}$. Then

$$* \circ (\alpha \otimes \mathrm{id} + \mathrm{id} \otimes \beta)(u \otimes v) = \alpha(u)v * \beta(v)u \\ = \alpha(u)v - \alpha(u)\beta(v)1 + \beta(v)u = u \cdot v$$

by the "fundamental relation" from Theorem 2.5. □

Iterating these morphisms and constructions, *higher algebra* naturally enters into the picture (cf. loc. cit).

4.2. **Functorial supercalculus.** A main motivation to develop the present functorial approach is that it is perfectly in keeping with known functorial approaches to *super*-calculus. For this purpose, it was necessary to introduce the source parameter $\mathbf{s}$ into differential calculus: the transition from usual to graded calculus is most natural in case $\mathbf{t} = -\mathbf{s}$ (*symmetric calculus*), since only in this case the groupoid inversion κ (which becomes the grading automorphism of superalgebras) is given by the simple formula $\kappa(v_0 + ev_1) = v_0 - ev_1$ (cf. Theorem 2.5). In this context, tensor products have to be replaced by *graded tensor products*.

4.3. **Coalgebras, and duality.** The reader may have noticed that our algebras carry more structure than has been used so far: they are *coalgebras*, and this structure should play a significant rôle in the theory, related to *duality* aspects, and, possibly, to the preceding item.

4.4. **Full iteration, and simplicial calculus.** As mentioned in Remark 1.3, *full iteration* leads to higher order "tangent maps" $f^{\{1,\ldots,n\}}$ having a very complicated structure. In principle, this structure can also be interpreted in terms of higher groupoids (see [Be15b]). In this setting, the analog of the tangent algebra category $\mathbf{talg}_{\mathbb{K}}$ will be some small higher order category, whose structure remains to be understood yet. However, restricting again variables to certain subspaces, one can obtain a suffiently simple calculus, called *simplicial* in [Be13], and corresponding to the classical concept of *divided differences*. It is certainly possible to put this simplicial calculus into a categorical form, essentially as done in this work for restricted iteration. The advantage should be a better compatibility of calculus with algebra in *positive* characteristic, but the drawback is that the close link with the tensor product, featured in the present approach, gets lost: iteration is no longer given by subsequent tensor products.

Appendix A. Algebraic scalar extensions

For convenience, we recall some basic facts concerning linear algebra over rings (cf. e.g., [Bou], see also on the n-lab, or here or there.)

Definition A.1. *Let $\mathbb{K}$ be a commutative ring, $\mathbb{A}$ an associative unital $\mathbb{K}$-algebra, and V a $\mathbb{K}$-module. Then we denote by $V_{\mathbb{A}}$ the $\mathbb{K}$-module $V \otimes_{\mathbb{K}} \mathbb{A}$, together with its right $\mathbb{A}$-module structure given by*

$$(v \otimes \lambda).\mu = v \otimes \lambda\mu.$$

If $f : V \to W$ is a $\mathbb{K}$-linear map, we define the $\mathbb{A}$-linear map

$$f_{\mathbb{A}} := f \otimes \mathrm{id}_{\mathbb{A}} : V_{\mathbb{A}} \to W_{\mathbb{A}}, \quad v \otimes \lambda \mapsto f(v) \otimes \lambda,$$

and if $\phi : \mathbb{A} \to \mathbb{B}$ is an algebra morphism, we define the $\mathbb{K}$-linear map

$$V_\phi := \mathrm{id}_V \otimes \phi : V_{\mathbb{A}} \to V_{\mathbb{B}}, \quad v \otimes \lambda \mapsto v \otimes \phi(\lambda).$$

Clearly, $f_{\mathbb{B}} \circ V_\phi = W_\phi \circ f_{\mathbb{A}} : V_{\mathbb{A}} \to W_{\mathbb{B}}$. This relation can be interpreted in terms of *functor categories*, see Appendix C: $\underline{V}$ is a functor from $\mathbf{Alg}_{\mathbb{K}}$ to **Sets**, and linear maps define natural transformations between such functors.

Remark A.1. More precisely, $V_{\mathbb{A}}$ is in fact an $\mathbb{A}$-*bimodule*: it is also a left $\mathbb{A}$-module, with left action given by $\mu.(v \otimes \lambda) = v \otimes \mu\lambda$, such that left and right $\mathbb{A}$-actions commute. Then $f_{\mathbb{A}}$ is a morphism of $\mathbb{A}$-bimodules.

For all $\mathbb{K}$-modules V, there is a canonical $\mathbb{K}$-linear map

$$\iota_V : V \to V_{\mathbb{A}}, \quad v \mapsto v \otimes 1, \tag{A.1}$$

and for all $\mathbb{A}$-(right) modules V, there is also a canonical $\mathbb{A}$-linear map

$$p_V : V_{\mathbb{A}} \to V, \quad v \otimes \lambda \mapsto v\lambda. \tag{A.2}$$

If V is an $\mathbb{A}$-module, then we write $V_{\mathbb{K}} = V$ as set and abelian group, but with scalar action by $\mathbb{K}$ only ("scalar restriction"). The following lemma formalizes that scalar restriction and scalar extension are *adjoint functors*:

Lemma A.2. *Assume V is a right $\mathbb{A}$-module and W a $\mathbb{K}$-module. Then the following two maps are mutually inverse $\mathbb{K}$-linear isomorphisms:*

$$\mathrm{Hom}_{\mathbb{K}}(W, V_{\mathbb{K}}) \to \mathrm{Hom}_{\mathbb{A}}(W_{\mathbb{A}}, V), \quad f \mapsto F = p_V \circ f_{\mathbb{A}},$$
$$\mathrm{Hom}_{\mathbb{A}}(W_{\mathbb{A}}, V) \to \mathrm{Hom}_{\mathbb{K}}(W, V_{\mathbb{K}}), \quad F \mapsto f = F \circ \iota_W.$$

Replacing V by $V_{\mathbb{A}}$ for some $\mathbb{K}$-module V, we get a canonical isomorphism

$$\boxed{\mathrm{Hom}_{\mathbb{K}}(W, (V_{\mathbb{A}})_{\mathbb{K}}) = \mathrm{Hom}_{\mathbb{A}}(W_{\mathbb{A}}, V_{\mathbb{A}})}.$$

Proof. We have $(p_V \circ f_{\mathbb{A}}) \circ \iota_W(w) = f(w)1 = f(w)$. Concerning the other direction,

$$p_V \circ (F \circ \iota_W)_{\mathbb{A}} = p_V \circ ((F \circ \iota_W) \otimes \mathrm{id}_{\mathbb{A}}),$$

whence, for $w \otimes \lambda \in W_{\mathbb{A}}$, and using $\mathbb{A}$-linearity of F,

$$\begin{aligned} p_V \circ (F \circ \iota_W)_{\mathbb{A}}(w \otimes \lambda) &= p_V(F(\iota_W(w)) \otimes \lambda) \\ &= F(w \otimes 1) \cdot \lambda = F((w \otimes 1)\lambda) = F(w \otimes \lambda), \end{aligned}$$

whence $p_V \circ (F \circ \iota_W)_{\mathbb{A}} = F$, as claimed. □

Corollary A.3. *Let V, W be $\mathbb{K}$-modules, and let $X_{\mathbb{A}} := \mathrm{Hom}_{\mathbb{A}}(W_{\mathbb{A}}, V_{\mathbb{A}})$ for a $\mathbb{K}$-algebra $\mathbb{A}$. Then $\mathbb{A} \mapsto X_{\mathbb{A}}$ is functorial in the sense that, for every algebra morphism $\phi : \mathbb{A} \to \mathbb{B}$, there is a canonical morphism $X_\phi : X_{\mathbb{A}} \to X_{\mathbb{B}}$.*

Proof. Identifying $\mathrm{Hom}_{\mathbb{K}}(W, (V_{\mathbb{A}})_{\mathbb{K}}) = \mathrm{Hom}_{\mathbb{A}}(W_{\mathbb{A}}, V_{\mathbb{A}})$, we let $X_\phi(h) := V_\phi \circ h = (\mathrm{id}_V \otimes \phi) \circ h$. Then $X_{\psi\phi}(h) = V_{\psi\phi} \circ h = V_\psi \circ V_\phi \circ h = X_\psi(X_\phi(h))$. □

Proposition A.4. *Assume $\mathbb{A}$ is commutative. Then, for all $\mathbb{K}$-modules V, W, we have a canonical isomorphism*

$$(V \otimes_{\mathbb{K}} W)_{\mathbb{A}} = V_{\mathbb{A}} \otimes_{\mathbb{A}} W_{\mathbb{A}},$$

and likewise for iterated finite tensor products $((\otimes_{\mathbb{K}})_{i=1}^m V_i)_{\mathbb{A}} = (\otimes_{\mathbb{A}})_{i=1}^m (V_i)_{\mathbb{A}}$.

Proof. In one direction, the morphism is given by

$$(v \otimes w) \otimes \lambda \mapsto ((v \otimes 1) \otimes (w \otimes 1)) \cdot \lambda = (v \otimes \lambda) \otimes (w \otimes 1) = (v \otimes 1) \otimes (v \otimes \lambda),$$

in the other, using commutativity of $\mathbb{A}$,

$$(v \otimes \lambda) \otimes (w \otimes \mu) \mapsto (v \otimes w) \otimes (\lambda\mu).$$

Clearly, both are inverse to each other. □

Concerning *polynomials* and *forms*, see Appendix A of [BGN04]. Polynomials do admit scalar extensions (see also [Ro63, Lo75]):

Definition A.5. *Let V, W be $\mathbb{K}$-modules and $m \geq 1$. A* polynomial (with domain V and codomain W) *is given a an m-times multilinear map $B : V^m \to W$, giving rise to a map $f(x) := B(x, \ldots, x)$. By definition, a* polynomial of degree 0 *is a non-zero constant $c \in W$. The zero constant polynomial has degree $-\infty$.*

Since B is not required to be symmetric, f is not uniquely determined by B.

Theorem A.6. *Let B as in the preceding definition, of degree m, and $\mathbb{A}$ a commutative $\mathbb{K}$-algebra. Then B admits a scalar extension $B_{\mathbb{A}}$.*

Proof. For $m = 0$, let $f_{\mathbb{A}}(x) = f(x) \otimes 1$ (constant map). For $m = 1$, take the scalar extension of the linear map f. Consider the case $m = 2$. Then $B : V \times V \to W$ gives rise to a $\mathbb{K}$-linear $\underline{B} : V \otimes_{\mathbb{K}} V \to W$, and this gives an $\mathbb{A}$-linear $\underline{B}_{\mathbb{A}} : (V \otimes_{\mathbb{K}} V)_{\mathbb{A}} \to W_{\mathbb{A}}$. By the preceding proposition, this map can be considered as $\mathbb{A}$-linear

$$\underline{B}_{\mathbb{A}} : V_{\mathbb{A}} \otimes_{\mathbb{A}} V_{\mathbb{A}} \to W_{\mathbb{A}},$$

which is a polynomial over $\mathbb{A}$ extending B. For general m, the claim follows by induction. □

Remark A.2. We may define, as usual: *a* polynomial map *between V and W is a finite sum of homogeneous polynomial maps.* Then the theorem can be stated for polynomial maps; however, we have to mention explicitly the degrees of the components, and require that the functor $B \mapsto B_{\mathbb{A}}$ be applied in each degree.

Appendix B. Hypercubic linear algebra

In this appendix, "linear spaces" are modules over a commutative ring $\mathbb{K}$.

Definition B.1. *A* hypercubic $\mathbb{K}$-linear space, based on $\mathcal{P}(N)$, *is a free $\mathbb{K}$-module V together with a basis $(E_A)_{A\in\mathcal{P}(N)}$ indexed by the hypercube $\mathcal{P}(N)$ (power set of N, where $N\subset\mathbb{N}$ is a set, supposed to be finite in this appendix, and of cardinality n. If necessary, we write E^N_A instead of E_A.) In other words, $V\cong\mathbb{K}^{\mathcal{P}(N)}$ together with its canonical basis. When $N=\mathsf{n}=\{1,\ldots,n\}$, then we just speak of a n*-hypercubic space, *or of a 2^n*-space. *For $n=1$, $N=\{k\}$, the two basis elements are also denoted by $(E_\emptyset,E_i)$ or $(E^i_\emptyset,E^i_i)$.*

Remark B.1. A hypercubic space carries *several* algebra structures: for instance, it is a Clifford algebra, or an exterior algebra, or a commutative algebra, in rather canonical ways. However, in this appendix, we forget about such algebra structures. Therefore, everything will apply to the underlying $\mathbb{K}$-module of any such algebra.

Lemma B.2. *When N and M are disjoint finite subsets of $\mathbb{N}$, then there is a canonical isomorphism of hypercubic spaces,*

$$\mathbb{K}^{\mathcal{P}(N)}\otimes\mathbb{K}^{\mathcal{P}(M)}\to\mathbb{K}^{\mathcal{P}(N\sqcup M)},\quad E^n_A\otimes E^M_B\mapsto E^{N\sqcup M}_{A\sqcup B}.$$

In particular, when $N=\{k_1,\ldots,k_n\}$, then there is a canonical isomorphism

$$\bigotimes_{i=1}^n\mathbb{K}^{\mathcal{P}(\{k_i\})}\to\mathbb{K}^{\mathcal{P}(N)},\quad \otimes_{i=1}^n E^{\{k_i\}}_{\{k_i\}\cap A}\mapsto E^N_A.$$

Proof. See proof of Lemma 2.22. □

When $f:V\to W$ is linear, for bases $(b_j)_{j\in J}$ in V and $(c_i)_{i\in I}$ in W, we denote by $f_{i,j}:=c^*_i(f(b_j))$ its *matrix coefficients* (where $(c^*_i)_{i\in I}$ is the dual basis of c). We write also $(\phi\otimes v)(x)=\phi(x)\cdot v$. Then

$$f=\sum_{(i,j)\in I\times J}f_{i,j}\,b^*_j\otimes c_i,\qquad f(b_k)=\sum_i f_{i,k}c_k.$$

Writing a matrix in the usual way as rectangular number array, we use the natural *total order* on the index set – that is, the *lexicographic order*; for instance,

$$\mathcal{P}(\{1,2\})=(\emptyset,\{1\},\{2\},\{1,2\}).$$

In the following, for an n-tuple $\mathbf{a}=(a_i)_{i\in N}\in\mathbb{K}^n$, we use the notation $\mathbf{a}_N:=\prod_{i\in N}a_i$, in the same way as we do for $\mathbf{t},\mathbf{s}\in\mathbb{K}^n$ in the main text. When N is considered to be fixed, and $A\subset N$, we denote by $A^c=N\setminus A$ its complement.

Theorem B.3. *Let $N=\{k_1,\ldots,k_n\}$ and $f_i:\mathbb{K}^{\mathcal{P}(\{k_i\})}\to\mathbb{K}^{\mathcal{P}(\{k_i\})}$ linear, with matrix*

$$f_i=\begin{pmatrix}a_i & b_i\\ c_i & d_i\end{pmatrix}:\qquad E^i_\emptyset\mapsto a_iE^i_\emptyset+c_iE^i_i,\quad E^i_i\mapsto b_iE^i_\emptyset+d_iE^i_i.$$

Then the matrix of the linear map $f:=\otimes_{i=1}^n f_i:\mathbb{K}^{\mathcal{P}(N)}\to\mathbb{K}^{\mathcal{P}(N)}$ is given by the matrix coefficients, for $(A,B)\in\mathcal{P}(N)^2$,

$$f_{A,B}=E^*_A\big(f(E_B)\big)=\mathbf{a}_{A^c\cap B^c}\cdot\mathbf{b}_{A^c\cap B}\cdot\mathbf{c}_{A\cap B^c}\cdot\mathbf{d}_{A\cap B}.$$

In other terms, $f(E_B^N) = \sum_{A\in\mathcal{P}(N)} \mathbf{a}_{A^c\cap B^c}\cdot \mathbf{b}_{A^c\cap B}\cdot \mathbf{c}_{A\cap B^c}\cdot \mathbf{d}_{A\cap B}\ E_A^N$, *or*

$$f = \sum_{(A,B)\in\mathcal{P}(N)^2} \mathbf{a}_{A^c\cap B^c}\cdot \mathbf{b}_{A^c\cap B}\cdot \mathbf{c}_{A\cap B^c}\cdot \mathbf{d}_{A\cap B}\ (E_B^N)^*\otimes E_A^N.$$

Proof. When the cardinality n of N is equal to one, then the claim is true, directly by definition of the matrix coefficients. For $n=2$, the matrix of $f_1\otimes f_2$ is

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1\end{pmatrix}\otimes\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2\end{pmatrix} = \begin{pmatrix} a_1a_2 & b_1a_2 & a_1b_2 & b_1b_2 \\ c_1a_2 & d_1a_2 & c_1b_2 & d_1b_2 \\ a_1c_2 & b_1c_2 & a_1d_2 & b_1d_2 \\ c_1c_2 & d_1c_2 & c_1d_2 & d_1d_2\end{pmatrix}$$

("Kronecker product"). For instance, when $B=\emptyset$, so $B^c=\{1,2\}$,

$$f(E_\emptyset^{\{12\}}) = a_{12}E_\emptyset + c_1a_2E_1 + a_1c_2E_2 + c_{12}E_{12},$$

in keeping with the claim. In the general case, we expand the expression

$$f = \otimes_i f_i = \otimes_i\Big(a_i(E_\emptyset^i)^*\otimes E_\emptyset^i + b_i(E_\emptyset^i)^*\otimes E_i^i + c_i(E_i^i)^*\otimes E_\emptyset^i + d_i(E_i^i)^*\otimes E_i^i\Big)$$

by distributivity: we get a sum of 4^n terms, which correspond exactly to the 4^n terms in the last formula of the claim. (E.g., for $n=2$, there are 16 terms, corresponding to expanding the product $(a_1+b_1+c_1+d_1)(a_2+b_2+c_2+d_2)$ by distributivity, giving the 16 matrix coefficients shown above. The first column contains the 4 terms from expanding $(a_1+c_1)(a_2+c_2)$, etc.) □

To memorise the formula: for 2×2-matrices and indices, the correspondence is

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d}\end{pmatrix} \quad : \quad \begin{pmatrix} A^c\cap B^c & A^c\cap B \\ A\cap B^c & A\cap B\end{pmatrix}.$$

Next, we give a formula for the *inverse* of f, when its determinant is invertible. From well-known properties of the Kronecker product it follows that

$$\det(f) = \det(\otimes_{i=1}^n f_i) = (\prod_{i=1}^n \det(f_i))^{2^{n-1}},$$

whence the first statement of the following theorem:

Theorem B.4. *Let N and $f=\otimes_{i=1}^n f_i$ be as in the preceding theorem. Then f is invertible if, and only if, all f_i are invertible, and then its inverse is given by the matrix coeffients, for $(A,B)\in\mathcal{P}(N)^2$ (recall $A\Delta B$ is the symmetric difference)*

$$(f^{-1})_{A,B} = \frac{(-1)^{|A\Delta B|}}{\prod_{i=1}^n \det(f_i)} f_{B^c,A^c} = \frac{(-1)^{|A\Delta B|}}{\prod_{i=1}^n \det(f_i)} \mathbf{a}_{A\cap B}\cdot \mathbf{b}_{A\cap B^c}\cdot \mathbf{c}_{A^c\cap B}\cdot \mathbf{d}_{A^c\cap B^c}.$$

Proof. Assume each f_i is invertible. For $n=1$, $N=\{k\}$, the inverse is

$$\begin{pmatrix} a_k & b_k \\ c_k & d_k\end{pmatrix}^{-1} = \frac{1}{(a_kd_k - b_kc_k)}\begin{pmatrix} d_k & -b_k \\ -c_k & a_k\end{pmatrix}. \tag{B.1}$$

For $n=2$, the matrix of the inverse is the Kronecker product of the inverses

$$\frac{1}{\det(f_1)\det(f_2)}\begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1\end{pmatrix}\otimes\begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2\end{pmatrix} =$$

$$\frac{1}{\det(f_1)\det(f_2)}\begin{pmatrix} d_1d_2 & -b_1d_2 & -d_1b_2 & b_1b_2 \\ -c_1d_2 & a_1d_2 & c_1b_2 & -a_1b_2 \\ -d_1c_2 & b_1c_2 & d_1a_2 & -b_1a_2 \\ c_1c_2 & -a_1c_2 & -c_1a_2 & a_1a_2 \end{pmatrix}$$

which is in keeping with the formula announced in the claim. To put this computation into a conceptual framework, note that the inverse in (B.1) is obtained by first taking the adjugate matrix, and then dividing by the determinant. The adjugate $X^\sharp$ of a 2×2-matrix X, in turn, is given by

$$X^\sharp = JX^\top J^{-1},$$

where $X^\top$ is the transposed matrix, $(X^\top)_{(A,B)} = X_{(B,A)}$, and

$$\text{(B.2)}\qquad I := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

i.e., J sends $E_\emptyset \mapsto E_1$, $E_1 \mapsto -E_\emptyset$ (so $X^\sharp$ is the adjoint of X with respect to the canonical symplectic form on $\mathbb{K}^2$; call it "symplectic adjoint"). For each 2×2-matrix M let

$$M_{\mathsf{n}} = \otimes_{i=1}^n M : \mathbb{K}^{\mathcal{P}(\mathsf{n})} \to \mathbb{K}^{\mathcal{P}(\mathsf{n})}.$$

Then, for the matrices I, J, K defined by (B.2), the effect on E_A is

$$\text{(B.3)}\qquad I_{\mathsf{n}}(E_A) = (-1)^{|A|}E_A, \quad K_{\mathsf{n}}(E_A) = E_{A^c}, \quad J_{\mathsf{n}}(E_A) = (-1)^{|A^c|}E_{A^c},$$

The inverse of J_{n} is $J_{\mathsf{n}}^{-1}(E_A) = K_{\mathsf{n}}I_{\mathsf{n}}(E_A) = (-1)^{|A|}E_{A^c} = (-1)^n J_{\mathsf{n}}(E_A)$. Using this, we compute

$$\begin{aligned} f^\sharp(E_A) &= J_{\mathsf{n}} \circ f^\top \circ J_{\mathsf{n}}^{-1}(E_A) = (-1)^{|A|} J_{\mathsf{n}} \circ f^\top(E_{A^c}) \\ &= (-1)^{|A|} J_n \sum_B f^\top_{A^c,B} E_B \\ &= (-1)^{|A|} \sum_B f_{B,A^c} (-1)^{|B^c|} E_{B^c} = (-1)^{|A|} \sum_B f_{B^c,A^c}(-1)^{|B|} E_B \\ &= \sum_B (-1)^{|A|}(-1)^{|B|} \mathbf{a}_{A\cap B} \cdot \mathbf{b}_{A\cap B^c} \cdot \mathbf{c}_{A^c\cap B} \cdot \mathbf{d}_{A^c \cap B^c} E_B \end{aligned}$$

which together with $|A| + |B| \equiv |A\Delta B| \mod (2)$, so $(-1)^{|A|}(-1)^{|B|} = (-1)^{|A\Delta B|}$, gives us the adjugate and the claim. □

Remark B.2. In the same way, it follows that, even if f is not invertible, we have

$$f \circ J_{\mathsf{n}} \circ f^\top \circ J_{\mathsf{n}}^{-1} = \prod_{i=1}^n \det(f_i) \cdot \mathrm{id}.$$

Appendix C. Functor categories

C.1. **Categories of anchored sets or algebras.** A *category* C is a pair (C^{obj}, C^{mor}) of collections of objects and of morphisms, together with certain structure operations, see [CWM, MM92].

Definition C.1. *We denote by* **Sets** *the (large) category of sets and set-maps, and (following notation from* [MM92], *p. 25) by* $\mathbf{Sets^2}$ *the (large) category of* anchored sets, *that is, objects* (M, Υ, M') *are maps* $\Upsilon : M \to M'$ *(we call* Υ *the* anchor map, *and often denote, by abuse of notation, the triple just by* Υ *or by* M, *according to context), and morphisms are* anchor-compatible pairs of maps $\Phi : M \to N$, $\Phi' : M' \to N'$, *i.e., they commute with anchors:*

$$\Upsilon_N \circ \Phi = \Phi' \circ \Upsilon_M.$$

We fix once and for all a commutative unital ring $\mathbb{K}$, *and denote by* $\mathbf{Alg}_{\mathbb{K}}$ *the (large) category of* all *(associative, unital)* $\mathbb{K}$*-algebras, and by* $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{2}}$ *the category all* anchored (associative, unital) $\mathbb{K}$-algebras, *that is, objects* $(\mathbb{A}, \Upsilon, \mathbb{A}')$ *are algebra morphisms* $\Upsilon : \mathbb{A} \to \mathbb{A}'$ *(called* anchor*), and morphisms are pairs of anchor-compatible algebra morphisms. The identity morphism* $\mathrm{id}_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$ *is called the* trivial anchor*. It is an initial object in* $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{2}}$.

We denote by suitable supercripts certain subcategories of $\mathbf{Alg}_{\mathbb{K}}$ *such as: all* commutative *associative algebras; in particular, we denote by* $\mathbf{Alg}_{\mathbb{K}}^{\mathrm{fin}}$ *the subcategory of all* $\mathbb{K}$*-algebras that are commutative, free and finite-dimensional as* $\mathbb{K}$*-modules. The corresponding subcategories of anchored algebras are then defined in the obvious way.*

C.2. **Functor categories.** A *functor* F from a category C to a category D is a pair (F^{obj}, F^{mor}) of arrows $(C^{obj} \to D^{obj}, C^{mor} \to D^{mor})$, all of this satisfying the usual axioms (see [CWM, MM92]). A *natural transformation* τ between two functors $S, T : C \to B$ is an arrow $\tau : C^{obj} \to B^{mor}$, again satisfying the usual axioms. Functors from a category C to a category B, together with their natural transformations, form a *functor category* $\mathbf{Fn}(C, B) = B^C$ (see e.g. [CWM], II.4, or [MM92]). Specifically, for the case of categories we are interested in:

Definition C.2. *Let* $\mathbf{c}_{\mathbb{K}}$ *be some (small) subcategory of* $\mathbf{Alg}_{\mathbb{K}}^{\mathbf{2}}$ *(containing at least the trivial anchor as object, and the unique morphism of the trivial anchor to any other object as morphism). A* $\mathbf{c}_{\mathbb{K}}$-space law *is a functor*

$$\underline{M} : \mathbf{c}_{\mathbb{K}} \to \mathbf{Sets^2},$$

from $\mathbf{c}_{\mathbb{K}}$ *to the category* $\mathbf{Sets^2}$ *of anchored sets: for very object* $\Upsilon : \mathbb{A} \to \mathbb{A}'$ *of* $\mathbf{c}_{\mathbb{K}}$, *we have sets* $M_{\mathbb{A}}$ *and* $M_{\mathbb{A}'}$ *and a set-map* $M_\Upsilon : M_{\mathbb{A}} \to M_{\mathbb{A}'}$, *and for every* $\mathbf{c}_{\mathbb{K}}$*-morphism* $(\phi : \mathbb{A} \to \mathbb{B}, \phi' : \mathbb{A}' \to \mathbb{B}')$, *we have a pair of maps* $(M_\phi : M_{\mathbb{A}} \to M_{\mathbb{B}}, M_{\phi'} : M_{\mathbb{A}'} \to M_{\mathbb{B}'})$ *commuting with anchors and satisfying the functorial rules* $M_{\mathrm{id}} = \mathrm{id}_{M_{\mathbb{A}}}$ *and* $M_{\phi\circ\psi} = M_\phi \circ M_\psi$, $M_{\phi'\circ\psi'} = M_{\phi'} \circ M_{\psi'}$. *The trivial anchor* $\mathrm{id}_{\mathbb{K}}$ *shall give rise to the "trivial anchored set"* $\mathrm{id} : M_{\mathbb{K}} \to M_{\mathbb{K}}$, *called the* underlying set of $\underline{M}$.

A morphism between $\mathbf{c}_{\mathbb{K}}$-space laws $\underline{M}$ and $\underline{N}$, *also called* $\mathbf{c}_{\mathbb{K}}$-mapping law, *is a natural transformation* $\underline{f} : \underline{M} \to \underline{N}$, *i.e., for each object* $\Upsilon : \mathbb{A} \to \mathbb{A}'$ *of* $\mathbf{c}_{\mathbb{K}}$, *there is an anchor-compatible pair*

$$f_{\mathbb{A}} : M_{\mathbb{A}} \to N_{\mathbb{A}}, \qquad f_{\mathbb{A}'} : M_{\mathbb{A}'} \to N_{\mathbb{A}'}$$

varying functorially with Υ*: for any* $\mathbf{c}_{\mathbb{K}}$*-morphism* $(\phi : \mathbb{A} \to \mathbb{B}, \phi' : \mathbb{A}' \to \mathbb{B}')$ *and space laws* $\underline{M}, \underline{N}$, *there are maps* M_ϕ, N_ϕ *such that* $N_\phi \circ f_{\mathbb{A}} = f_{\mathbb{B}} \circ M_\phi$ *and*

$N_{\phi'} \circ f_{\mathbb{A}'} = f_{\mathbb{B}'} \circ M_{\phi'}$

$$\begin{array}{ccc} M_{\mathbb{A}} & \xrightarrow{f_{\mathbb{A}}} & N_{\mathbb{A}} \\ M_\phi \downarrow & & \downarrow N_\phi, \\ M_{\mathbb{B}} & \xrightarrow{f_{\mathbb{B}}} & N_{\mathbb{B}} \end{array} \qquad \begin{array}{ccc} M_{\mathbb{A}'} & \xrightarrow{f_{\mathbb{A}'}} & N_{\mathbb{A}'} \\ M_{\phi'} \downarrow & & \downarrow N_{\phi'}. \\ M_{\mathbb{B}'} & \xrightarrow{f_{\mathbb{B}'}} & N_{\mathbb{B}'} \end{array}$$

The map $f_{\mathbb{K}} : M_{\mathbb{K}} \to N_{\mathbb{K}}$ *is called the* underlying set-map of $\underline{f}$. *The* functor category of $\mathbf{c}_{\mathbb{K}}$-space laws and their morphisms, *denoted by*

$$\mathbf{Fn}(\mathbf{c}_{\mathbb{K}}, \mathbf{Sets^2})$$

is the (large) category whose objects are functors $\underline{M}$ *and morphisms natural transformations* $\underline{f}$, *which are* composed *"pointwise", i.e., for two laws* $\underline{f} : \underline{M} \to \underline{N}$, $\underline{g} : \underline{N} \to \underline{P}$ *and all* $\Upsilon : \mathbb{A} \to \mathbb{A}' \in \mathbf{c}_{\mathbb{K}}$, *we have* $(\underline{g} \circ \underline{f})_{\mathbb{A}} := g_{\mathbb{A}} \circ f_{\mathbb{A}} : M_{\mathbb{A}} \to P_{\mathbb{A}}$, *and* $(\underline{g} \circ \underline{f})_{\mathbb{A}'} := g_{\mathbb{A}'} \circ f_{\mathbb{A}'} : M_{\mathbb{A}'} \to P_{\mathbb{A}'}$.

Remark C.1. By definition, for each object $\Upsilon : \mathbb{A} \to \mathbb{A}'$ of $\mathbf{c}_{\mathbb{K}}$, *evaluation at level* Υ,

$$\mathrm{ev}_\Upsilon : \underline{M} \mapsto (M_\Upsilon : M_{\mathbb{A}} \to M_{\mathbb{A}'}),\ \underline{f} \mapsto (f_{\mathbb{A}}, f_{\mathbb{A}'}),$$

is a functor from $\mathbf{Fn}(\mathbf{c}_{\mathbb{K}}, \mathbf{Sets^2})$ to $\mathbf{Sets^2}$. The natural morphism $\iota_{\mathbb{A}} : \mathbb{K} \to \mathbb{A}$ induces a map $M_{\iota_{\mathbb{A}}} : M_{\mathbb{K}} \to M_{\mathbb{A}}$.

Remark C.2. One can define the *direct product* of $\mathbf{c}_{\mathbb{K}}$-space laws $\underline{M}_i$, $i \in I$, by

$$(\times_{i\in I} \underline{M}_i)_{\mathbb{A}} := \times_{i\in I}((M_i)_{\mathbb{A}}), \qquad (\times_{i\in I} \underline{M}_i)_\phi := \times_{i\in I}((M_i)_\phi).$$

Definition C.3. *An* algebraic $\mathbb{K}$-space law *is a functor* $\underline{M}$ *from the whole category* $\mathbf{Alg}^{\mathbf{2}}_{\mathbb{K}}$ *to* $\mathbf{Sets^2}$.

Example C.1 (Linear sets). Every $\mathbb{K}$-module V gives rise to a functor $\underline{V}$ from $\mathbf{Alg}^{\mathbf{2}}_{\mathbb{K}}$ to $\mathbf{Sets^2}$ (algebraic space law), via *algebraic scalar extension* (Appendix A)

$$V_{\mathbb{A}} = V \otimes_{\mathbb{K}} \mathbb{A}, \quad V_\Upsilon = \mathrm{id}_V \otimes \Upsilon : V_{\mathbb{A}} \to V'_{\mathbb{A}}.$$

Natural transformations are given, e.g. by linear maps $f : V \to W$: letting $f_{\mathbb{A}} = f \otimes \mathrm{id}_{\mathbb{A}} : V_{\mathbb{A}} \to W_{\mathbb{A}}$, we get a morphism $\underline{f} : \underline{V} \to \underline{W}$. We get other natural transformations if we restrict our functors to categories of *commutative* algebras: then every m-multilinear map $f : V^m \to W$ gives rise to a natural transformation $\underline{f}_{\mathbb{A}}$ (Theorem A.6). Thus multilinear maps give rise to *polynomial laws* in the sense of Roby [Ro63] (cf. Appendix of [Lo75]).

Example C.2. For every $m \in \mathbb{N}$, there is an algebraic space law $\underline{M} = \mathrm{Gl}(m, \mathbb{K})$, by letting $M_{\mathbb{A}} = \mathrm{Gl}(m, \mathbb{A})$, $M_\Upsilon = \mathrm{id} \otimes \Upsilon$. For instance, one can define its "tangent group" by $TM := M_{\mathbb{K}[\varepsilon]} = \mathrm{Gl}(m, \mathbb{K}[\varepsilon])$, and use this to develop a kind of algebraic differential calculus – this approach is used in [DG], see also [Be08, Be14].

The following is an "axiom" which is characteristic to our setting, compared with the general theory of functor categories. It rules out that "model spaces" give rise to trivial functors (like the functor sending every algebra to a point), and will force that all constructions from functorial differential calculus reduce to familiar linear algebra constructions, when applied to linear spaces and linear maps.

Definition C.4. *By definition, the* $\mathbf{c}_{\mathbb{K}}$-law $\underline{V}$ of a $\mathbb{K}$-module V *shall be given by algebraic scalar extension (as in Example C.1), and likewise for the mapping law corresponding to a* $\mathbb{K}$*-linear map. In other words, the category of* $\mathbb{K}$*-linear spaces shall be imbedded into the functor category* $\mathbf{Fn}(\mathbf{c}_{\mathbb{K}}, \mathbf{Sets^2})$ *in the natural way.*

C.3. **Further properties.** It is not our aim here to develop the general theory of the functor categories in question. In practice, the categories $\mathbf{c}_{\mathbb{K}}$ will have further properties; in particular, the most interesting additional property is the one of a *(strict) monoidal categories* with respect to the tensor product of algebras.

Definition C.5. *We say that some subcategory* C *of* $\mathbf{Alg}^{\mathbf{2}}_{\mathbb{K}}$ *is*

(1) closed under direct products *if it contains, along with objects* $\Upsilon_{\mathbb{A}}, \Upsilon_{\mathbb{B}}$*, also the object*
$$\Upsilon_{\mathbb{A}} \oplus \Upsilon_{\mathbb{B}} : \mathbb{A} \times \mathbb{B} \to \mathbb{A}' \times \mathbb{B}'$$
(we write also $\mathbb{A} \oplus \mathbb{B}$ *for the direct product algebra* $\mathbb{A} \times \mathbb{B}$*, with product* $(a, a') \cdot (b, b') = (ab, a'b')$*), and likewise for morphisms;*

(2) closed under tensor products *if it contains, along with objects* $\Upsilon_{\mathbb{A}}, \Upsilon_{\mathbb{B}}$*, also their tensor product*
$$\Upsilon_{\mathbb{A}} \otimes \Upsilon_{\mathbb{B}} : \mathbb{A} \otimes \mathbb{B} \to \mathbb{A}' \otimes \mathbb{B}'$$
(recall the tensor product algebra $\mathbb{A} \otimes \mathbb{B}$ *has product* $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$*), and likewise for morphisms;*

When $\underline{M} = \underline{V}$ is a $\mathbb{K}$-module, then, by linear algebra,
$$V_{\mathbb{A}\oplus\mathbb{B}} = V_{\mathbb{A}} \times V_{\mathbb{B}}, \qquad V_{\mathbb{A}\otimes\mathbb{B}} = (V_{\mathbb{A}})_{\mathbb{B}},$$
and likewise $V_{\Upsilon_{\mathbb{A}} \oplus \Upsilon_{\mathbb{B}}} = V_{\Upsilon_{\mathbb{A}}} \times V_{\Upsilon_{\mathbb{B}}}$, $V_{\Upsilon_{\mathbb{A}} \otimes \Upsilon_{\mathbb{B}}} = (V_{\Upsilon_{\mathbb{A}}})_{\Upsilon_{\mathbb{B}}}$. These rules should serve as model, when generalizing functor categories beyond the realm of linear algebra.

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