# JORDAN GEOMETRIES BY INVERSIONS

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Abstract. *Jordan geometries* are defined as spaces $\mathcal{X}$ equipped with point reflections $J_a^{xz}$ depending on triples of points $(x, a, z)$, exchanging $x$ and $z$ and fixing $a$. In a similar way, symmetric spaces have been defined by Loos ([Lo69]) as spaces equipped with point reflections $S_x$ fixing $x$, and therefore the theories of Jordan geometries and of symmetric spaces are closely related to each other – in order to describe this link, the notion of *symmetry actions* of torsors and of symmetric spaces is introduced. Jordan geometries give rise both to symmetry actions of certain abelian torsors and of certain symmetric spaces, which in a sense are dual to each other. By using an algebraic differential calculus generalizing the classical Weil functors, we attach a tangent object to such geometries, namely a *Jordan pair*, resp. a *Jordan algebra*. The present approach works equally well over base rings in which 2 is not invertible (and in particular over $\mathbb{Z}$), and hence can be seen as a globalization of *quadratic Jordan pairs*; it also has a very transparent relation with the theory of *associative geometries* developped in [BeKi09a].

## Introduction

Symmetries of order two – according to context, called *reflections, inversions* or *involutions* – play a basic rôle in all of geometry, and some parts of geometry can be entirely reconstructed by using them (cf. the "Aufbau der Geometrie aus dem Spiegelungsbegriff", [Ba73]). In the present work, we will use the term "inversion" since the involutions we use can be interpreted as *(generalized) inverses in rings or algebras*: geometrically, the inversion map $x \mapsto x^{-1}$ in a unital associative algebra behaves like a *reflection through a point*, with respect to the "isolated" fixed point 1, the unit element of the algebra. This choice of terminology should not lead to conflict with the common one from Inversive Geometry, where the term "inversion" refers to reflections with respect to circles or spheres (cf. [Wi81]).

The inversion map of an associative algebra is a "Jordan feature", *i.e.*, it depends only on the symmetric part ("Jordan product") $x \bullet z = \frac{1}{2}(xz + zx)$ of the associative product, and it contains the whole information of the Jordan product. The approach to Jordan algebras given in the book [Sp73] by T. Springer is based on this observation. In the present work, we extend this approach to the *geometries* corresponding to Jordan algebraic structures. We have defined such geometries, called *generalized projective geometries*, in another way in [Be02] – the approach given there was not based on inversions, but rather on the various actions of a scalar ring $\mathbb{K}$ on the geometry (a point of view already introduced by Loos in [Lo79]); it relied in a crucial way on *midpoints*, and thus on the existence of a scalar $\frac{1}{2}$ in $\mathbb{K}$.

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The present approach does not have this drawback, and at the same time is simpler and more natural. Another advantage is its close relation with the *associative case*: since symmetrization of associative algebras leads to Jordan algebras, there is a similar link of Jordan geometries with the *associative geometries* defined and studied in [BeKi09a, BeKi09b]. Finally, another feature, simplifying the approach compared to the one of [Be02], is a more conceptual use of "algebraic differential calculus", keeping fairly close both to the usual language of differential geometry and to the use of scalar extensions in algebraic geometry. Let us explain these items in some more detail.

0.1. **Jordan structure map.** A *Jordan structure map* on a space $\mathcal{X}$ is given by a family of bijections of order two, $J_a^{xz} : \mathcal{X} \to \mathcal{X}$, parametrized by certain triples $(x, a, z)$ of points from $\mathcal{X}$ and satisfying the following identities:

(IN) *involutivity*: $J_a^{xz} \circ J_a^{xz} = \mathrm{id}_{\mathcal{X}}$

(IP) *idempotency*: $J_c^{ab}(c) = c$, $J_c^{ab}(a) = b$, $J_c^{ab}(b) = a$

(A) *associativity:* $J_c^{xz} J_c^{uv} J_c^{ab} = J_c^{J_c^{xa}(v), J_c^{bz}(u)}$

(D) *distributivity:* $J_c^{xz} \circ J_b^{uv} \circ J_c^{xz} = J_{J_c^{xz}(b)}^{J_c^{xz}(u), J_c^{xz}(v)}$

(C) *commutativity:* $J_c^{ab} = J_c^{ba}$

(S) *symmetry:* $J_a^{xx} = J_x^{aa}$

(T) *tangent map at fixed point:* $T_a(J_a^{xz}) = (-\mathrm{id}_{T_a \mathcal{X}})$.

These axioms all have a very clear geometric interpretation: the first two axioms say that triples of points define involutions exchanging two of the points and fixing the third; axioms (A), (D), (C) mean that the inversions $J_a^{xz}$ play a double rôle:

(D): for fixed $(x, z)$, the map $(a, b) \mapsto J_a^{xz}(b)$ defines a *reflection space* $U_{xz}$, and (T) says that this reflection space is in fact a *symmetric space*[1],

(A): for fixed $a$, the map $(x, y, z) \mapsto J_a^{xz}(y)$ defines a *torsor* (see below) $U_a$, and (C) means that this torsor is *commutative*.

Moreover, these symmetric spaces, resp. torsors, "act" on $\mathcal{X}$ by *symmetry actions*. Finally, (S) says that these two points of view are compatible.

0.2. **Symmetry actions.** By *torsor* we mean a group of which we forget the origin by considering the ternary law $(xyz) = xy^{-1}z$ instead of the binary law $xz$. (Other terms such as *heap, groud*, or *principal homogeneous space* are used in the literature – see [BeKi09a] for some remarks concerning terminology.) A torsor has not only the familiar *left and right translations*, but also *middle multiplication operators*:

$$\ell_{xy}(z) = (xyz) = m_{xz}(y) = r_{zy}(x). \tag{0.1}$$

For $x = z$, $m_{xx}$ is "inversion with respect to the origin $x$", and then is of order two; but in general $m_{xz}$ is not of order two (but it is so if $G$ is commutative). One can write the defining identities of a torsor in terms of the $m$-operators (Lemma A.2):

(SA) $m_{xy} \circ m_{uv} \circ m_{rs} = m_{m_{xr}(v), m_{sy}(u)}$,

(IP) $m_{xz}(x) = z$, $m_{xz}(z) = x$.

[1]The concepts of *reflection space* and of *symmetric space*, for general smooth spaces, are defined in Appendix A, following the approach of Loos [Lo67, Lo69] from the real finite-dimensional case.

The reader may recognize the form of the identities (A) and (IP) given above. In a second step, just as the properties of left- or right translations give rise to left- or right actions of a group $G$, by allowing one argument to be in a set $X$ different from $G$, the same can be done for the $m$-operators, thus leading to the notion of *symmetry action of a torsor* (Appendix A). Such symmetry actions are "rich" structures, in the sense that they contain strictly more information than a simple left- or right action, and they do appear "in nature": on the one hand, every torsor has a *regular symmetry representation* on itself (in Appendix A we give some arguments why this point of view could be useful in general Lie theory) and on the other hand, such actions appear precisely in the context of Jordan geometries and of associative geometries: for fixed $a$, the operator $m_{xz} = J_a^{xz}$ describes the *commutative symmetry action of the commutative torsor* $U_a$, and for fixed $(x, z)$, the same formula describes the symmetry action of the (possibly non-abelian) *symmetric space* $U_{xz}$. When $x = z$, the identity (S) means that both points of view coincide.

0.3. **The main examples.** Let us now look briefly at the main examples.

0.3.1. *Grassmannians.* Let $\mathrm{Gras}_E^F(W)$ be the Grassmannian of subspaces of a $\mathbb{K}$-module $W$, isomorphic to $E$ and having a complement isomorpic to $F$ (in finite dimension over a field, these are the Grassmannians of $p$-spaces in $\mathbb{K}^n$), and let $\mathcal{X} = \mathrm{Gras}_E^F(W) \cup \mathrm{Gras}_F^E(W)$. Two elements $x, a \in \mathcal{X}$ are called *transversal* if their sum is direct: $W = x \oplus a$, and the projection with kernel $a$ and image $x$ is then denoted by $P_x^a : W \to W$. Now, if $x$ and $z$ are transversal to $a$, define a linear map

$$\mathrm{J}_a^{xz} := P_a^x - P_z^a : \quad W \to W. \tag{0.2}$$

This map acts as identity on $a$; on $x$ it has the effect of projection onto $z$ along $a$, and (because of the relation $\mathrm{J}_a^{xz} = -\mathrm{J}_a^{zx}$) similarly for $x$ and $z$ exchanged. From these observations it follows easily that the family of maps $J_a^{xz} : \mathcal{X} \to \mathcal{X}$ defined by $J_a^{xz}(y) := \mathrm{J}_a^{xz}(y)$ satisfies the axioms listed above.

0.3.2. *Lagrangian Grassmannians.* The structure described in the preceding example can be restricted to fixed point sets of antiautomorphisms – e.g., of *orthocomplementation maps* $x \mapsto x^\perp$ (induced by a quadratic or symplectic form), having as fixed point spaces the *Lagrangian Grassmannians* (see [BeKi09b]), which thus are Jordan geometries.

0.3.3. *Projective quadrics.* Assume $\mathcal{X} = Q$ is a projective quadric in the projective space $\mathbb{P}(W)$ of a vector space $W$. Two elements of $Q$ are called *transversal* if the line joining them in $\mathbb{P}(W)$ is a secant, i.e., not a tangent line of $Q$. If $x = [v]$ and $z = [w] \in Q$ are transversal to $a = [u] \in Q$, then there exists a unique orthogonal map $I_a^{xz} : W \to W$ exchanging $[x]$ and $[z]$, fixing $u$, and acting as $-1$ on the orthogonal complement of $\mathrm{Span}(u, v, w)$. We define $J_a^{x,z}$ to be the restriction to $Q$ of the projective map induced by $I_a^{x,z}$. The family of maps thus defined satisfies the properties given above (and has some more specific properties which we we intend to investigate in more detail in subsequent work).

0.3.4. *An exceptional geometry.* In [BeKi12], we have defined geometrically a family of *Moufang torsors* on a *Moufang projective plane*; this structure can be used to define a structure map $J^{ab}_x$ (work in progress).

0.4. **Special Jordan geometries.** Let us look again at the example of the Grassmannian (see 0.3.1). Given four subspaces $a, b, x, z \subset W$ such that $x$ and $z$ are common complements of $a$ and $b$, we define a linear operator, similarly as in (0.2),

$$\mathrm{M}^{xz}_{ab} := P^a_x - P^z_b : \quad W \to W, \tag{0.3}$$

inducing a map $M^{xz}_{ab} : \mathbb{P}W \to \mathbb{P}W$. Obviously, the $J$-operators are obtained from the $M$-operators by restriction to the diagonal $a = b$: $J^{xz}_a = M^{xz}_{aa}$. The $M$-operators satisfy certain axioms, similar to those of a Jordan structure map $J$ (see Chapter 4), and which – in a slightly different form – have been used in [BeKi09a] to define *associative geometries*. In a certain sense, these axioms have an even more symmetric form than those of $J$: the high symmetry of the $M$-operators is reflected, among others, by the identity $M^{xz}_{ab} = M^{ab}_{xz} = M^{ba}_{zx}$, which implies that restriction to the diagonal $a = b$ is equivalent to restriction to $x = z$. Jordan geometries obtained as subgeometries from associative geometries by restriction to a diagonal are called *special*. In this context, the diagonal $a = b$ ($J^{xz}_a = M^{aa}_{xz}$) corresponds to the point of view of commutative torsors, and the diagonal $x = z$ ($J^{xz}_a = M^{xz}_{aa}$) to the point of view of symmetric spaces. This is the geometric and base point free analog of restricting the product $xz$ of an associative algebra to the diagonal $x = z$.

0.5. **Unit elements, idempotents, and self-duality.** Geometrically, there is an important difference between associative or Jordan algebras with or without unit element. In a similar way, there is an important difference between Grassmannians $\mathrm{Gras}^F_E(W)$ of the "first kind", where $E$ and $F$ are isomorphic to each other, and of "second kind", where this is not the case: the first are *self-dual* (equal to their *dual geometry* $\mathrm{Gras}^E_F(W)$), the second are not; and for the first kind, there exist *pairwise transversal triples* $(a, b, c)$ of subspaces, for the second kind, this is not the case. For instance, the *projective line* $\mathrm{Gras}^{\mathbb{K}}_{\mathbb{K}}(\mathbb{K}^2)$ is of the first kind, and higher dimensional projective spaces are not. Whenever we have a pairwise transversal triple $(a, b, c)$, the inversions $J^{ab}_c$, $J^{ba}_c$, $J^{ac}_b$ are all defined and generate a permutation group $S_3$; if we add $J^{aa}_b$ to the set of generators, they generate a group which is a homomorphic image of $\mathbb{P}\mathrm{GL}(2, \mathbb{Z})$ (Theorem 6.2) and hence the three points $a, b, c$ generate a subgeometry that is a homomorphic image of the projective line $\mathbb{ZP}^1$ with its canonical base triple $(o, 1, \infty)$ (Theorem 6.3). Pairwise transversal triples $(a, b, c)$ in a Jordan geometry correspond to *unital Jordan algebras*; classification of such triples in a given geometry is related to the classical *Maslov index* – the Jordan algebras associated to a geometry are *isotopic* to each other, but in general not isomorphic.

In a geometry of the second kind, there are no transversal triples; a substitute is given by *idempotent quadruples* which are defined by relations obtained by "dissociating" the projective line $\mathbb{ZP}^1$ into two copies, and leading to a homomorphism $\mathrm{GL}(2, \mathbb{Z}) \to \mathrm{Aut}(\mathcal{X})$ (Theorem 6.6). From an algebraic point of view, spaces of the first kind correspond the situation where $J^{ab}_c$ can be interpreted as an *inversion of an algebra with respect to a unit element*, whereas spaces of the second kind lead to the situation where $J_a^{xz}$ can be understood as *the (negative of) the quasi-inverse in a Jordan pair*. In order to explain the relation with algebraic structures, we need, however, to digress on additional structures, such as *smoothness* and *Lie groups*, which are needed to define *tangent algebras*.

0.6. **Algebraic differential calculus.** In the second half of this work, our aim is to attach a tangent object to a geometry. This cannot be done in a completely abstract setting: we have to assume some *regularity* of our structure map $J : (x, a, z, y) \mapsto J_a^{xz}(y)$. Such assumptions can be formulated in various ways; in order to stress that the theory developed up to know is independent of such assumptions, we have divided the text into two parts – only the second part depends on the regularity assumptions. The regularity assumptions used in the present work are simpler than those used in [Be02, BeKi09a] (at the beginning of Part two we discuss this in some more detail): essentially, we assume that $\mathcal{X}$ is a *smooth manifold* and that $J$ is a smooth map. However, since we wish to keep the framework as general as possible, we have to define "smoothness" in a purely algebraic setting. In [Lo79], the language of schemes and algebraic groups has been used; here we propose a more elementary use of "algebraic differential calculus", motivated by the classical *Weil functors* (see [KMS93]) and their generalization in the framework of differential caculus over topological fields or rings ([BeS11, Be08]). The principles are explained in Appendix B: one requires that our geometry behaves functorially with respect to scalar extensions that are *Weil algebras* $\mathbb{A}$ over the base ring $\mathbb{K}$, that is, to our geometry $(\mathcal{X}, J)$ we can attach a "generalized tangent geometry" $(T^{\mathbb{A}}\mathcal{X}, T^{\mathbb{A}}J)$. For instance, the Weil algebra $T\mathbb{K} := \mathbb{K}[X]/(X^2) = \mathbb{K} \oplus \varepsilon\mathbb{K}$ ($\varepsilon^2 = 0$) of dual numbers over $\mathbb{K}$ gives rise to the "usual" tangent functor $T$, and the scalar extension by dual numbers, $(T\mathcal{X}, TJ)$, behaves in all respects like a tangent bundle of $(\mathcal{X}, J)$. Thus the "tangent map axiom" (T) mentioned above makes sense in a purely algebraic way.

This "covariant approach" to differential calculus is well-adapted to Jordan- and associative geometries since they come with a *natural atlas*: chart domains and transition functions are intrinsically defined by the axiomatic algebraic data; it thus suffices to simply assume that they are "smooth laws", i.e., that they behave functorially with respect to Weil algebras. This should be compared to Lie groups and symmetric spaces: one may define them in a purely algebraic way, but then an atlas is an additional datum, independent of the algebraic structure – over general fields or rings, there is no exponential map; thus in general Lie theory there are no "canonical coordinates". On the other hand, in Jordan theory there are "canonical charts" – this fact partially explains why Jordan theory is much better adapted than Lie theory to infinite dimensional situations.

0.7. **The Jordan pair associated to a Jordan geometry.** With the framework just described, we can associate a "tangent algebra" to a Jordan geometry with base point in a similar way as in [Be02]: the Lie algebra of *infinitesimal automorphisms* or *derivations* of the geometry is a 3-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, and it is well-known that such a Lie algebra corresponds to a *linear Jordan pair* $(V^+, V^-) = (\mathfrak{g}_1, \mathfrak{g}_{-1})$. As already mentioned, our theory works even in case that the scalar 2 is not invertible in $\mathbb{K}$, and then we need to work with *quadratic Jordan pairs* as defined in [Lo75] – we define quadratic maps $Q^{\pm}$ that contain more information then the trilinear bracket derived from the Lie algebra; the proof that these maps satisfy the Jordan identities (JP1) – (JP3) from [Lo75] follows closely the lines of work by O. Loos ([Lo79]).

0.8. **The Jordan geometry associated to a Jordan pair.** The Jordan geometry can be reconstructed from a Jordan pair – in case 2 is invertible in $\mathbb{K}$, this follows quite easily from the corresponding result in [Be02], by letting $J_a^{xz} := (-1)^a_{\mu(x,a,z)}$ (reflection at the midpoint $\mu(x,a,z) = \frac{x+z}{2}$ of $x$ and $z$ in the affine space $U_a$), see Theorem 7.3. If 2 is not invertible in $\mathbb{K}$, the definition is less direct (Theorem 10.1).

Summing up, just as in classical Lie theory, we can go back and forth from Jordan geometries to Jordan pairs and -algebras. We will study this correspondence more thoroughly in subsequent work.

## Contents



We have divided the text into two main parts, taking account of the organization of the axiomatic structure on four levels:

(1) a rather weak (non-)incidence structure, called *transversality*,
(2) the general datum of one or several *structure maps*,
(3) certain *identities* (associative, Jordan,...) satisfied by the structure maps,
(4) a *regularity hypothesis*, such as smoothness of structure maps.

The first part deals with the axiomatic structures of levels (1), (2) and (3), and in the second part, which is more algebraic in nature, the consequences of (4) are developped. Appendix A belongs to the first part and Appendix B to the second part; both appendices have more general applications and may be of independent interest.

# FIRST PART: GEOMETRIES WITH SYMMETRY ACTIONS

## 1. Transversality relations, splittings, dissociations

1.1. **Transversality relations.** A *transversality relation* on a set $\mathcal{X}$ is a binary relation on $\mathcal{X}$, that is, a subset $\mathcal{D}_2 \subset (\mathcal{X} \times \mathcal{X})$; we write also $x \top a$ if $(x,a) \in \mathcal{D}_2$, and this relation is assumed to be

– *symmetric* : $x \top a$ iff $a \top x$,
– *irreflexive*: $x$ is never transversal to itself.

For the sets of elements transversal to one, resp. to two given elements, we write

$$U_x := x^\top := \{a \in \mathcal{X} \mid a \top x\}, \qquad U_{ab} := U_a \cap U_b. \tag{1.1}$$

The relation $\top$ is called *non-degenerate* if $U_x = U_y$ always implies $x = y$.

*Homomorphisms* of sets with transversality relation are maps $f : \mathcal{X} \to \mathcal{Y}$ preserving transversality: $x \top a$ implies $f(x) \top f(y)$.

1.2. **Grassmannians.** The standard example of transversality is given by the *Grassmannian* $\mathrm{Gras}(W)$ of all submodules of some $\mathbb{A}$-right module $W$ with the relation: $x \top a$ iff $V = x \oplus a$ (here $\mathbb{A}$ may be a possibly non-commutative ring). The *Grassmannian of type $E$ and co-type $F$* is the space

$$\mathrm{Gras}_E^F(W) := \big\{x \in \mathrm{Gras}(W) \mid x \cong E,\, W/x \cong F\big\} \tag{1.2}$$

of submodules isomorphic to $E$ and such that $W/x$ is isomorphic to $F$ (as modules), where $W = E \oplus F$ is some fixed decomposition. Then the space

$$\mathcal{X} = \mathrm{Gras}_E^F(W) \cup \mathrm{Gras}_F^E(W) \tag{1.3}$$

has non-trivial transversality relation. In particular, we get the *projective geometries*

$$\mathbb{A}\mathbb{P}^n \cup (\mathbb{A}\mathbb{P}^n)' := \mathrm{Gras}_{\mathbb{A}}^{\mathbb{A}^n}(\mathbb{A}^{n+1}) \cup \mathrm{Gras}_{\mathbb{A}^n}^{\mathbb{A}}(\mathbb{A}^{n+1}). \tag{1.4}$$

If $\mathbb{A}$ is a field or skew-field, this is a “usual” projective space together with its dual space of hyperplanes (and “transversal” means the same as “non-incident”, and the relation $\top$ is non-degenerate); however, if $\mathbb{A}$ is a ring, such as $\mathbb{A} = \mathbb{Z}$, then these geometries show some rather unusual features (cf. the article of Veldkamp on Ring Geometries in [Bue]).

1.3. **Transversal chains and connectedness.** Let $n \in \mathbb{N}$, $n > 1$. A *transversal chain of length $n$ in $\mathcal{X}$* is a sequence $(x_1, \ldots, x_n)$ of elements of $\mathcal{X}$ such that $x_{i+1} \top x_i$ for $i = 1, \ldots, n-1$ (equivalently, $x_i \in U_{x_{i-1}, x_{i+1}}$). A transversal chain is called *closed* if $x_n \top x_1$. We denote by

$$\begin{aligned} \mathcal{D}_n &= \{(x_1, \ldots, x_n) \in \mathcal{X}^n \mid \forall i = 1, \ldots, n-1 : \, x_{i+1} \top x_i\}, \\ \mathcal{D}'_n &= \{(x_1, \ldots, x_n) \in \mathcal{D}_n \mid x_n \top x_1\} \end{aligned} \tag{1.5}$$

the set of transversal chains, resp. of closed transversal chains, of length $n$ in $\mathcal{X}$. A chain of length two is also called a *transversal pair*, a chain of length three is a *transversal triple*, and a closed transversal chain of length three is a *pairwise transversal triple*. A *chain joining two elements $x, y \in \mathcal{X}$* is a finite chain $(x_1, \ldots, x_n) \in \mathcal{D}_n$ such that $x_1 = x$, $x_n = y$. We say that $\mathcal{X}$ is *connected* if, for each $x, y \in \mathcal{X}$, there is a chain joining $x$ and $y$. We may also define *connected* *components*: the relation defined by "$x \sim y$ iff there is a chain joining $x$ and $y$" is an equivalence relation; its equivalence classes are the *connected components of* $\mathcal{X}$.

For instance, Grassmannians $\mathcal{X} = \mathrm{Gras}(V)$ are in general not connected; if $\mathbb{K}$ is a field and $V = \mathbb{K}^n$, then its connected components are of the form (1.3).

1.4. **Duality: splitting, and antiautomorphisms.** Assume $\top$ is a transversality relation on $\mathcal{X}$. A *splitting of* $\mathcal{X}$ is a decomposition into a disjoint union $\mathcal{X} = \mathcal{X}^+ \dot\cup \mathcal{X}^-$ such that for all $a \in \mathcal{X}^-$, we have $a^\top \subset \mathcal{X}^+$, and for all $x \in \mathcal{X}^+$, we have $x^\top \subset \mathcal{X}^-$. Equivalently, chains with odd length end up in the same part ($\mathcal{X}^+$ or $\mathcal{X}^-$) they started in, and chains with even length end up in the other. We then say that $\mathcal{X}^+$ and $\mathcal{X}^-$ are *dual to each other*.

We say that $(\mathcal{X}^+, \mathcal{X}^-)$ is *connected of stable rank one* if for each $(x, y) \in (\mathcal{X}^\pm)^2$ there is $a \in \mathcal{X}^\mp$ such that $x, y \in U_a$; equivalently, $a \in U_{xy}$, so $U_{xy}$ is not empty.

Spaces with splitting $(\mathcal{X}^+, \mathcal{X}^-)$ form a category: morphisms $g$ preserve transversality and the given splitting (that is, $g(\mathcal{X}^\pm) \subset \mathcal{Y}^\pm$), so we have well-defined restrictions

$$g^\pm : \mathcal{X}^\pm \to \mathcal{Y}^\pm.$$

In presence of a splitting, we may also define *anti-homomorphisms*: these are pairs of maps exchanging the components, $\mathcal{X}^\pm \to \mathcal{Y}^\mp$, i.e., morphisms from $(\mathcal{X}^+, \mathcal{X}^-)$ to the *opposite splitting* of $\mathcal{Y}$.

1.5. **Self-dual geometries and closed transversal triples.** We say that a connected geometry $(\mathcal{X}, \top)$ is *self-dual* if it does not admit any non-trivial splitting. This is the case if in $\mathcal{X}$ there is a closed transversal chain of odd length (at least three); the converse is true as well. We say that $\mathcal{X}$ is *strongly self-dual* if there is a closed chain of length three, that is, a pairwise transversal triple $(a, b, c)$.

In the example of a Grassmannian (1.3), the indicated decomposition is a splitting if $E$ and $F$ are not isomorphic as modules. Typical antiautomorphisms are then given by orthocomplementation maps. On the other hand, if $E \cong F$ as modules, then there exists a pairwise symmetric triple $(E, F, D)$ where $D$ is the diagonal of $E \oplus F$, after some fixed identification of $E$ and $F$, and hence $\mathrm{Gras}_E(E \oplus E)$ does not admit any splitting (this is the case, in particular, for the *projective line* $\mathbb{AP}^1$). In this case one may introduce an "artificial splitting", as follows.

1.6. **Duality: dissociation.** A *dissociation* of a space $(\mathcal{Y}, \top)$ is the disjoint union $\mathcal{X}$ of two copies $\mathcal{X}^+$ and $\mathcal{X}^-$ of $\mathcal{Y}$, where we define a transversality relation on $\mathcal{X}$ by declaring, for $x \in \mathcal{X}^\pm$, the set $x^\top$ to be the set of elements $a$ in $\mathcal{X}^\mp$ such that $a$ and $x$ are transversal in $\mathcal{Y}$. Obviously, this defines a transversality relation on $\mathcal{X}$, and $\mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^-$ is a splitting.

## 2. Jordan structure maps

2.1. **Structure maps in general.** Assume $(\mathcal{X}, \top)$ is a space with transversality relation, and let $n \in \mathbb{N}$. An $n+1$-*ary structure map (with domain $\mathcal{D}_n$)* is a map

$$S : \mathcal{D}_n \to \mathrm{End}(\mathcal{X}, \top)$$

attaching to each chain $x = (x_1, \ldots, x_n)$ a map $S(x) : \mathcal{X} \to \mathcal{X}$ preserving transversality. In the sequel we will mainly consider structure maps such that $S(x)$ is a bijection, and the case of ternary and quaternary structure maps will be most important: for a ternary structure map we use also the notation

$$S_x^a := S(x, a),$$

and for a quaternary structure map

$$S_a^{xz} := S(x, a, z).$$

Sometimes we view $S$ as a map of $n+1$ arguments, defined by

$$S(x_1, \ldots, x_n, x_{n+1}) := (S(x_1, \ldots, x_n))(x_{n+1}) = S_{x_2 x_4 \ldots}^{x_1 x_3 \ldots}(x_{n+1})$$

*Structure maps with domain* $\mathcal{D}_n'$ are defined similarly.

*Morphisms of spaces with structure map* are maps preserving transversality and commuting with structure maps in the obvious sense. The group of automorphisms of $(\mathcal{X}, \top, S)$ is denoted by $\mathrm{Aut}(\mathcal{X})$, $\mathrm{Aut}(\mathcal{X}, S)$, or $\mathrm{Aut}(\mathcal{X}, \top, S)$, according to the context. Other categorial notions can be defined for spaces with structure maps, such as *subspaces, direct products...*

2.1.1. *Structure maps and duality.* If $\mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^-$ is a splitting of $\mathcal{X}$, then let $\mathcal{D}_n^\pm$ be the set of chains of length $n$ starting in $\mathcal{X}^\pm$, so that $\mathcal{D}_n = \mathcal{D}_n^+ \cup \mathcal{D}_n^-$. Then, by restriction to $\mathcal{D}_n^\pm$, a structure map $S$ gives rise to two parts of $S^\pm$ of the structure map. Thus one recovers the notation used in [Be02].

2.2. **Jordan structure map.** A *Jordan structure map* on a space $(\mathcal{X}, \top)$ is a quaternary structure map

$$J : \mathcal{D}_3 \to \mathrm{End}(\mathcal{X}, \top), \qquad (x, a, z) \mapsto J_a^{xz}$$

such that the following *Jordan identities* hold:

(IN) *involutivity*: $J_a^{xz} \circ J_a^{xz} = \mathrm{id}_{\mathcal{X}}$

(IP) *idempotency*: $J_c^{ab}(c) = c$, $J_c^{ab}(a) = b$, $J_c^{ab}(b) = a$

(A) *associativity:* $J_c^{xz} J_c^{uv} J_c^{ab} = J_c^{J_c^{xa}(v), J_c^{bz}(u)}$

(D) *distributivity:* $J_c^{xz} \circ J_b^{uv} \circ J_c^{xz} = J_{J_c^{xz}(b)}^{J_c^{xz}(u), J_c^{xz}(v)}$, that is, $J_c^{xz} \in \mathrm{Aut}(\mathcal{X}, J)$,

(C) *commutativity:* $J_c^{ab} = J_c^{ba}$

(S) *symmetry:* $J_a^{xx} = J_x^{aa}$.

Recall (introduction) that such structures "arise in nature", among others, for Grassmannians, Lagrangians, and projective quadrics.

2.3. **First properties: torsor and symmetric space actions.**

**Lemma 2.1** (Torsor action). *Given a Jordan structure map, the set* $U_a$ *with product*

$$(xyz)_a := J_a^{xz}(y). \tag{2.1}$$

*is a commutative torsor, and the map* $U_a \times U_a \to \mathrm{Bij}(\mathcal{X})$, $(x, z) \mapsto J_a^{xz}$ *is an inversive symmetry torsor action.*

*Proof.* If $x, y, z \top a$, then also $J_a^{xz}(y) \top J_a^{xz}(a) = a$, hence $U_a$ is stable under $(xyz)_a$. Now the idempotent identity of a torsor is the first identity of (I), and para-associativity follows from lemma A.2. The properties of an inversive symmetry torsor action are precisely the axioms (A) and (C). $\square$

As a useful application of the lemma, by Appendix A.5, we have the following *transplantation formula* for the symmetries: for all $x, o, z \top a$,

$$J_a^{xz} = J_a^{xo} J_a^{oo} J_a^{zo} = J_a^{J_a^{xz}(o),o}. \tag{2.2}$$

**Lemma 2.2** (Symmetric space action)**.** *For all $x \in U_{ab}$, the restricted maps*

$$J_x^{ab} : U_a \to U_b, \quad y \mapsto J_x^{ab}(y), \qquad J_x^{ab} : U_b \to U_a, \quad y \mapsto J_x^{ab}(y),$$

*are $\mathbb{Z}$-affine (i.e., a homomorphism of abelian torsors), and the set $U := U_{ab}$ is stable under the map*

$$\mu := \mu_{ab} : U \times U \to U, \quad (x, y) \mapsto \mu(x, y) := s_x(y) := J_x^{ab}(y)$$

*which turns it into a reflection space, and this reflection space has a symmetry action on $\mathcal{X}$ given by $S_x := J_x^{ab}$.*

*Proof.* The restricted maps are well-defined: this is shown as in the preceding proof. As mentioned in the statement of axiom (D), distributivity means that $g := J_x^{ab}$ is an automorphism of $J$, and since it exchanges $a$ and $b$, we get

$$g\big((uvw)_a\big) = (gu\, gv\, gw)_{Ja} = (gu\, gv\, gw)_b$$

and similarly for $a$ and $b$ exchanged. The same argument shows that $J_x^{ab}$ is an automorphism of $\mu$. Summing up, $U_{ab}$ is a reflections space, and it acts on $\mathcal{X}$ by a symmetry action. $\square$

We say that $U_{ab}$ with product $\mu = \mu_{ab}$ is the *symmetric space associated to the pair* $(a, b)$ (there is a slight abuse of language: we anticipate implicitly that property (T) of a symmetric space is also valid; this will be ensured by regularity assumptions to be introduced later, see Definition 7.1).

**Lemma 2.3** (Compatibilty)**.** *The symmetric space $U_{aa}$ is the same as the abelian group $U_a$ with its usual inversion maps.*

*Proof.* This follows directly from the symmetry property (S): in $U_{aa}$ the symmetric element of $y$ with respect to $x$ is $s_x(y) = J_x^{aa}(y)$, and in $U_a$ it is $(xyx)_a = J_a^{xx}(y)$. $\square$

**Theorem 2.4** (The polarized symmetric space)**.** *The set $\mathcal{D}_2$ of transversal pairs becomes a symmetric space with the law*

$$s_{(x,a)}(y, b) := \big(J_a^{xx}(y), J_x^{aa}(b)\big).$$

*The same formula defines a symmetry action of the symmetric space $\mathcal{D}_2$ on $\mathcal{X}^2$. The exchange map $\tau : \mathcal{X}^2 \to \mathcal{X}^2$, $(x, a) \mapsto (a, x)$ is an automorphism of $\mathcal{D}^2$ and of the action.*

*Proof.* Everything follows easily from the axioms (In), (IP, (D). $\square$

The symmetric space $\mathcal{D}_2$ contains flat subspaces for $a = b$ fixed (or $x = y$ fixed), but it is not flat itself. It corresponds to the *twisted polarized symmetric spaces* from [Be00]: it has a local product structure, but not a global one.

2.4. **Some categorial notions.** Most categorial notions are defined in an obvious way – cf. [Be02, BeL08], and we refer to loc. cit. for more details:

2.4.1. *Morphisms.* It follows directly from the definitions that $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of Jordan structure maps if, and only if, for all $a \in \mathcal{X}$, the restriction $U_a \to U_{f(a)}$, $x \mapsto f(x)$, is a $\mathbb{Z}$-affine map (= morphism of abelian torsors), if, and only if, for all $x, z \in \mathcal{X}$, the restriction $U_{xz} \to U_{f(x),f(z)}$ is a homomorphism of symmetric spaces. Therefore, if we denote, for $x \in \mathcal{X}$, resp. $(x, a) \in \mathcal{D}_2$, by

$$P_x := \{g \in \mathrm{Aut}(\mathcal{X}) \mid g(x) = x\}, \qquad H := H_{x,a} := P_x \cap P_a \tag{2.3}$$

the stabilizer of $x$, resp. of $(x, a)$, then $P_a$ acts $\mathbb{Z}$-affinely on $U_a$, and $H_{x,a}$ acts $\mathbb{Z}$-linearly on $(U_a, x)$ and on $(U_x, a)$.

2.4.2. *Duality:* In presence of a splitting, we define structure maps $J^\pm$, see above.

2.4.3. *Direct products:* direct product of transversality and of structure maps

2.4.4. *Subspaces:* subsets stable under structure maps

2.4.5. *Intrinsic subspaces (inner ideals):* subsets $\mathcal{Y} \subset \mathcal{X}$ such that $J_a^{xz}(y) \in \mathcal{Y}$ whenever $x, y, z \in \mathcal{Y}$ and $a \in \mathcal{X}$; this can be interpreted in two ways: $U_a \cap \mathcal{Y}$ is an affine subspace, for all $a \in \mathcal{X}$ (point of view taken in [BeL08]), or: $\mathcal{Y}$ is an invariant subspace of the symmetry action of $U_{xz}$, for all $x, z \in \mathcal{Y}$.

2.4.6. *Flat geometries:* given by two abelian groups $(V_1, +), (V_{-1}, +)$, $\mathcal{X} = V_1 \cup V_{-1}$ (disjoint union), $a \top x$ iff $a \in V_{\pm 1}$ and $x \in V_{\mp 1}$, $J_a^{xz}(y) = x - y + z$, $J_a^{xz}(b) = 2a - b$ for $x, y, z \in V_{\pm 1}$, $a, b \in V_{\mp 1}$.

2.4.7. *Congruences and quotient spaces:* defined as in [Be02], following [Lo69], III.2.

2.4.8. *Polarities.* A *polarity* is an automorphism $p \in \mathrm{Aut}(\mathcal{X})$ which is of order two: $p^2 = \mathrm{id}_{\mathcal{X}}$, and having *non-isotropic elements*: there is $x$ such that $p(x) \top x$. In other words, $(x, p(x)) \in \mathcal{D}_2$, so the graph of $p$ has non-empty intersection with $\mathcal{D}_2$.

**Theorem 2.5.** *Assume $p$ is a polarity of $(\mathcal{X}, \top, J)$. Then the set*

$$\mathcal{X}^{(p)} = \{x \in \mathcal{X} \mid p(x) \top x\},$$

*is stable under the law $(x, y) \mapsto J_{p(x)}^{xx}(y)$, which turns it into a symmetric space.*

*Proof.* Indeed, this space can be realized as subsymmetric space of the polarized space $\mathcal{D}_2$ (Theorem 2.4) fixed under the involution $p\tau = \tau p$, by the imbedding $x \mapsto (x, p(x))$. (This is the analog of [Be02], Theorem 4.2). $\square$

The symmetric spaces $\mathcal{X}^{(p)}$ are in general not homogeneous under $\mathrm{Aut}(\mathcal{X})$: the *symmetric spaces with base point* $(\mathcal{X}^{(p)}, x)$, $(\mathcal{X}^{(p)}, y)$ are called *isotopic*; in general they are not isomorphic to each other. Note that the *space of polarities*,

$$\mathrm{Pol}(\mathcal{X}) := \{p \in \mathrm{Aut}(\mathcal{X}) \mid p^2 = \mathrm{id}_{\mathcal{X}}, \exists x \in \mathcal{X} : (x, p(x)) \in \mathcal{D}_2\},\tag{2.4}$$

is itself a reflection space: it is stable under $(p, q) \mapsto pqp$. If $(a, b, c)$ is a pairwise transversal triple, then $p = J_c^{ab}$ is a polarity, called an *inner polarity* (cf. [Be03]).

## 3. Translations

3.1. **Definition of translations.** Fix $a \in \mathcal{X}$. We have seen that the torsor $U_a$ acts on $\mathcal{X}$ by a symmetry action, and according to Lemma A.4, it acts therefore also by a left- and by a right action on $\mathcal{X}$; since the symmetry action is commutative, left- and right action coincide (Lemma A.4): for all $x, z \in V_a$, the map

$$L_a^{xz} := J_a^{xu} J_a^{uz} = J_a^{ux} J_a^{zu} \tag{3.1}$$

called *(left) $a$-translation*, does not depend on the choice of $u \in a^\top$ (in particular, we may choose $u = x$ or $u = z$). We can speak of "the" *$a$-translation group*

$$V_a := \{L_a^{xz} \mid x, z \in a^\top\}. \tag{3.2}$$

Since $L_a^{xz}(a) = a$, the $a$-translations form a commutative subgroup of $P_a$, isomorphic to $(U_a, o)$, for any origin $o \in U_a$. Note that, if $x, y, z \in U_a$, then we have usual properties of torsors, such as "Chasles relation", and the link with the symmetries:

$$L_a^{xy} L_a^{yz} = L_a^{xz}, \quad (L_a^{xy})^{-1} = L_a^{yx}, \quad L_a^{xy}(z) = (xyz)_a = J_a^{xz}(y). \tag{3.3}$$

3.2. **Base points, quasi-translations, and exact sequence of stabilizers.** A *base point* is a fixed transversal pair $(x, a)$; we then often write $(o, o')$ or $(o^+, o^-)$. Since $P_a$ acts affinely, and $H_{x,a}$ linearly on $V_a$, we may decompose $P_a$ into a semidirect product of $H_{a,x}$ with $V_a$. More formally, this can be stated by saying that $V_a$ is normal in $P_a$ (since $g \circ L_a^{xz} \circ g^{-1} = L_{ga}^{gx,gz}$), and hence the sequence

$$0 \to V_a \to P_a \to H_{x,a} \to 0, \qquad g \mapsto L_a^{x,gx} \circ g \tag{3.4}$$

associating to $g$ its "linear part", is well-defined and splits via the natural inclusion of $H_{x,a}$ in $P_a$. For a fixed base point $(o, o')$, and following a standard terminology from Jordan theory, the elements $L_o^{ao'}$, are said to act as *quasi inverses* on $V = V_{o'}$; the notation $x^a := L_o^{ao'}(x)$ is often used in Jordan theory.[2]

3.3. **Triple decomposition in $G$, and denominators.** Fix a base point $(x, a) = (o, o') \in \mathcal{D}_2$ and let $(V, V') = (U_{o'}, U_o)$. The conditions $g.o\top o'$ and $g(o) \in V$ are equivalent. We define the *big cell of* $\operatorname{Aut}(\mathcal{X})$

$$\Omega := \Omega^{o,o'} := \{g \in \operatorname{Aut}(\mathcal{X}) \mid g(o) \top o'\} \tag{3.5}$$

(then $o \top g^{-1}(o')$, whence $(\Omega^{o',o} = (\Omega^{o,o'})^{-1})$. For $g \in \Omega$, we define the *denominator (with respect to $(o, o')$)* by

$$D(g) := D^{o,o'}(g) := L_{o'}^{o,g(o)} \circ g \circ L_o^{g^{-1}(o'),o'} \in \operatorname{Aut}(\mathcal{X}). \tag{3.6}$$

The element $D(g)$ stabilizes $o$ and $o'$: $D(g).o = L_{o'}^{o,g(o)}(g(o)) = o$, $D(g)o' = L_{o'}^{o,g(o)} g(g^{-1}(o')) = o'$, and therefore $D(g)$ acts $\mathbb{Z}$-linearly on $V$ and on $V'$. (We will see later that it is a substitute of a "differential of $g$ at $o$".) By (3.6), every element $g \in \Omega$ admits a *triple decomposition* into a translation, a $\mathbb{Z}$-linear part, and a quasi-translation:

$$g = L_{o'}^{t,o} h L_o^{t',o'}, \qquad \text{with } t \in V, h \in H_{o,o'}, t' \in V'. \tag{3.7}$$

[2] From our point of view, it would seem more appropriate to call $L_o^{ao'}(x)$ a *quasi-translation*, and to reserve the term *quasi-inverse* for $J_o^{ao'}(x)$, which is of order two, as an inverse should be.

This decomposition is unique: necessarily, $t = g(o)$, $t' = -g^{-1}(o')$ and hence $h = D(g)$. The denominators satisfy the *cocyle relation*

$$D(gh) = D(gL_{o'}^{ho,o})D(h) \tag{3.8}$$

Indeed, if $h = sD(h)s'$ and $gs = tD(gs)t'$, where $s = L_{o'}^{ho,o}$, then

$$gh = gsD(h)s' = tD(gs)D(h)s'$$

whence, by uniqueness of the decomposition, $D(gh) = D(gs)D(h)$.

3.4. **Transitivity, and generators of $G$.** The projective group $\mathbb{P}\mathrm{GL}(p+q,\mathbb{K})$ acts transitively on $\mathrm{Gras}_p(\mathbb{K}^{p+q})$, but not on $\mathrm{Gras}_p(\mathbb{K}^{p+q}) \cup \mathrm{Gras}_q(\mathbb{K}^{p+q})$ (since it preserves dimension) – unless $p = q$, in which case the geometry is self-dual. The following result generalizes these observations:

**Theorem 3.1** (Transitivity)**.** *Assume $(\mathcal{X}, \top, J)$ is connected and fix a base point $(o, o')$. If $\mathcal{X}$ is self-dual, then the action of $G(\mathcal{X})$ on $\mathcal{D}_2$ and on $\mathcal{X}$ is transitive:*

$$\mathcal{D}_2 = G(\mathcal{X})/H_{o,o'}, \qquad \mathcal{X} = G(\mathcal{X})/P_o \cong G(\mathcal{X})/P_{o'}\,,$$

*and every element $g \in \mathrm{Aut}(\mathcal{X})$ has a decomposition*

$$g = L_{o'}^{x_1 o} L_o^{a_1 o'} \ldots L_{o'}^{x_n o} L_o^{a_n o'} h$$

*with $x_1, \ldots, x_n \in V = U_{o'}$, $a_1, \ldots, a_n \in V' = U_o$, and $h \in H_{o,o'}$. This decomposition is in general not unique. If $\mathcal{X}$ admits a splitting $(\mathcal{X}^+, \mathcal{X}^-)$, then the action of $G(\mathcal{X})$ on $\mathcal{D}_2^+$, on $\mathcal{X}^+$ and on $\mathcal{X}^-$ is transitive:*

$$\mathcal{D}_2^+ = G(\mathcal{X})/H_{o,o'}, \qquad \mathcal{X}^+ = G(\mathcal{X})/P_{o'}, \qquad \mathcal{X}^- = G(\mathcal{X})/P_o,$$

*and every element of $\mathrm{Aut}(\mathcal{X}^+, \mathcal{X}^-)$ has a decomposition as above.*

*Proof.* Both claims are proved by induction on length of chains joining two points. Assume that $(x, a, y, b)$ is a chain, so $x \top a$, $a \top y$, $y \top b$. Then the element

$$\Lambda := \Lambda_{yx}^{ba} := L_y^{ba} \circ L_a^{yx} \tag{3.9}$$

has the properties $\Lambda(a) = L_y^{ba}(a) = b$ and $\Lambda(x) = L_y^{ba}(y) = y$, and hence maps $(a, x)$ to $(b, y)$. Now the transitivity result follows by induction on the length of chains. Note, moreover, that $\Lambda$ may be rewritten in the form

$$\Lambda = L_y^{ba} \circ L_a^{yx} = L_y^{ba} \circ L_x^{L_y^{ab} y, x} = L_y^{ba} \circ L_x^{L_y^{ab} y, x} \circ L_y^{ab} \circ L_y^{ba} = L_b^{y, L_y^{ba}(x)} \circ L_y^{ba}, \tag{3.10}$$

thus (if $(y, b) = (o, o')$ is the base point) expressing $\Lambda$ by an element of the desired form. Again, the general decomposition now follows by induction. □

It follows that, if $\mathcal{X}$ is conntected, all involutions $J_a^{xx}$ are conjugate to each other under $\mathrm{Aut}(\mathcal{X})$. See Remark 5.3 for a sufficient condition that ensures that also all $J_a^{xz}$ are conjugate to each other.

*Remark.* If $(\mathcal{X}^+, \mathcal{X}^-)$ is connected of stable rank one, the decomposition from the theorem exists with $n = 2$ and $a_2 = o'$ (so we can write $G = VV'VH$; in the case of Hermitian symmetric spaces this is called "Harish-Chandra decomposition").

*Remark.* In the preceding statements and proofs, we could have replaced the letter "$L$" by "$J$"; for instance, we have

$$\begin{aligned} \Lambda^{ab}_{xy} = L^{ab}_x L^{xy}_b &= J^{ab}_x J^{bb}_x J^{xx}_b J^{xy}_b = J^{ab}_x J^{xy}_b \\ &= J^{ab}_x J_b^{J^{ab}_x(x),y} = J_a^{x,J^{ab}_x(y)} J^{ab}_x \\ &= J_x^{aJ^{xy}_b(b)} J^{xy}_b = J^{xy}_b J_y^{J^{xy}_b(a),b} . \end{aligned} \tag{3.11}$$

3.5. **Bergman operator.** A *quasi-invertible quadruple* is a closed chain of length four, $(a,x,b,y) \in \mathcal{D}'_4$. By closedness, we can define the element $\Lambda^{yx}_{ba} = L^{yx}_b L^{ba}_x$, having the same effect on $(a,x)$ as the element $\Lambda^{ba}_{yx}$ from the preceding proof. It follows that $(\Lambda^{yx}_{ba})^{-1}\Lambda^{ba}_{yx}$ stabilizes $(a,x)$, i.e., belongs to $H_{x,a}$. This leads us to define, for $(a,x,b,y) \in \mathcal{D}'_4$, the *Bergman operator*

$$B^{xa}_{yb} := (\Lambda^{yx}_{ba})^{-1}\Lambda^{ba}_{yx} = L^{ab}_x L^{xy}_b L^{ba}_y L^{yx}_a = \Lambda^{ab}_{xy}\Lambda^{ba}_{yx} \in H_{xa} . \tag{3.12}$$

According to (3.11), we have also the expression

$$B^{xa}_{yb} = J^{ab}_x J^{xy}_b J^{ba}_y J^{yx}_a . \tag{3.13}$$

Note that

$$(B^{yb}_{xa})^{-1} = B^{by}_{ax}. \tag{3.14}$$

Obviously, the fourfold map $B$ is invariant under automorphisms $g \in \mathrm{Aut}(\mathcal{X}, J)$. If $(x,a) = (o,o')$ is chosen as base point, we also use the notation from Jordan theory

$$\beta(y,b) := B^{o,o'}_{yb} . \tag{3.15}$$

**Lemma 3.2.** *Fix $(x,a) =: (o,o')$ as base point. Then $\beta(y,b)$ is a denominator, namely $\beta(y,b) = D(L^{o'b}_o L^{yo}_{o'})$. In other terms, we have the relation*

$$L^{o'b}_o \circ L^{yo}_{o'} = L_{o'}^{L^{o'b}_o(y),o} \circ \beta(y,b) \circ L_o^{o',L^{oy}_{o'}(b)} .$$

*In a similar way,*

$$J^{o'b}_o \circ J^{yo}_{o'} = J_{o'}^{J^{o'b}_o(y),o} \circ \beta(y,-b) \circ J_o^{o',J^{oy}_{o'}(b)} .$$

*Proof.* As in (3.11), $\beta(y,b) = L^{o'b}_o L^{oy}_b L^{bo'}_y L^{yo}_{o'} = L_{o'}^{o,L^{o'b}_o(y)} L^{o'b}_o L^{yo}_{o'} L_o^{L^{oy}_{o'}(b),o'} = D(g)$ for $g = L^{o'b}_o L^{yo}_{o'}$. □

3.6. **The canonical atlas.** Every Jordan geometry comes with a canonical atlas (for the general notion of atlas, see subsection B.5). The data of an atlas require a fixed (and not only: "fixed up to isomorphism") model space, so we have to fix a base point $(o,o') \in \mathcal{D}_2$. For simplicity, let us assume that $\mathcal{X}$ is connected (if not, decompose $\mathcal{X}$ first into connected components). Then we call *model space* the pair of abelian groups

$$V := (U_{o'}, o), \qquad V' := (U_o, o'). \tag{3.16}$$

We define the *index set* to be the group $G := G(\mathcal{X})$, and for every $g \in G$ we define the *chart domains* in $\mathcal{X}$ by

$$U_g := g(V), \qquad U'_g := g(V').$$

Then $\mathcal{X}$ is covered by chart domains, in the following sense: if $\mathcal{X}$ admits a splitting, $\mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^-$, then, by the transitivity result,

$$\mathcal{X}^+ = \bigcup_{g \in G} U_g, \qquad \mathcal{X}^- = \bigcup_{g \in G} U_g'.$$

If $\mathcal{X}$ does not admit a splitting, then $\bigcup_{g\in G} U_g = \bigcup_{g \in G} U_g' = \mathcal{X}$. In both cases, define subsets of $V$ by

$$V_{g,h} := V \cap hg^{-1} V,$$

let $\phi_g : U_g \to V$, $x \mapsto g^{-1}(x)$, and $\phi_g' : U_g' \to V'$, $x \mapsto g^{-1}(x)$. Then the data

$$\mathcal{A} := (\mathcal{X}, U_g, \phi_g)_{g \in G}$$

form an *atlas of $\mathcal{X}^+$, with model space $V$*, in the sense of subsection B.5. The transition functions are

$$\phi_{g,h} : V_{h,g} \to V_{g,h}, \quad v \mapsto gh^{-1}(v).$$

This atlas depends in an inessential way on the choice of $(o, o')$, since the action of $G$ is transitive on $\mathcal{D}_2$.

3.7. **Transvections.** For a closed quadruple $(x, a, z, b)$ we define the *transvection*

$$Q^{ab}_{xz} := J^{ab}_x J^{ab}_z \,. \tag{3.17}$$

This is the transvection $Q_{xz} = S_x S_z$ associated to the symmetry action of the symmetric space $U_{ab}$ on $\mathcal{X}$ (see Definition A.11). It maps $a$ to $a$ and $b$ to $b$, hence acts affinely both on $U_a$ and on $U_b$. These operators satisfy the fundamental formula (A.5), and we have $(Q^{ab}_{xz})^{-1} = Q^{ab}_{zx} = Q^{ba}_{zx}$. For $a = b$, $Q^{aa}_{xz}$ is a (square of) translations for the torsor $U_a$:

$$Q^{aa}_{xz} = J^{aa}_x J^{aa}_z = J^{xx}_a J^{xz}_a J^{xz}_a J^{zz}_a = (L^{xz}_a)^2.$$

Fixing $(z, b) = (o, o')$ as base point, one may thus think of $Q^{ab}_{xz}$ as a sort of "deformation" of the translation operator $(L^{x,o}_{o'})^2(y) = 2x + y$.

## 4. Associative structure maps

4.1. **Spaces with associative structure map.** An *associative structure map* on a set $\mathcal{X}$ with transversality relation $\top$ is a pentary structure map

$$M : \mathcal{D}_4' \to \mathrm{End}(\mathcal{X}), \qquad (x, a, z, b) \mapsto M^{ab}_{xz}$$

(we write also $(xyz)_{ab} := M^{ab}_{xz}(y)$) such that the following identities hold

1. *symmetry*: $M^{ab}_{xz} = M^{xz}_{ab} = M^{zx}_{ba}$,
2. *idempotency*: $M^{ab}_{xz}(x) = z$, $M^{ab}_{xz}(z) = x$, $M^{ab}_{xz}(b) = a$ and $M^{ab}_{xz}(a) = b$,
3. *inverse:* $M^{xz}_{ab} \circ M^{zx}_{ab} = \mathrm{id}_{\mathcal{X}}$,
4. *associativity:* $M^{xz}_{ab} M^{uv}_{ab} M^{rs}_{ab} = M^{(xvr)_{ab},(suz)_{ab}}_{ab}$,
5. *distributivity:* $M^{ab}_{xz} \circ M^{cd}_{uv} \circ (M^{ab}_{xz})^{-1} = M^{(xcz)_{ab},(xdz)_{ab}}_{(xuz)_{ab},(xvz)_{ab}}$ (i.e., $M^{ab}_{xz} \in \mathrm{Aut}(M)$).

From idempotency and associativity, it follows that the set $U_{ab}$ is stable under the ternary map $(x, y, z) \mapsto (xyz)_{ab}$, and that it forms a torsor with this law; the symmetry law implies that $U_{ba} = U_{ab}$ as sets, but with torsor structures opposite to each other. In particular, $U_a = U_{aa}$ is commutative. Associativity now says that the map

$$U_{ab} \times U_{ab} \to \mathrm{Bij}(\mathcal{X}), \qquad (x, z) \mapsto M^{ab}_{xz}$$

is a symmetry action of $U_{ab}$ on $\mathcal{X}$, and hence, by Lemma A.4, we have associated left and right actions of the torsor $U_{ab}$ on $\mathcal{X}$ given by

$$L^{ab}_{xy} := M^{ab}_{xu} \circ M^{ab}_{uy}, \qquad R^{ab}_{yz} := M^{ab}_{uy} \circ M^{ab}_{zu}. \tag{4.1}$$

Spaces with associative structure map form a category in the obvious way. Many categorial notions can be defined exactly as in the case of Jordan structure maps, see 2.4 above. The most important difference is that now, at several places, we have to distinguish between "left" and "right": besides the inner ideals (intrinsic subspaces), we also have *left and right ideals*, that is, subspaces $\mathcal{Y}$ that are invariant under the left, resp. right actions of the torsors $U_{ab}$, whenever $a, b \in \mathcal{Y}$; and besides homomorphisms, we also have *antihomomorphisms* (see [BeKi09b]).

4.2. **Grassmannian geometries.** The main example of an associative geometry is given by *Grassmannians*: we equip the Grassmannian $\mathrm{Gras}(W)$ with the transversality relation described in section 1.2.

**Theorem 4.1.** *The structure map $M$ defined on the Grassmannian by Equation (0.3): $M^{xz}_{ab}(y) = (P^a_x - P^z_b)(y)$, is an associative structure map. Here, one may also consider the Grassmannian of submodules in an a module over a possibly non-commuative ring.*

The (elementary) proof has been given in [BeKi09a].

4.3. **Special Jordan geometries.**

**Lemma 4.2** (The associative-to-Jordan functor)**.** *If $(\mathcal{X}, \top, M)$ is a geometry with associative structure map, then $(\mathcal{X}, \top, J)$ is a geometry with Jordan structure map, where*

$$J^{ab}_x := M^{ab}_{xx},$$

*and the correspondence $(\mathcal{X}, M) \mapsto (\mathcal{X}, J)$ is functorial.*

*Proof.* Easy check of definitions. □

Note that the functor is defined by restriction to the diagonal $x = z$, just like the functor from Lie torsors to symmetric spaces or from associative to Jordan algebras. A *special Jordan structure map* is the restriction of the map defined by the lemma to some subspace $\mathcal{Y} \subset \mathcal{X}$ which is stable under $J$. For instance, Lagrangian geometries (cf. Introduction, 0.3.2) are of this form; using Clifford algebras, one can show that the structure map defined for projective quadrics (cf. 0.3.3) is also special.

4.4. **Self-dual geometries, and link with associative algebras.** A geometry with associative structure map is called *(strongly) self-dual* if it contains a closed transversal triple $(a,b,c)=(o,o',e)$. Then let $V:=V_{o'}$, $V':=V_o$ and $V^\times := U_{oo'} = V\cap V'$; this set is a group with origin $e$ and group law

$$xz = (xez)_{oo'} = M^{oo'}_{xz}(e) = L^{oo'}_{xe}(z) = R^{oo'}_{ez}(x). \tag{4.2}$$

The left translation operator $L^{ab}_{xy}$ defined by (4.1) maps $a$ to $a$ and $b$ to $b$, hence defines by restriction affine bijections of $U_a$, resp. of $U_b$, onto itself, and hence the group law defined by (4.2) is $\mathbb{Z}$-linear with respect to both arguments $x$ and $z$. Using algebraic infinitesimal calculus (under assumption of smoothness), we will show below (Theorem 9.7) that this group law extends to a bilinear associative algebra structure on $V$ (and every associative algebra arises in this way, see [BeKi09a]).

## 5. Scalar action and major dilations

From now on, we will fix a *base ring* $\mathbb{K}$: this ring is assumed to be commutative and unital, with unit element denoted by 1.

5.1. **Scalar action.** Let $(\mathcal{X},\top,J)$ be a geometry with Jordan structure map. A $\mathbb{K}$-*scalar action* on these data is given by a *compatible* $\mathbb{K}$-*module structure* on the abelian group $(U_a,y)$, for every pair $(y,a)\in\mathcal{D}_2$, that is: $U_a$ is a $\mathbb{K}$-module with origin $y$ and underlying abelian group structure given by $x+z=J^{xz}_a(y)=(xyz)_a$, and scalar multiplication denoted by

$$\mathbb{K}\times U_a\to U_a,\quad (r,z)\mapsto r^a_y(z),$$

and which is *compatible with* $J$ in the sense that, for invertible scalars $r$, the following properties connecting addition and multiplication by scalars in a $\mathbb{K}$-module *extend* to the whole of $\mathcal{X}$: for every $r\in\mathbb{K}^\times$ and $(y,a)\in\mathcal{D}_2$ there is a bijection $r^a_y:\mathcal{X}\to\mathcal{X}$; in other words, there is a *scalar action structure map*

$$\mathbf{r}:\mathbb{K}^\times\times\mathcal{D}_2\to \mathrm{Bij}(\mathcal{X}),\qquad (r;y,a)\mapsto r^a_y \tag{5.1}$$

which is supposed to satisfy:

- (C) *compatibility:* for $x,y\in U_a$, $r^a_y(x)=rx$ is multiplication by the scalar $r$ in $(U_a,y)$,
- (A) *associativity:* for $y$ fixed, the map $\mathbb{K}^\times\times\mathcal{X}\to\mathcal{X}$, $(r,x)\mapsto r^a_y(x)$ is an action: for all $r,s\in\mathbb{K}^\times$, we have $r^a_y\circ s^a_y=(rs)^a_y$ and $1^a_y=\mathrm{id}_{\mathcal{X}}$,
- (Du) *duality:* $(r^{-1})^a_y = r^y_a$
- (Di) *distributivity:* $r^a_y$ is an automorphism of $J$: $r^a_y\circ J^{xz}_b\circ(r^a_y)^{-1}=J^{r^a_y x, r^a_y z}_{r^a_y b}$, and similarly, $J^{xz}_a$ is an automorphism of $\mathbf{r}$: $J^{xz}_a\circ r^b_y\circ J^{xz}_a = r^{J^{xz}_a(b)}_{J^{xz}_a(y)}$,
- (Tr) *link with translations:* $(-1)^a_x = J^{aa}_x$ and $r^a_x(r^a_y)^{-1}=L^{x,r^a_y x}_a = L^{r^a_x y,y}_a$.

We can change base points in $U_a$ by usual formulas from affine geometry: if $o$ is an origin in $U_a$, and $rx:=r^a_o(x)$ multiplication by $r$ in the $\mathbb{K}$-module $(U_a,o)$, then

$$r^a_y(x)=(1-r)y+rx.$$

Property (Du) is motivated by the example of *projective spaces* (cf. introduction, 0.3.1): multiplication by the scalar $r$ in the linear part $(U_a, x)$ in a projective space $\mathrm{Gras}_1(V)$ is given by $r_x^a = P_x^a + rP_a^x$, where $a$ is a hyperplane in $V$ and $x$ a line complementary to $a$. Multiplying by the scalar $r^{-1}$ gives the same projective map, whence $r_x^a = r^{-1}P_x^a + P_a^x = (r^{-1})_a^x$. (Property (Du) is the "Fundamental Identity (PG1)" from [Be02]; the identity (PG2) from loc. cit. does not appear in the axiomatics given here since it concerns possibly non-invertible maps.)

5.2. **Major and minor dilations.** There are a lot of identies relating the "major" dilations $r_x^a$ with the "minor" dilations (translations $L_c^{yz}$). Most of them, such as (Tr), just rephrase and globalize relations from usual affine geometry over $\mathbb{K}$ (cf. [Be04]). In the sequel, we will focus on the relation between scalar action and "usual" translations, on the one hand, and "quasi-translations", on the other hand: fix a base point $(o, o') \in \mathcal{D}_2$; then the usual scalar action in the linear space $(V, o) = (U_{o'}, o)$ is given by $rv = r_o^{o'}(v)$, and the one in the linear space $(V', o') = (U_o, o')$ by $ra = r_{o'}^o(a)$. For $v \in V$ we have by (Di)

$$r_o^{o'} \circ L_{o'}^{vo} \circ (r_o^{o'})^{-1} = L_{o'}^{rv,o}, \tag{5.2}$$

which coresponds to the semidirect product structure of the usual affine group of $V$. For $a \in V'$ we have, again by (Di),

$$r_o^{o'} \circ L_o^{ao'} \circ (r_o^{o'})^{-1} = L_o^{r^{-1}a,o'}, \tag{5.3}$$

which means that the "quasi-translation" $x^a := L_o^{ao'}(x)$ for $x \in V$, $a \in V'$ satisfies the homogeneity relation $(rx)^a = rx^{ra}$.

5.3. **Remark on midpoints.** Assuming that 2 is invertible in $\mathbb{K}$, *midpoints*

$$\mu(y, a, x) := (2_y^a)^{-1}(x) = \frac{x+y}{2} \tag{5.4}$$

in the affine space $U_a$ have been extensively used in [Be02]: relation (Tr) implies

$$\begin{aligned} 2_x^a (2_y^a)^{-1} &= L_a^{x, 2_y^a(x)} = L_a^{x, 2x-y} = L_a^{yx}, \\ J_a^{\mu(x,a,z), \mu(x,a,z)} &= (2_x^a)^{-1} J_a^{zz} (2_x^a) = (2_x^a)^{-1} 2^a_{J_a^{zz}(x)} J_a^{zz} = L_a^{x,z} J_a^{zz} = J_a^{xz}, \end{aligned} \tag{5.5}$$

so translations and inversions $J_a^{xz}$ can be expressed by major dilations. Moreover, by (5.5), $J_a^{xz} = J^{aa}_{\mu(x,a,z)}$, thus every inversion is of the form $J_v^{aa}$ for some $(a, v) \in \mathcal{D}_2$; it follows that (if $\mathcal{X}$ is connected) all inversions $J_a^{xz}$ are conjugate to each other under $\mathrm{Aut}(\mathcal{X})$. In the present approach, we avoid using midpoints, and we consider the map $J$ as a primary object and $\mathbf{r}$ as secondary.

5.4. **Remark on the base ring $\mathbb{Z}$.** A geometry with Jordan structure map $J$ always carries a $\mathbb{Z}$-scalar action: indeed, an abelian group $(U_a, y)$ is automatically a $\mathbb{Z}$-module, and since $\mathbb{Z}^\times = \{\pm 1\}$, the scalar action structure map can be defined by letting $1_y^a = \mathrm{id}_{\mathcal{X}}$ and $(-1)_x^a = J_x^{aa}$. It is easily checked that this satisfies the properties (C) – (Tr). Moreover, by (Tr), any $\mathbb{Z}$-scalar action is necessarily given by these formulae. Thus a geometry with Jordan structure map is the same as one with compatible $\mathbb{Z}$-action.

## 6. Idempotents and the modular group

Let $\mathcal{X}$ be a geometry with Jordan structure map $J$. By a *configuration of points in* $\mathcal{X}$ we just mean a subset $P \subset \mathcal{X}$. In this chapter, we study some simple configurations:

(1) $P = \{x, a\}$, with $(a, x) \in \mathcal{D}_2$ (transversal pair),
(2) $P = \{o, a, z\}$, with $(o, a, z) \in \mathcal{D}_3$ (transversal triple), but not closed,
(3) $P = \{a, b, c\}$, where $(a, b, c) \in \mathcal{D}_3'$ (pairwise transversal triple),
(4) $P = \{a, x, b, y\}$, where $(a, x, b, y)$ is an idempotent quadruple.

For any configuration, consider the "group generated by inversions from $P$"

$$G_{(P)} := \Big\langle J_a^{xz} \mid x, a, z \in P,\, x \top a,\, z \top a \Big\rangle \subset \mathrm{Aut}(\mathcal{X}) \tag{6.1}$$

and the smallest subgeometry $\langle P \rangle \subset \mathcal{X}$ containing $P$. For configuration (1), $G_{(P)} = \{J_x^{aa}, \mathrm{id}\}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$; for configuration (2), $G_{(P)}$ contains a subgroup that is a quotient of $\mathbb{Z}$, generated by $L_a^{zo} = J_a^{zo} \circ J_a^{oo}$, and the whole group $G_{(P)}$ is a quotient of $\mathbb{Z} \ltimes \mathbb{Z}/2\mathbb{Z}$. Then $\langle P \rangle$ is a flat geometry (see 2.4.6). Configuration (3) is more interesting:

**Theorem 6.1.** *Assume $(a, b, c)$ is a pairwise transversal triple. Then*

$$\big\{\mathrm{id}_{\mathcal{X}}, J_c^{ab}, J_b^{ac}, J_a^{cb}, J_c^{ab} \circ J_b^{ac}, J_b^{ac} \circ J_c^{ab}\big\} \subset \mathrm{Aut}(\mathcal{X}, \top, J) \tag{6.2}$$

*is a group, isomorphic to the permutation group $S_3$.*

*Proof.* We claim that the following correspondences are group homomorphisms, where we list first the elements of $S_3$, then the corresponding element of (6.2), a corresponding element of $\mathbb{P}\mathrm{GL}(2, \mathbb{Z})$, and the fractional linear transformation (in the variable $z$) corresponding to the element from the precedig line:

| | | | | | |
|---|---|---|---|---|---|
| $(1)$ | $(12)$ | $(23)$ | $(13)$ | $(123) = (12)(23)$ | $(132) = (23)(12)$ |
| $\mathrm{id}_{\mathcal{X}}$ | $J_c^{ab}$ | $J_a^{bc}$ | $J_b^{ac}$ | $C_{abc} := J_c^{ab} \circ J_a^{bc}$ | $C_{bac} = J_a^{bc} \circ J_c^{ab}$ |
| $1$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ | $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ |
| $z$ | $z^{-1}$ | $(1 - z^{-1})^{-1}$ | $1 - z$ | $1 - z^{-1}$ | $(1 - z)^{-1}$ |

Indeed, it is checked by direct computation that all of these correspondences are group homomorphisms: since the elements $J_c^{ab}, J_b^{ac}, J_a^{cb}$ are of order two, it suffices to show that the composition of two of them is a 3-cycle, e.g., that $(J_c^{ab} \circ J_b^{ac})^3 = \mathrm{id}_{\mathcal{X}}$:

$$(J_c^{ab} \circ J_b^{ac})^3 = \big(J_c^{ab} J_b^{ac} J_c^{ab}\big)\big(J_b^{ac} J_c^{ab} J_b^{ac}\big) = J_a^{bc} J_a^{bc} = \mathrm{id}_{\mathcal{X}} \tag{6.3}$$

by using (IN), (IP), (D), and (C). □

*Remark.* The action of matrices from $\mathrm{GL}(2, \mathbb{Z})$ defined by this and the following tables corresponds to its "usual" action by fractional linear transformations on a Jordan algebra with unit 1, as indicated. See Section 9.5 for more on this.

**Theorem 6.2.** *Assume $(a,b,c)$ is a pairwise transversal triple and $P = \{a,b,c\}$. Then $G_{(P)}$ is a quotient of $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$. More precisely, define the matrices*

$$S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \, T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \, F := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \, I := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}), \tag{6.4}$$

*and by $[S]$, etc., denote the corresponding element in $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$. Then there is a unique group epimorphism*

$$\phi : \mathbb{P}\mathrm{GL}(2,\mathbb{Z}) \to G_{(P)} \subset \mathrm{Aut}(\mathcal{X}),$$

*defined by the correspondences $[S] \mapsto J_a^{bb} J_c^{ab}$ and $[T] \mapsto L_b^{ca} = J_b^{ca} J_b^{aa}$ and $[I] \mapsto J_a^{bb}$. Moreover, we then have the following correspondences (notation as above):*

| $J_b^{aa}$ | $J_c^{bb}$ | $J_a^{cc}$ | $L_c^{ba} = J_c^{ba} J_c^{aa}$ | $L_b^{ca} = J_b^{ca} J_b^{aa}$ | $L_a^{bc} = J_a^{bb} J_a^{bc}$ |
|---|---|---|---|---|---|
| $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ |
| $-z$ | $2-z$ | $(2-z^{-1})^{-1}$ | $(z^{-1}-1)^{-1}$ | $z+1$ | $(1-z^{-1})^{-1}$ |

*Proof.* Recall that the modular group $\Gamma := \mathbb{P}\mathrm{SL}(2,\mathbb{Z})$ is presented by generators and relations

$$\Gamma = \Big\langle [S],\, [T] \quad | \quad [S]^2 = 1,\, [ST]^3 = 1 \Big\rangle. \tag{6.5}$$

We prove the relation corresponding to $[S]^2 = 1$, that is, $(J_a^{bb} J_c^{ab})^2 = \mathrm{id}$:

$$(J_a^{bb} J_c^{ab})^2 = J_a^{bb} (J_c^{ab} J_a^{bb} J_c^{ab}) = J_a^{bb} J_b^{aa} = (J_a^{bb})^2 = \mathrm{id}.$$

Next, we prove the relation corresponding to $[ST]^3 = 1$: note that $J_a^{bb} J_c^{ab} J_b^{ca} J_b^{aa} = J_a^{bb} J_c^{ab} J_b^{ca} J_a^{bb} = J_c^{ab} J_a^{cb}$, and according to (6.3), this is a 3-cycle. According to the presentation (6.5), this defines a homomorphism $\mathbb{P}\mathrm{SL}(2,\mathbb{Z}) \to G(\mathcal{X})$. The remaining correspondences are checked by similar computations, and they establish a homomorphism $\mathbb{P}\mathrm{GL}(2,\mathbb{Z}) \to G_{(P)}$ which is obviously surjective. □

**Theorem 6.3.** *Recall that the geometry $\mathbb{Z}\mathbb{P}^1 = \mathrm{Gras}_1^1(\mathbb{Z}^2)$ has been defined in 0.3.1 and in 1.2. We denote its canonical pairwise transversal triple by $o = [e_1]$, $\infty = [e_2]$, $e = [e_1 + e_2]$. Assume $(a,b,c)$ is a pairwise transversal triple in $\mathcal{X}$ and $P = \{a,b,c\}$. Then the geometry $\langle P \rangle$ is a quotient of $\mathbb{Z}\mathbb{P}^1$. More precisely, there is a unique morphism of geometries*

$$\Phi : \mathbb{Z}\mathbb{P}^1 \to \mathcal{X}$$

*which preserves the pairwise transversal triples: $\Phi(o) = a$, $\Phi(\infty) = b$, $\Phi(e) = c$. This map is equivariant with respect to the homomorphism $\phi$ from the preceding theorem in the sense that $\Phi(g.x) = \phi(g)\Phi(x)$ for all $g \in \mathbb{P}\mathrm{GL}(2,\mathbb{Z})$.*

*Proof.* The projective line $\mathbb{Z}\mathbb{P}^1$ is homogeneous under the group $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$. As base point in the set $\mathcal{D}_2(\mathbb{Z}\mathbb{P}^1)$ of transversal pairs we take $(o,\infty) = ([e_2],[e_1])$. The stabilizer $H$ of this pair in $\mathrm{GL}(2,\mathbb{Z})$ is the group of diagonal matrices. Since $\phi(I) = J_a^{bb}$, and $J_a^{bb}$ preserves the pair $(a,b)$, the map $\phi$ from the theorem induces a well-defined and base point preserving map $\Phi_2 : \mathcal{D}_2(\mathbb{Z}\mathbb{P}^1) \to \mathcal{D}_2(\mathcal{X})$. Let $\mathrm{pr}_1 : (x,a) \mapsto x$ the projections from $\mathcal{D}_2$ to $\mathbb{Z}\mathbb{P}^1$ and to $\mathcal{X}$, respectively. Since the group $\mathrm{GL}(2,\mathbb{Z})$

and its image group under $\phi$ preserve the respective transversality relations, there is a well-defined map $\Phi : \mathbb{ZP}^1 \to \mathcal{X}$ such that $\mathrm{pr}_2 \circ \Phi_2 = \Phi \circ \mathrm{pr}_2$. It maps $o$ to $a$, and by equivariance, it maps then $\infty$ to $b$ and $e$ to $c$. This map is a morphism of geometries, i.e., we have $\Phi(J^{uv}_w(y)) = J^{\Phi(u)\Phi(v)}_{\Phi(w)} \Phi(y)$ whenever this makes sense. Indeed, for $\{u, v, w\} \subset \{a, b, c\}$, this follows from the equivariance property, and for general $u, v, w$ and then follows by induction on "word length". □

*Remark.* Of the many relations that are valid in the setting of the preceding theorems, let us just mention the following: the involution $J^{ca}_b$ has, besides $b$, another fixed point given by $J^{aa}_c(b)$:

$$J^{ca}_b(J^{aa}_c(b)) = J^{aa}_c(b). \tag{6.6}$$

Indeed, $J^{ca}_b J^{aa}_c(b) = J^{ca}_b J^{aa}_c J^{ca}_b(b) = J^{cc}_a(b) = J^{aa}_c(b)$. Another non-trivial relation is

$$J^{ac}_b = J^{ac}_{J^{aa}_c(b)}, \tag{6.7}$$

coming from $J^{aa}_c J^{ac}_b J^{aa}_c J^{ac}_b = 1$. In order to get a visual image of such and other relations, the best realization of $\mathbb{ZP}^1$ is not a "line" but rather a tessalation of the hyperbolic plane of type $(2, 3, \infty)$; such images can be found on the internet, see e.g., http://upload.wikimedia.org/wikipedia/commons/thumb/0/04/H2checkers_23i.png/1024px-H2checkers_23i.png. In this image, the points $a, b, c$ may be chosen as points on the boundary circle such that the triangle $(a, b, c)$ contains as its "center" a point of rotational symmetry with order 3. The symmetries $J^{ab}_c$ are then easily visible, but the orbit of $a, b, c$ (the set $\langle P \rangle$) will be on the boundary circle; thus this visualisation gives only a partial image, but at least it may give an idea of how complicated the corresponding geometry really is. In particular, the orbits of the translations groups defined by $a, b$, resp. $c$ correspond to limits of $\mathbb{Z}$-points of horocycles touching the boundary circle at $a, b$, resp. $c$.

The projective line $\mathbb{ZP}^1$ and its quotients are the most elementary building blocks for analyzing the structure of a general geometry. It is important that $\mathbb{ZP}^1$ appears not only in the context of a self-dual geometry, where pairwise transversal triples exist, but also for certain geometries "of the second kind", namely those having *idempotents*. The following definition arises when retaining the properties of the quadruple $(a, b, c, a)$, where $(a, b, c)$ is pairwise transversal, but then allowing some pairs to be not necessarily transversal:

**Definition 6.4.** *We say that* $(a, x, b, y) \in \mathcal{D}_4$ *is an* idempotent *if it satisfies*

$$J^{aa}_x(y) = y, \;\; J^{xy}_b J^{aa}_x(b) = J^{aa}_x(b), \;\; J^{aa}_x J^{xy}_b(a) = J^{xy}_b(a), \;\; J^{xy}_b J^{ab}_x(y) = J^{ab}_x(y) \tag{6.8}$$

$$J^{bb}_y(a) = a, \;\; J^{ab}_x J^{yy}_b(x) = J^{yy}_b(x), \;\; J^{yy}_b J^{ab}_x(y) = J^{ab}_x(y), \;\; J^{ab}_x J^{xy}_b(a) = J^{xy}_b(a). \tag{6.9}$$

*A* strong idempotent *is an idempotent* $(a, x, b, y)$ *such that, moreover,*

$$J^{yx}_{J^{aa}_x(b)} = J^{a, J^{xy}_b(a)}_{J^{ab}_x(y)}. \tag{6.10}$$

**Lemma 6.5.** *If* $(a, b, c)$ *is a pairwise transversal triple, then* $(a, x, b, y) := (a, c, b, a)$ *is a strong idempotent.*

*Proof.* Easy check – cf. (6.6); the "strong" relation (6.10) boils down to (6.7). □

Conditions (6.8) and (6.9) are dual to each other in the sense that they imply that $(a,x,b,y)$ is an idempotent if and only if so is $(y,b,x,a)$. Another way to formulate this definition is to define 4 new points

$$c := J_x^{aa}(b), \quad z := J_b^{yy}(x), \quad d := J_b^{xy}(a), \quad w := J_x^{ab}(y), \tag{6.11}$$

(thinking of $(x,y,z,w)$ and $(a,b,c,d)$ as two harmonic quadruples on two "dissociated" projective lines $\ell, \ell'$, in the sense of 1.6) and to require that

$$J_x^{aa}(y) = y,\ J_b^{xy}(c) = c,\ J_x^{aa}(d) = d,\ J_b^{xy}(w) = w\,, \tag{6.12}$$

$$J_y^{bb}(a) = a,\ J_x^{ab}(z) = z,\ J_b^{yy}(w) = w,\ J_x^{ab}(d) = d\,. \tag{6.13}$$

Geometrically, this means that certain fixed points of our involutions on the lines $\ell, \ell'$ are determined in a definite way. Fixed points of $J_x^{bb}$ are then given by

$$J_x^{bb}(w) = J_x^{bb}J_x^{ab}(y) = J_x^{ab}J_x^{aa}(y) = w, \qquad J_b^{xx}(d) = d. \tag{6.14}$$

**Theorem 6.6.** *Assume $(a,x,b,y) \in \mathcal{D}_4$ is a strong idempotent. Then there is a homomorphism* $\mathrm{GL}(2,\mathbb{Z}) \to \mathrm{Aut}(\mathcal{X})$ *defined by the following correspondences:*

| | | | | | |
|---|---|---|---|---|---|
| $J_x^{bb}$ | $J_b^{xy}$ | $L_b^{yx}$ | $J_y^{bb}$ | $L_x^{ab}$ | $J_x^{ab}$ |
| $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ |
| $J_x^{aa}$ | $\Lambda_{xy}^{ab} = L_x^{ab}L_b^{xy}$ | $(\Lambda_{xy}^{ab})^3$ | $W_{ab}^{xy} = L_x^{ab}L_b^{xy}L_x^{ab}$ | $J_b^{xy}J_x^{aa}$ | $(J_b^{xy}J_x^{aa})^2$ |
| $\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$ | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ |

*If $(a,x,b,y)$ is an idempotent (not necessarily strong), then a similar statement still holds, but* $\mathrm{GL}(2,\mathbb{Z})$ *has to be replaced by the universal central extension* $\widetilde{\mathrm{GL}(2,\mathbb{Z})}$ *(that is, by an extended braid group).*

*Proof.* Let $P = \{a,x,y,b\}$ and $G := G_{(P)}$. The group $G$ is clearly generated by the three elements $A := L_x^{ab}$, $B := L_b^{xy}$ and $J := J_x^{bb}$. We show that these elements satisfy the following relations defining $\mathrm{GL}(2,\mathbb{Z})$

$$(ABA)^4 = 1, \quad ABA = BAB, \quad J^2 = 1, \quad (JA)^2 = 1 = (JB)^2.$$

Indeed, the proof of the last three relations is immediate. In order to prove the first relation, we start by proving that the following element $Z$ is central in $G$:

$$Z := (J_b^{xy}J_x^{aa})^2 = J_b^{xy}J_x^{aa}J_b^{xy}J_x^{aa} = J_b^{xy}J^{x,J_x^{aa}(y)}_{J_x^{aa}(b)} = J_b^{xy}J^{xy}_{J_x^{aa}(b)}\,. \tag{6.15}$$

It is obvious that $Z(x) = x$ and $Z(y) = y$; using (6.8), it follows that also $Z(a) = a$ and $Z(b) = b$. Therefore $Z$ commutes with all generators $J_w^{uv}$ of $G$: $ZJ_w^{uv}Z^{-1} = J_{Zw}^{Zu,Zv} = J_w^{uv}$, and hence is central in $G$. Moreover, $Z$ is of order 2, since

$$J_b^{xy}J^{xy}_{J_x^{aa}(b)}J_b^{xy} = J^{xy}_{J_b^{xy}J_x^{aa}(b)} = J^{xy}_{J_x^{aa}(b)},$$

and hence $Z$ is a product of two commuting involutions. Now consider the "Weyl-element" (cf. [Lo95], 6.1) $W := W_{ab}^{xy} = ABA = L_x^{ab}L_b^{xy}L_x^{ab} = J_x^{ab}\,J_b^{xy}J_x^{aa}\,J_x^{ab}$. The last expression shows that it is conjugate to $J_b^{xy} J_x^{aa}$, and hence $W^2$ is conjugate to $(J_b^{xy} J_x^{aa})^2 = Z$. Since $Z$ is central, it follows that $W^2 = Z$, and so $W^4 = Z^2 = \mathrm{id}_{\mathcal{X}}$.

Next, we prove that $Z' := (AB)^3$ is a central element of order 2. Indeed, the proof is very similar to the one given above: we have

$$Z' = J_x^{ab} J_b^{xy} J_x^{ab} J_b^{xy} J_x^{ab} J_b^{xy} = J_x^{ab} J_{J_b^{xy}(a)}^{y, J_x^{ab}(y)} \tag{6.16}$$

As above, it is checked that $Z'$ fixes $a, x, b, y$, and hence is central; it is a product of two commuting involutions, hence of order 2 (and hence $(AB)^6 = 1$).

Since $Z = W^2 = ABAABA$ and $Z' = ABABAB$, the relation $ABA = BAB$ is equivalent to $Z = Z'$ or to $ZZ' = 1$. But, by an easy computation,

$$ZZ' = J_{J_x^{aa}(b)}^{yx} \, J_{J_x^{ab}(y)}^{a, J_b^{xy}(a)} \tag{6.17}$$

so $ZZ' = 1$ is equivalent to (6.10). This proves the claim for a strong idempotent. If the idempotent is not strong, then, as we have seen, all relations from $\mathrm{GL}(2,\mathbb{Z})$ a satisfied, possibly up to central elements. Therefore the homomorphism may be defined on the level of the universal central extension. □

*Remark.* It is not true that the homomorphism always factorizes via $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$: the central element $Z$ (or $Z'$) acts trivially on the $G$-orbit of $x, a, b, z$, but in general it will act non-trivially on the whole of $\mathcal{X}$ (cf. the following example) – this action is precisely described by the *Peirce-decomposition* associated to the idempotent. In a similar way, the geometry $\langle P \rangle$ is not always a quotient of $\mathbb{Z}\mathbb{P}^1$, but rather of the *dissociation of* $\mathbb{Z}\mathbb{P}^1$.

*Example.* Let $\mathcal{X} = \mathrm{Gras}(W)$ be the Grassmannian geometry a $\mathbb{K}$-module $W$ (see 0.3.1) with a direct sum decomposition $W = E \oplus F \oplus H$, and (non-zero) subspaces $u, v, w \subset F$ such that $u \oplus v = F = u \oplus w = v \oplus w$ (so $(u, v, w)$ is a pairwise transversal triple in $\mathrm{Gras}(F)$). Let

$$a := w \oplus H, \quad x := E \oplus u, \quad b := H \oplus v, \quad y := E \oplus w. \tag{6.18}$$

Then $(a, x, b, y)$ is a chain in $\mathcal{X}$, but $a \cap y = w$, so $a$ and $y$ are not transversal. It can be shown that $(a, x, b, y)$ is a (strong) idempotent in $\mathrm{Gras}(W)$. Instead of checking the defining properties, it is easier to exhibit directly the corresponding realization of $\mathrm{GL}(2,\mathbb{Z})$ in $\mathrm{Aut}(\mathcal{X}) = \mathbb{P}\mathrm{GL}(W)$: we decompose $W = E \oplus u \oplus v \oplus H$, and write elements of $\mathrm{GL}(W)$ accordingly as $4 \times 4$-matrices. Then, considering $w$ as diagonal in $u \oplus v$, all four middle blocks are square matrices, so that a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2,\mathbb{Z})$ may be identified with the class of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{in } \mathbb{P}\mathrm{GL}(W)\,.$$

# SECOND PART: TANGENT ALGEBRAS

Our aim in this part is to associate to a Jordan- or associative structure map, at a given base point $(o, o')$, a "tangent object", namely a Jordan pair or -algebra, resp. an associative pair or -algebra. Doing this amounts to "extend" our structure maps to more singular situations, and it requires additional assumptions. There are two different ways to formulate such assumptions:

(A) For associative geometries ([BeKi09a]), regularity assumptions are introduced by extending $M_{xz}^{ab}(y)$, in some canonical way, to "all" 5-tuples $(x, a, y, b, z)$; however, the existence of such an extension is a highly non-trivial fact, and, for the time being, we do not know how this could carry over to the Jordan setting.

(B) In the present work, we follow a more "classical" approach, by introducing an algebraic formulation of smoothness and infinitesimal calculus (Appendix B) which is fairly close to the approach to "usual" differential calculus via so-called *Weil functors* (see [KMS93] for the classical theory, and [BeS11] for its extension to general differential calculus over topological fields or rings).

The strategy used in [Be02], although closer to (B) than to (A), remains somewhat in between: from an algebraic point of view, (A) requires to work (as in [Be02]) with *non-invertible operators*: on their extended domain of definition, the operators $M_{xz}^{ab}$ and $J_a^{xz}$ (and the "middle multiplication operators" from [Be02]) are no longer invertible operators , and hence cannot be automorphisms; but they still are endomorphisms, in the category having *structural transformations* as morphisms (not in the "usual" category), see [Be02, BeKi09a]. Moreover, the extended maps $M$ and $J$ will no longer be smooth on the largest domain of extension: they have very "strong" singularities, in the sense that the singular values are "at very far infinity", outside of the connected component. Therefore the theory can no longer be formulated in terms of the connected geometry only. One may expect that these singularities contain a lot of information on the geometry – this aspect should be very interesting; to our knowledge, it has never been studied so far.

## 7. Smooth geometries

7.1. **Jordan geometries over $\mathbb{K}$.** If we have a Jordan geometry with scalar action by a ring $\mathbb{K}$, then the canonical atlas from subsection 3.6 is an atlas modelled on a pair of $\mathbb{K}$-modules $(V, V')$. As explained in Appendix B.5, it makes thus sense to speak of a *smooth atlas* in a purely algebraic way, by requiring that *the transition functions are smooth laws* – essentially, this means that, by scalar extension with Weil algebras, they admit "tangent maps": we say that our geometry $(\underline{\mathcal{X}}, \underline{\top}, \underline{J}, \underline{\mathbf{r}})$ is *smooth over* $\mathbb{K}$ (or, more precisely, a *smooth Jordan geometry law over* $\mathbb{K}$) if

1. the canonical atlas (subsection 3.6) of the Jordan geometry is smooth in the sense of B.5, and
2. the Jordan structure map $J$ and the scalar action structure map $\mathbf{r}$ are given by *smooth laws* $\underline{J}$ and $\underline{\mathbf{r}}$ in the sense of B.5.

In the same way, smoothness of associative structure maps is defined (where the underlying atlas is the same as the one of the corresponding Jordan geometry).

**Definition 7.1.** *A* Jordan geometry over $\mathbb{K}$ *is a* $\mathbb{K}$*-smooth geometry* $(\underline{\mathcal{X}}, \underline{\top})$ *with smooth Jordan structure law* $\underline{J}$ *and smooth scalar action law* $\underline{\mathbf{r}}$ *satisfying the following* symmetric space axiom *(or* tangent space axiom*):*

(T) *for all* $(a, x, b) \in \mathcal{D}_3$*, the tangent map of* $J_x^{ab}$ *at its fixed point* $x$ *is* $-\mathrm{id}_{T_x\mathcal{X}}$*:*

$$T_x(J_x^{ab}) = -\mathrm{id}_{T_x\mathcal{X}}$$

*Thus each* $U_{ab}$ *is a symmetric space in the sense of B.6.* Morphisms of Jordan geometries over $\mathbb{K}$ *are smooth laws* $\underline{f} : \underline{\mathcal{X}} \to \underline{\mathcal{Y}}$ *such that* $T^{\mathbb{A}} f : \mathcal{X}_{\mathbb{A}} \to \mathcal{Y}_{\mathbb{A}}$ *is a homomorphism of geometries over* $\mathbb{A}$*, for each Weil algebra* $\mathbb{A}$*. A* polarity *of a Jordan geometry* $\underline{\mathcal{X}}$ *is an automorphism* $\underline{p}$ *such that there exists* $x \in \mathcal{X}$ *with* $p(x) \top x$*, and a* polar Jordan geometry *is a Jordan geometry together with a fixed polarity.*

Existence of large classes of Jordan geometries is ensured by the following two constructions:

7.2. **Associative geometries.** An *associative geometry over* $\mathbb{K}$, $(\underline{\mathcal{X}}, \underline{\top}, \underline{\mathbf{r}}, \underline{M})$, is a geometry such that the scalar action $\mathbf{r}$ and the associative structure map $M$ are smooth laws over $\mathbb{K}$. The symmetric space axiom is a consequence of these:

**Theorem 7.2.** *If* $(\underline{\mathcal{X}}, \underline{\top}, \underline{M})$ *is an associative geometry over* $\mathbb{K}$*, then* $(\underline{\mathcal{X}}, \underline{\top}, \underline{J})$*, with* $J_x^{ab}$ *defined as in lemma 4.2, is a Jordan geometry over* $\mathbb{K}$*.*

*Proof.* In view of the preceding remarks, it only remains to show that the tangent map of $J_x^{ab}$ at $x$ is minus identity on $T_x\mathcal{X}$. But, in the associative setting, $(U_{ab}, x)$ is a Lie group, and as usual for Lie groups, it is shown that the tangent map of inversion is minus identity on the tangent space of the origin (Theorem B.3). □

A *special Jordan geometry* is a subgeometry of $(\underline{\mathcal{X}}, \underline{\top}, \underline{J})$, where $\underline{J}$ comes from an associative structure map as in the preceding theorem. For instance, Grassmann geometries are smooth (the scalar extensions exist for all commutative $\mathbb{K}$-algebras $\mathbb{A}$, not only for Weil algebras, by tensoring everything with $\mathbb{A}$ over $\mathbb{K}$), and therefore its Jordan-subgeometries, such as Lagrangians, are Jordan geometries over $\mathbb{K}$.

7.3. **Generalized projective geometries.** These geometries have been defined in [Be02], based solely on properties of the scalar action map $\mathbf{r}$, by assuming that 2 is invertible in $\mathbb{K}$. We refer to loc. cit. for the precise formulation of the axioms. Any such geometry is a Jordan geometry over $\mathbb{K}$:

**Theorem 7.3.** *Assume* 2 *is invertible in* $\mathbb{K}$*. If* $(\mathcal{X}, \top)$ *is a generalized projective geometry, with scalar action denoted by* $r_x^a$ *for* $r \in \mathbb{K}^\times$*,* $x \top a$*, then* $(\mathcal{X}, \top, J, \mathbf{r})$ *is a Jordan geometry over* $\mathbb{K}$*, where* $J$ *is defined by* $J_x^{aa} := (-1)_x^a$ *and*

$$J_a^{xz} := (-1)^a_{\mu(x,a,z)} = J^{aa}_{\mu(x,a,z)}$$

*and* $\mu(x, a, z) := (2_x^a)^{-1} z$ *is the midpoint of* $x$ *and* $z$ *in the affine space* $U_a$*.*

*Proof.* In the theorem, and in the proof, we suppress the superscripts $\pm$ used in [Be02] (formally, this can be justified by working in the "dissociation" of the geometry $(\mathcal{X}^+, \mathcal{X}^-)$). Using this notation, we check the defining identities of $J$: Involutivity follows from $(-1)^2 = 1$, commutativity from the fact that $\mu(x, a, z) = \mu(z, a, x)$, symmetry from the "fundamental identity" $(r_x^a)^{-1} = r_a^x$ (which implies

$(-1)^a_x = (-1)^x_a)$, distributivity holds since all maps $s^a_x$ for $s \in \mathbb{K}^\times$ are automorphisms of the scalar action map $\mathbf{r}$, and idempotency follows from the following computation in the affine space $U_a$:

$$J^{xz}_a(y) = (-1)^a_{\frac{x+z}{2}}(y) = 2\frac{x+z}{2} - y = x - y + z.$$

Associativity is proved by establishing first that, in a generalized projective geometry, for all $x, y, z\top a$, with the usual torsor structure $x - y + z$ on $U_a$,

$$(-1)^x_a \circ (-1)^y_a \circ (-1)^z_a = (-1)^{x-y+z}_a.$$

This identity is not among the defining identities given in [Be02], but it follows by combining the "translation identity" (T) from loc. cit. with the properties of scalar actions. Using this, associativity follows in a straightforward way:

$$\begin{aligned} J^{xz}_a J^{uv}_a J^{pq}_a &= (-1)^{\mu(x,a,z)}_a \circ (-1)^{\mu(u,a,v)}_a \circ (-1)^{\mu(p,a,q)}_a = (-1)_a^{\frac{x+z}{2} - \frac{u+v}{2} + \frac{p+q}{2}} \\ &= (-1)_a^{\frac{(x-v+p)+(z-u+q)}{2}} = J_a^{J^{xp}_a(v), J^{zq}_a(u)}. \end{aligned}$$

The regularity assumption on the geometry is satisfied: as shown in [Be02], every generalized projective geometry is associated to a Jordan pair; the scalar extension of a Jordan pair by $\mathbb{A}$ is a Jordan pair over $\mathbb{A}$, and as shown in [Be02] this implies that the structure maps of a generalized projective geometry admit scalar extensions from $\mathbb{K}$ to $\mathbb{A}$. Finally, the symmetric space axiom (T) holds since $J^{xz}_a = (-1)^a_\mu = (-1)^\mu_a$ acts by multiplication by $-1$ in the $\mathbb{K}$-module $V_\mu$ with origin $a$ (where $\mu = \mu(x, a, z)$). □

Combined with the existence theorem for generalized projective geometries ([Be02], Th. 10.1), this implies an existence result for Jordan geometries over rings in which 2 is invertible (cf. Theorem 10.1 below).

## 8. Tangent geometries and infinitesimal automorphisms

8.1. **Tangent geometries.** Every smooth manifold law $\underline{M}$ admits tangent bundles $T^{\mathbb{A}}M$, for each Weil algebra $\mathbb{A}$ (see Theorem B.1). This construction is compatible with additional structure, such as Lie groups, or Jordan geometries (Theorem B.2).

**Theorem 8.1** ($\mathbb{A}$-tangent bundles). *Assume $(\underline{\mathcal{X}}, \underline{\top}, \underline{J}, \underline{\mathbf{r}})$ is a smooth Jordan geometry over $\mathbb{K}$, modelled on $(\underline{V}, \underline{V}')$. Then, for any $\mathbb{K}$-Weil algebra $\mathbb{A}$, we obtain a Jordan geometry over $\mathbb{A}$, modelled on $(V \otimes_{\mathbb{K}} \mathbb{A}, V' \otimes_{\mathbb{K}} \mathbb{A})$ and called the* $\mathbb{A}$-tangent geometry of *$\mathcal{X}$, by applying the Weil functor $T^{\mathbb{A}}$ to these data. This correspondence is functorial. The canonical projection and the zero section,*

$$\pi : T^{\mathbb{A}}\mathcal{X} \to \mathcal{X}, \qquad z : \mathcal{X} \to T^{\mathbb{A}}\mathcal{X},$$

*are homomorphisms. Analogous statements hold for associative geometries.*

*Proof.* The arguments are the same as those proving the corresponding theorem on Lie groups and symmetric spaces (Theorem B.2): the transversality relation in $\mathcal{X}_{\mathbb{A}}$ is given by applying $T^{\mathbb{A}}$ to the one of $\mathcal{X}$, that is, $\mathcal{D}^{\mathbb{A}}_2 = T^{\mathbb{A}}(\mathcal{D}_2)$, or quivalently, $u\top w$ iff $\pi(u)\top\pi(w)$. All defining identities of Jordan structure maps and of structure maps of scalar multiplications can be expressed by commutative diagrams; applying the functor $T^{\mathbb{A}}$ to such diagrams yields diagrams of the same form, over $\mathbb{A}$, and hence the space $(\mathcal{X}_{\mathbb{A}}, \top_{\mathbb{A}}, J_{\mathbb{A}}, \mathbb{A})$ satisfies the same identies over $\mathbb{A}$, so it is again a Jordan geometry. $\square$

For $\mathbb{A} = T\mathbb{K} = \mathbb{K}[\varepsilon]$ we get the *tangent geometry* $T\mathcal{X} = T^{\mathbb{K}[\varepsilon]}\mathcal{X}$. Recall that $T\mathbb{K}$ is a *vector algebra*: the nilpotent ideal carries the zero product. For such algebras, the fibers of $\pi$ (tangent spaces) carry a canonical $\mathbb{K}$-module structure (see B.7).

**Theorem 8.2** (Tangent spaces as intrinsic subspaces)**.** *Assume $(\underline{\mathcal{X}}, \underline{J})$ is a Jordan geometry over $\mathbb{K}$ and let $x \in \mathcal{X}$. Then, for all $a \top x$, the natural $\mathbb{K}$-module structure in the tangent space $T_x\mathcal{X}$ (defined in B.7) coincides with its $\mathbb{K}$-module structure as a submodule of $(U_a, x)$ in the geometry $T\mathcal{X}$ (and hence is independent of the choice of $a \in x^\top$).*

*Proof.* By definition, the linear structure in $T(U_a) = \{v \in T\mathcal{X} \mid v \top a\}$ is given by the structure maps of $T\mathcal{X}$ (and hence depends on $a$). On the other hand, for given $x$, chart changes in the geometry $T\mathcal{X}$ are given by tangent maps of those in $\mathcal{X}$, which are linear in fibers; therefore the linear structure in the fiber over $x$, induced by $a$, does not depend on the choice of $a$. $\square$

**Lemma 8.3** (Linear isotropy)**.** *The translation group $V_a$ belongs to the kernel of the* linear isotropy representation *$P_a \to \mathrm{GL}(T_a\mathcal{X})$, $g \mapsto T_a g$, i.e., the tangent map of $L_a^{xz}$ at the fixed point $a$ is the identity:*

$$T_a(L_a^{xz}) = \mathrm{id}_{T_a\mathcal{X}}.$$

*Proof.* Using the symmetric space axiom (S),

$$T_a(L_a^{xz}) = T_a(J_a^{xx} J_a^{xz}) = T_a(J_a^{xx}) \circ T_a(J_a^{xz}) = (-\mathrm{id}_{T_a\mathcal{X}})^2 = \mathrm{id}_{T_a\mathcal{X}}.$$ $\square$

Note that the lemma furnishes an interpretation of the split exact sequence (3.4) in terms of the linear isotropy representation.

8.2. **Infinitesimal automorphisms.** Let $(\underline{\mathcal{X}}, \underline{J}, \underline{\mathbf{r}})$ be a Jordan geometry over $\mathbb{K}$. An *infinitesimal Jordan automorphism* of $T\mathcal{X}$ is an automorphism $\xi$ of $T\mathcal{X}$ over $T\mathbb{K}$ such that $\pi \circ \xi = \pi$, i.e., $\xi$ preserves tangent spaces.

**Theorem 8.4.** *There is a canonical bijection between infinitesimal Jordan automorphisms $\xi$ and* derivations *$X : \mathcal{X} \to T\mathcal{X}$, i.e., morphisms $\mathcal{X} \to T\mathcal{X}$ such that $\pi \circ X = \mathrm{id}_{\mathcal{X}}$. This bijection is given by*

$$X := X_\xi := \xi \circ z : \mathcal{X} \to T\mathcal{X}, \qquad \xi := \xi_X := \Phi \circ TX : T\mathcal{X} \to T\mathcal{X},$$

*where $z : \mathcal{X} \to T\mathcal{X}$ is the zero section and $\Phi : TT\mathcal{X} \to T\mathcal{X}$ the natural map induced by $TT\mathbb{K} \to T\mathbb{K}$, $(x, a, b, c) \mapsto (x, a+b)$ (see B.7). The infinitesimal Jordan automorphisms form a subgroup of $\mathrm{Aut}(T\mathcal{X})$, denoted by $\mathfrak{g}(\mathcal{X})$, and under the isomorphism with $\mathrm{Der}(\mathcal{X})$ the group structure of $\mathfrak{g}(\mathcal{X})$ corresponds to pointwise addition of vector fields (derivations).*

*Proof.* The bijection between vector fields and infinitesimal automorphisms follows from the general theory of Weil functors (see B.9). This bijection is compatible with additional structure (Lie groups, symmetric spaces, Jordan- or associative geometries). $\square$

We call $X = X_\xi$ also the *associated vector field* of $\xi$. By the theorem, $\mathfrak{g}(\mathcal{X})$ acts by translations in each fiber:

$$\xi(v) = v + X(\pi(v)),$$

therefore its group law of $\mathfrak{g}(\mathcal{X})$ will be written additively. Since $\mathrm{Der}(\mathcal{X})$ is a $\mathbb{K}$-module, so is $\mathfrak{g}(\mathcal{X})$, by transport of structure. Over $V$, and writing $TV = V \oplus \varepsilon V$, an infinitesimal automorphism $\xi$ has the chart representation

$$\xi(x + \varepsilon v) = x + \varepsilon(v + X(x)) \tag{8.1}$$

where $X|_V : V \to \varepsilon V \cong V$ is the chart representation of a vector field. The group $\mathfrak{g}(\mathcal{X})$ is normal in $\mathrm{Aut}_{T\mathbb{K}}(T\mathcal{X})$, and the group $\mathrm{Aut}(\mathcal{X})$ acts on $\mathfrak{g}(\mathcal{X})$ by conjugation ("adjoint representation")

$$\mathrm{Aut}(\mathcal{X}) \times \mathfrak{g}(\mathcal{X}) \to \mathfrak{g}(\mathcal{X}), \qquad (g, \xi) \mapsto g.\xi := Tg \circ \xi \circ Tg^{-1}. \tag{8.2}$$

**Lemma 8.5.** *Assume $v, w \in T_p\mathcal{X}$ and $p \top a$. Then the following are infinitesimal automorphisms:*

(1) *the* vertical translation $L_a^{vw}$,
(2) *the* Euler field $(1+\varepsilon)_p^a$.

*Proof.* For the vertical translation: $\pi(L_a^{vw} x) = L_{\pi(a)}^{\pi(v),\pi(w)} \pi(x) = L_a^{p,p}(\pi(x)) = \pi(x)$, whence $\pi \circ L_a^{vw} = \pi$. Concerning the Euler field, note first that from functoriality of the law $\mathbf{r}$ we get

$$\pi \circ r_x^a = (\pi r)_{\pi x}^{\pi a} \circ \pi$$

for all scalars $r \in T\mathbb{K}^\times$. Apply this to the invertible scalar $r = 1 + \varepsilon$ (whose inverse is $1 - \varepsilon$): since $\pi(1 + \varepsilon) = 1$, we get $\pi \circ (1 + \varepsilon)_p^a = 1_{\pi p}^{\pi a} \circ \pi = \pi$. □

Since $(1 - \varepsilon)_x^a (1 + \varepsilon)_y^a = L_a^{(1-\varepsilon)_x^a y, y}$, one could deduce (1) also from (2).

8.3. **Triple decomposition, and chart representation.** Let us fix a base point $(o, o') \in \mathcal{D}_2$ and the model space $(V, V') = (U_{o'}, U_o)$. By identifying $o$ and $o'$ with their images under the zero section, we consider $(o, o')$ also as base point in $T\mathcal{D}_2$.

**Lemma 8.6.** *Fix a transversal pair $(o, o') \in \mathcal{D}_2$ and define subgroups of $\mathfrak{g} = \mathfrak{g}(\mathcal{X})$*

$$\begin{aligned}
\mathfrak{g}_0 &:= \mathfrak{g} \cap H_{o,o'} = \{\xi \in \mathfrak{g} \mid \xi(x) = x, \xi(o') = o'\}, \\
\mathfrak{g}_{-1} &:= \mathfrak{g} \cap L_{o'}^{TV,o} = \{L_{o'}^{vo} \mid v \in T_o\mathcal{X}\}, \\
\mathfrak{g}_1 &:= \mathfrak{g} \cap L_o^{TV',o} = \{L_o^{wo'} \mid w \in T_{o'}\mathcal{X}\}.
\end{aligned}$$

*Then we have a direct sum decomposition of the $\mathbb{K}$-module $\mathfrak{g}$*

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

*The group $\mathbb{K}^\times = \{r_o^{o'} \mid r \in \mathbb{K}^\times\}$ acts, via the adjoint representation, on these spaces diagonally with eigenvalues $r, 1, r^{-1}$.*

*Proof.* Let $\xi$ be an infinitesimal automorphism, then $\xi(o)$ lies in the fiber over $o$, hence is also transversal to $o'$, so we can decompose

$$\xi = L_{o'}^{\xi(o),o} \circ D(\xi) \circ L_o^{-\xi(o'),o'} = L_{o'}^{\xi(o),o} + D(\xi) + L_o^{-\xi(o'),o'}$$

which is simply the decomposition (3.7), written additively (where we note that each factor is fiber preserving, i.e., belongs to $\mathfrak{g}$). To prove the claim on the eigenvalues of the scalar action, just read equations (5.2) and (5.3) for infinitesimal arguments $\varepsilon v$ and $\varepsilon a$. □

**Theorem 8.7.** *For all $\xi \in \mathfrak{g}(\mathcal{X})$, the vector field $X|_V : V \to V$ representing $\xi$, is a quadratic polynomial. More precisely, this polynomial is constant if $\xi \in \mathfrak{g}_{-1}$, linear if $\xi \in \mathfrak{g}_0$, and homogeneous of degree 2 if $\xi \in \mathfrak{g}_1$. Thus $\mathfrak{g}$ is represented over $V$ by the Lie algebra of quadratic polynomials*

$$\big\{ X : V \to V \mid X(x) = v + Hx + Q(x)a, \quad v \in V, H \in \mathfrak{g}_o, a \in V' \big\},$$

*where the polynomial*

$$Q : V \times V' \to V, \quad (x,a) \mapsto Q(x)a, \tag{8.3}$$

*quadratic in $x$ and linear in $a$, is defined by*

$$L_o^{\varepsilon a, o'}(x + \varepsilon v) = x + \varepsilon(v + Q(x)a). \tag{8.4}$$

*Similarly, $\mathfrak{g}$ is also represented by quadratic polynomial vector fields over $V'$.*

*Proof.* If $\xi \in \mathfrak{g}_{-1}$, then $\xi = L_{o'}^{\varepsilon v, o}$ for some $v \in V$, hence $\xi(x) = x + \varepsilon v$, and the corresponding vector field is $X(x) = v$, which is a constant function. If $\xi \in \mathfrak{g}_0$, then $\xi$ acts linearly on $V$ (and on $V'$). It remains to show that $X$ is homogeneous quadratic polynomial if $\xi \in \mathfrak{g}_1$.

In the chart formula, the adjoint action (8.2) is described as follows: using (8.1) together with $Tg(x + \varepsilon v) = g(x) + \varepsilon df(x)v$, we get, whenever $g^{-1}.x \in V$,

$$(g.\xi)(x) = dg(g^{-1}.x) \cdot X(g^{-1}.x). \tag{8.5}$$

If $g = L_{o'}^{v,o}$ is a translation with $v \in V$, (8.5) gives

$$(L_{o'}^{v,o}.\xi)(x) = X(x - v), \tag{8.6}$$

and for a major dilation $g = r_o^{o'}$ it gives

$$(r_o^{o'}.\xi)(x) = rX(r^{-1}x)\,. \tag{8.7}$$

Specializing (8.7) to vector fields $X$ from the three parts $\mathfrak{g}_i$, $i = -1, 0, 1$, we get that $X : V \to V$ is homogeneous of degree 0, 1 or 2, respectively, since $\xi$ is eigenvector for the $\mathbb{K}^\times$-action for the eigenvalues $r, 1, r^{-1}$, respectively, by the preceding lemma.

Now let $\xi = L_o^{o', \varepsilon a} \in \mathfrak{g}_1$ ($a \in V'$) and $X$ its associated vector field. In order to show that $X(x)$ is quadratic polynomial, it remains to show that the map

$$X_v : V \to V, \qquad x \mapsto X_v(x) = X(x - v) - X(x),$$

is affine, for all $v \in V$. (Equivalently, that $x \mapsto X(x + v) - X(x) - X(v)$ is linear.) According to (8.6), the field $X(x - v)$ represents the infinitesimal automorphism $Tg \circ \xi \circ Tg^{-1}$ where $g = L_{o'}^{v,o}$ is translation by $v$, and $-X(x)$ represents $\xi^{-1}$, whence $X_v$ represents the infinitesimal automorphism $Tg \circ \xi \circ Tg^{-1} \circ \xi^{-1}$, that is, it represents

$$\xi_v := L_{o'}^{v,o} L_o^{o', \varepsilon a} L_{o'}^{o,v} - L_o^{o', \varepsilon a} = L_{o'}^{v,o} L_o^{o', \varepsilon a} L_{o'}^{o,v} L_o^{\varepsilon a, o'}.$$

Saying that $X_v$ is affine amounts to saying that $\xi_v$ fixes $o'$. Now,

$$\xi_v(o') = (L_{o'}^{v,o} L_o^{o', \varepsilon a} L_{o'}^{o,v} - L_o^{o', \varepsilon a})o' = L_{o'}^{v,o} L_o^{o', \varepsilon a} - (-\varepsilon a) = L_{o'}^{v,o}(-\varepsilon a) - (-\varepsilon a)$$

But $L_{o'}^{v,o}(-\varepsilon a) = T_{o'}(L_{o'}^{v,o})(-\varepsilon a)$ is the tangent map of $L_{o'}^{v,o}$ at its fixed point $o'$, applied to $-\varepsilon a$. According to Lemma 8.3, this tangent map is the identity, and hence it follows that $\xi_v(o') = o'$, hence $\xi_v$ is affine for all $v \in V$ and thus $\xi$ is quadratic. The map $Q(x)a$ is defined by $Q(x)a = X(x)$ for $\xi$ as above, and hence is homogeneous quadratic in $x$. It is linear in $a$ since the map $a \mapsto L_o^{\varepsilon a, o'}$ is a linear isomorphism from $V'$ to $\mathfrak{g}_1$. □

8.4. **The quadratic map.** With respect to a fixed origin $(o, o')$ and model space $(V, V')$, we define a map $Q : V \to \mathrm{Hom}(V', V)$ as above. The map

$$Q(x,v)a := D(x,a)v := Q(x+v)a - Q(x)a - Q(v)a \tag{8.8}$$

is bilinear symmetric in $(x, v)$ linear in $a$. The maps $Q' : V' \times V' \to \mathrm{Hom}(V, V')$ and $D' : V' \times V \to \mathrm{End}(V')$ are defined dually.

8.5. **Numerators and denominators.** We fix an origin $(o, o')$ and denote by

$$\mathbf{v} := X_{L_{o'}^{\varepsilon v, o}}, \ \mathbf{v}(x) = v, \qquad E := X_{(1+\varepsilon)_o^{o'}}, \ E(x) = x, \tag{8.9}$$

the "constant vector fields" $\mathbf{v} : V \to V$, resp. the Euler field, associated to $(o, o')$. Let $g \in \mathrm{Aut}(\mathcal{X})$. Recall that $g$ acts on $\mathfrak{g}$ by the adjoint representation, and in the chart $V$ this action is described by (8.5), when $g^{-1}.x \in V$. By the preceding theorem, the vector field corresponding to $g.\xi$ is again quadratic; in particular, applying this to constant fields $\mathbf{v}$, it follows that there is a unique quadratic polynomial $\delta_g : V \times V \to V$, $(x, v) \mapsto \delta_g(x)v$, linear in $v$, called *denominator of* $g$, such that

$$(g.\mathbf{v})(x) = \delta_g(x)v, \tag{8.10}$$

and, according to (8.5), if $g^{-1}.x \in V$,

$$\delta_g(x)v = dg(g^{-1}x)v = (dg^{-1}(x))^{-1}v, \qquad \text{whence } (\delta_g(x))^{-1} = dg^{-1}(x). \tag{8.11}$$

The same argument, applied to the Euler field $E$, shows that there is a unique quadratic polynomial $\nu_g : V \to V$, called the *numerator of* $g$, such that

$$(g.E)(x) = \nu_g(x). \tag{8.12}$$

According to (8.5), if $g^{-1}.x \in V$,

$$\nu_g(x) = dg(g^{-1}x)E(g^{-1}x) = dg(g^{-1}x)\,(g^{-1}x) = \delta_g(x)(g^{-1}x). \tag{8.13}$$

According to (8.11), if $g^{-1}x \in V$, the endomorphism $\delta_g(x) : V \to V$ is invertible, and hence we can multiply the last equality by its inverse. This proves:

**Theorem 8.8** (Automorphisms as quadratic fractional maps)**.** *Fix a base point $(o, o')$ and let $g \in \mathrm{Aut}(\mathcal{X})$ and $x \in V$ such that $g^{-1}(x) \in V$. Then the endomorphism $\delta_g(x) : V \to V$ is invertible, and using the differential $dg^{-1}(x)$, the value $g^{-1}(x)$ is given by*

$$g^{-1}(x) = (\delta_g(x))^{-1}\nu_g(x) = dg^{-1}(x)\nu_g(x).$$

*Moreover, we have $\delta_g(0) = D(g^{-1})^{-1}$ where $D(g)$ is defined by equation (3.6).*

*Proof.* Only the link with the triple decomposition (3.6) $g = tD(g)\tilde{t}$ remains to be proved: indeed, using that $d(\tilde{t})(0) = \mathrm{id}_V$,

$$(g^{-1}.\mathbf{v})(0) = (\tilde{t}^{-1}D(g)^{-1}t^{-1}.\mathbf{v})(0) = (\tilde{t}^{-1}D(g)^{-1}\mathbf{v})(0) = D(g)^{-1}v.$$

Replacing $g$ by $g^{-1}$ gives the claim. □

*Remark.* Reconstructing the geometry from a Jordan pair, it can be shown ([BeNe04]) that the condition that $\delta_g(x)$ is invertible is also sufficient to ensure that $g^{-1}x \in V$.

## 9. From Jordan geometries to Jordan pairs and Jordan algebras

9.1. **Double tangent bundle and Lie bracket.** The *Lie bracket of vector fields (or of infinitesimal automorphisms)* is defined in theorem B.3 by taking a commutator in the group of infinitesimal automorphisms of $TT\mathcal{X}$ over $\mathcal{X}$. Equipped with this bracket, the vector fields form a Lie algebra (or, more formally, a Lie algebra law over $\mathbb{K}$).

**Theorem 9.1** (The 3-graded Lie algebra)**.** *The space $\mathfrak{g} = \mathfrak{g}(\mathcal{X})$ of derivations of the Jordan geometry $(\underline{\mathcal{X}}, \underline{J}, \underline{\mathbf{r}})$ is a Lie-subalgebra of the algebra of all vector fields. The decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to a given base point $(o, o')$ is a 3-grading of the Lie algebra $\mathfrak{g}$, that is, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. In particular, the bracket of a constant field $\mathbf{v}(x) = v$ $(v \in V)$ and a homogeneous quadratic field $\xi_a(x) = Q(x)a$ $(a \in V')$ is given by the linear operator $D(v, a)x = Q(v, x)a$:*

$$[\mathbf{v}, \xi_a] = D(v, a).$$

*Proof.* Since the translation groups are abelian, their group commutators are trivial, and hence $[\mathfrak{g}_1, \mathfrak{g}_1] = 0 = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$. A standard computation in the semidirect product $P_o$, resp. $P_{o'}$, shows that $[\mathfrak{g}_0, \mathfrak{g}_{\pm 1}] \subset \mathfrak{g}_{\pm 1}$.

Let us now show that $[\mathfrak{g}_1, \mathfrak{g}_{-1}] \subset \mathfrak{g}_0$. The computation in the proof of theorem 8.7 shows that the group commutator of $L^{v,o}_{o'}$ and $L^{\varepsilon a, o'}_{o}$ in $\mathrm{Aut}(T\mathcal{X})$ is the infinitesimal automorphism given by the affine vector field

$$[L^{v,o}_{o'}, L^{\varepsilon a, o'}_{o}](x) = \varepsilon(Q(x - v)a - Q(x)a) = \varepsilon(Q(v)a - Q(x, v)a)\,.$$

We apply this relation with $v$ replaced by $\varepsilon_2 v$, and use that $Q(\varepsilon_2 v) = \varepsilon_2^2 Q(v) = 0$

$$[L^{\varepsilon_2 v,o}_{o'}, L^{\varepsilon_1 a, o'}_{o}](x) = -\varepsilon_1\varepsilon_2 Q(x, v)a = -\varepsilon_1\varepsilon_2 D(v, a)x\,.$$

This vector field is linear in $x$, and hence belongs to $\mathfrak{g}_0$; moreover, we have proved that this linear field is given by the linear operator $-D(v, a) : V \to V$. □

*Remark.* If $\mathbb{K}^\times$ contains scalars different from $\pm 1$, then another proof is obtained by noticing that the automorphisms $g := r^{o'}_{o}$, $r \in \mathbb{K}^\times$, act diagonally on $\mathfrak{g}$ by automorphisms with eigenvalue $r^i$ on $\mathfrak{g}_i$, and that the eigenspaces $E_\lambda$ satisfy $[E_\lambda, E_\mu] \subset E_{\lambda\mu}$.

9.2. **Linear Jordan pairs and 3-graded Lie algebras.** For every 3-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$, the pair of $\mathbb{K}$-modules $V^\pm := \mathfrak{g}_{\pm 1}$ becomes a *linear Jordan pair* with trilinear maps

$$V^\pm \times V^\mp \times V^\pm \to V^\pm, \qquad (x, a, z) \mapsto \{xaz\} := [[x, a], z]\,, \tag{9.1}$$

i.e., the trilinear map satisfies the identities

(1) $\{xaz\} = \{zax\}$
(2) $\{uv\{xyz\}\} = \{\{uvx\}yz\} - \{x\{vuy\}z\} + \{xy\{uvz\}\}$

Combining this with the preceding theorem gives immediatly:

**Theorem 9.2** (The linear Jordan pair of a Jordan geometry)**.** *Assume $\underline{\mathcal{X}}$ is a Jordan geometry over $\mathbb{K}$ with base point $(o,o') \in \mathcal{D}_2$. Then the pair of $\mathbb{K}$-modules $(V^+,V^-) = (U_{o'},U_o)$ becomes a linear Jordan pair with respect to $\{xaz\} = Q(x,z)a$, where $Q$ is the quadratic map defined by (8.8). The quadratic map, and hence the Jordan pair, depend functorially on the geometry with base point.*

9.3. **Quadratic Jordan pairs.** A *quadratic Jordan pair* is a pair $(V^+,V^-)$ of $\mathbb{K}$-modules together with quadratic maps $Q_\pm : V^\pm \to \mathrm{Hom}(V^\mp,V^\mp)$ such that the following identities hold in all scalar extensions (see [Lo75]; superscripts $\pm$ are omitted)[3]

(JP1) $D(x,y)Q(x) = Q(x)D(y,x)$
(JP2) $D(Q(x)y,y) = D(x,Q(y)x)$
(JP3) $Q(Q(x)y) = Q(x)Q(y)Q(x)$

Here, $\{xyz\} = D(x,y)z = Q(x,z)y = Q(x+z)y - Q(x)y - Q(z)y$, so $\{xyx\} = 2Q(x)y$. It is shown in loc. cit. that every quadratic Jordan pair is linear, and that the converse is true if $V$ has no 6-torsion.

**Theorem 9.3** (The quadratic Jordan pair of a Jordan geometry)**.** *Assume $\underline{\mathcal{X}}$ is a Jordan geometry over $\mathbb{K}$ with base point $(o,o') \in \mathcal{D}_2$. Then the pair of $\mathbb{K}$-modules $(V^+,V^-) = (U_{o'},U_o)$ becomes a quadratic Jordan pair with respect to the maps $Q_+ = Q$ and $Q_- = Q'$ defined by (8.4). The quadratic map, and hence the Jordan pair, depend functorially on the geometry with base point.*

*Proof.* If $V$ has no 6-torsion, by the preceding remarks, the Jordan pair is linear, and hence the claim follows from Theorem 9.2. In the general case, one can adapt to our framework the arguments given in the proof of [Lo79], Th. 4.1; however, since the computations are fairly long and involved, we will not reproduce them here in full detail. The main ingredients used in loc. cit. are the relations between "usual" translations and quasi-translations (in our framework: Lemma 3.2), and the behavior of (quasi-) translations with respect to scalars ((5.2), (5.3)); these relations furnish a description of the elementary projective group by generators and relations, from which the Jordan pair identities are deduced by using algebraic differential calculus in the setting of algebraic geometry ([Lo79], page 40). All of these arguments carry over to our setting; we only have to replace the argument of Zariski-density used repeatedly in loc. cit. by the following more general argument, which in turn is a geometric version of "Koecher's principle on identitities" saying that *a Jordan polynomial which vanishes in all quasi-invertible Jordan pairs is zero* (see [Lo95], p. 97, for this formulation).

**Lemma 9.4** ("Koecher's principle")**.** *A Jordan polynomial which vanishes on all quasi-invertible quadruples of Jordan geometries is zero. More formally, this means: assume $P = P_{\mathcal{X}}$ is a smooth law, depending functorially on Jordan geometries $\mathcal{X}$, such that $P_{\mathcal{X}}$ is defined for quadruples $(a,o,o',x) \in \mathcal{D}_4(\mathcal{X})$ and is polynomial in $(a,x)$ for all fixed base points $(o,o') \in \mathcal{D}_2(\mathcal{X})$; if $P$ vanishes for all quasi-invertible quadruples $(a,o,o',x) \in \mathcal{D}_4'(\mathcal{X})$ in all Jordan geometries $\mathcal{X}$, then $P = 0$.*

[3] As Loos remarks in loc. cit., p.1.3, it suffices to consider scalar extensions that are free and finite dimensional over $\mathbb{K}$; in particular, it suffices to consider Weil algebras.

*Proof of the lemma.* Considering $(o, o')$ as fixed, we suppress it in the notation. Let $P(a, x) = 0$ be a polynomial identity of degree $k$, valid for all quasi-inverible quadruples $(a, o, o', x) \in \mathcal{D}'_4$ in Jordan geometries $\mathcal{X}$. Let $J^k\mathcal{X}$ be the scalar extension of $\mathcal{X}$ by the *jet ring* $J^k\mathbb{K} := \mathbb{K}[X]/(X^{k+1}) = \mathbb{K}[\delta]$ (see (B.2)). Just as in case $k = 1$ (tangent bundle), $J^k\mathcal{X}$ is a bundle over $\mathcal{X}$, alled the *k-th order jet bundle of* $\mathcal{X}$: in every chart of the canonical atlas, it has a product structure, and likewise, the set $\mathcal{D}'_4(J^k\mathcal{X})$ is a bundle over $\mathcal{D}'_4(\mathcal{X})$. Fixing $(o, o')$ as base point, all elements $(\delta a, \delta x)$ with $(x, a) \in V^+ \times V^-$ are hence quasi-invertible (i.e., $(\delta a, o, o', \delta x)$, lying in the fiber over $(o', o, o', o)$, belongs to $\mathcal{D}'_4(J^k\mathcal{X})$). By assumption, we thus have $P(\delta x, \delta a) = 0$. Expanding this polynomial and ordering according to powers $\delta, \delta^2, \ldots, \delta^k$, shows that all homogeneous parts of the polynomial $P$ vanish, and hence $P = 0$. This proves the lemma and the theorem. □

The proof of the lemma takes up the idea that, geometrically, "modules" of Jordan pairs should correspond to *bundles in the category of Jordan geometries*. Vector bundles then correspond to *representations* in the sense of [Lo75], 2.3; they are scalar extensions by vector algebras (cf. B.7). In this context, the proof of the lemma leads to a geometric version of the *permanence principle* [Lo75], 2.8.

9.4. **Jordan geometries with polarities and Jordan triple systems.**

**Theorem 9.5** (The Jordan triple system of a Jordan geometry with polarity)**.** *Assume $\underline{\mathcal{X}}$ is a Jordan geometry over $\mathbb{K}$ with polarity $\underline{p} : \underline{\mathcal{X}} \to \underline{\mathcal{X}}$ and base point $(o, o') \in \mathcal{D}_2$ such that $o' = p(o)$. Then the $\mathbb{K}$-module $V = U_{o'}$ becomes a quadratic Jordan triple system with respect to the map $Q(x)y = pQ^{\pm}(x)p(y)$.*

*Proof.* The polarity $p$ defines an involution of the Jordan pair $(V^+, V^-)$ from the preceding theorem, and a Jordan pair with involution is the same as a Jordan triple system (cf. [Lo75]). □

9.5. **Unital Jordan and associative algebras.** Assume $(a, b, c) = (o, o', e)$ is a pairwise transversal triple. According to Lemma 2.2, the set $U = U_{oo'}$ is a symmetric space with product $s_x(y) = J^{oo'}_x(y)$. Since $J^{oo'}_x$ exchanges $o$ and $o'$, it induces a $\mathbb{Z}$-linear bijection of $V = U_{o'}$ onto $V' = U_o$. Fix the point $e$ as base point in $U$, and define, for $x \in U$, a linear map

$$Q_x := Q^{oo'}_{xe} = J^{oo'}_x \circ J^{oo'}_e|_V : V \to V. \tag{9.2}$$

Since $Q_x(Q_y)^{-1}y = J^{oo'}_x J^{oo'}_e J^{oo'}_e J^{oo'}_y y = J^{oo'}_x(y) = s_x(y)$, the structure of $U$ can be entirely described in terms of the map $U \times V \to V$, $(x, y) \mapsto Q_x(y)$.

**Theorem 9.6** (The Jordan algebra of a Jordan geometry with pairwise transversal triple)**.** *Assume $(\underline{\mathcal{X}}, \underline{J})$ is a Jordan geometry over $\mathbb{K}$ with pairwise transversal triple $(a, b, c)$. Choose $(a, b) =: (o, o') \in \mathcal{D}_2$ as base point. Then the $\mathbb{K}$-module $V = U_{o'}$ becomes a quadratic Jordan algebra with quadratic map $U_x(y) = Q(x)Q(e)^{-1}y$ and with unit element $e = c$. The set $V^\times$ of invertible elements agrees with the symmetric space $U = V \cap V'$, and the quadratic map $Q(x)$ agrees with $Q_x$ defined by (9.2).*

*Proof.* We have to show that the Jordan pair $(V, V')$ associated to the base point $(o, o')$ has invertible elements (cf. [Lo75]); more precisely, we show that every element $x$ from $U = U_{oo'}$ is invertible. Indeed, this follows from the fact that $j := J^{oo'}_e$

is an automorphism of $\mathfrak{g}$ exchanging $o$ and $o'$, hence exchanging also $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$: using numerators and denominators, it is shown exactly as in [BeNe04], Section 5.1, that, for all $x \in V^\times$, we have the formula

$$j(y) = Q(e)Q(y)^{-1}Q(e)y.$$

In particular, since $j(e) = e$, it follows that $e$ is an invertible element, thus $(V, e)$ is a quadratic Jordan algebra with Jordan inversion $j$ ([Lo75]). From this it follows is in [BeNe04] that $Q(x) = Q_x$ and that $V^\times = U$. $\square$

**Theorem 9.7** (The associative algebra of an associative geometry with transversal triple)**.** *Assume $(o, o', e)$ is a closed transversal triple in an associative geometry $(\underline{\mathcal{X}}, \underline{M})$. Then the group law of $U_{oo'}$ extends to an associative algebra structure on $V = U_{o'}$, with bilinear product induced by the second tangent law $TTU_{oo'}$.*

*Proof.* Let $V := V_{o'}$, $V' := V_o$ and $V^\times := U_{oo'} = V \cap V'$. From the properties of an associative geometry, it follows that $(V^\times, e)$ is a Lie group with group law

$$xz = (xez)_{oo'} = M^{oo'}_{xz}(e) = L^{oo'}_{xe}(z) = R^{oo'}_{ez}(x),$$

which is bilinear for the linear structure $(V, o)$. Let $m : U \times U \to U$ be the group law of the Lie group $U = V^\times$; then the group law of $TTU$ is given by $TTm$ which is scalar extension of $m$ by the ring $TT\mathbb{K} = \mathbb{K}[\varepsilon_1, \varepsilon_2]$. The map

$$\varepsilon_1 V \times \varepsilon_2 V \to \varepsilon_1\varepsilon_2 V, \qquad (\varepsilon_1 u, \varepsilon_2 v) \mapsto (\varepsilon_1 u)(\varepsilon_2 v) = TTm(\varepsilon_1 u, \varepsilon v)$$

is bilinear, since, for one of the arguments fixed, the remaining map is a tangent map. Thus a bilinear product $uv$ on $V$ is defined by requiring

$$\varepsilon_1\varepsilon_2(uv) := (\varepsilon_1 u)(\varepsilon_2 v).$$

The group law $T^3m$ on $T^3U$ is associative, thus, in particular, $\varepsilon_1 u(\varepsilon_2 v \cdot \varepsilon_3 w) = (\varepsilon_1 u \cdot \varepsilon_1 v)\varepsilon_3 w$, which, by definition of the product, yields $u(vw) = (uv)w$. Thus $V$ with product $uv$ is an associative algebra. Moreover, if $u, v \in U$, then left and right multiplications $L_u : V \to V$ and $R_v : V \to V$, are linear maps, hence agree with their tangent maps, implying that the products $uv$ taken in $U$ and in $V$ agree. $\square$

*Remark.* If the geometry is not self-dual, then, for a fixed base point $(o, o') \in \mathcal{D}_2$, the pair $(V, V')$ becomes an *associative pair* (see [BeKi09a] for relevant definitions).

9.6. **Explicit Jordan theoretic formulas for the Jordan structure map.** We give Jordan theoretic formulae expressing $J^{xz}_a(y)$ and $J^{xz}_a(b)$ in terms of the Jordan pair $(V^+, V^-)$ associated to a fixed base point $(o, o')$. Notation for Jordan pairs is as in subsection 9.3; in order to simplify formulas, we suppress subscripts $\pm$, by assuming always that $o, v, x, y, z \in V^\pm$ and $o', a, b, c \in V^\mp$. We use the following standard definitions from Jordan pair theory (cf. [Lo75]):

(1) the *Bergman operator* is defined by $B(y, b)x := x - D(y, b)x + Q(y)Q(b)x$,
(2) $(x, a)$ is called *quasi-invertible* if $B(x, a)$ is invertible, and then one defines

$$x^a := B(x, a)^{-1}\big(x - Q(x)a\big)\,,$$

and then the *inner automorphism defined by* $(x, b)$ is given by

$$\beta(x, a) := \big(B(x, a), B(a, x)^{-1}\big).$$

In order to obtain the general formulae for $J_a^{xz}$, we proceed in three steps:

Step 1: two of the points are base points (case $(z,a)=(o,o')$, Lemma 9.8)
Step 2: one point among $x,z$ is the base point $o$ (Lemma 9.9)
Step 3: general case – all three points different from base points (Theorem 9.10).

**Lemma 9.8.** *For $x,v\in V^+=U_{o'}$ and $a\in V^-=U_o$, the following holds: the affine space structure of $V^+$ (and dually of $V^-$) is described by the translations: $L_{o'}^{vo}(x)=v+x$ and the major dilations: $r_x^{o'}(y)=(1-r)x+ry$, and in particular, $J_{o'}^{oo}(x)=-x=(-1)_o^{o'}(x)$. The Jordan theoretic Bergman operator coincides with the "geometric Bergman operator" defined in (3.12) : $\beta(x,a)=B_{xa}^{o,o'}$, and $x\top a$ if, and only if, $(x,a)$ is quasi-invertible, and then*

$$L_o^{ao'}(x)=x^a=B(x,a)^{-1}\big(x-Q(x)a\big).$$

*If $(-v,a)$ is quasi-invertible, then*

$$J_o^{ao'}(v)=L_o^{ao'}(-v)=(-v)^a=-B(-v,a)^{-1}\big(Q(v)a+v\big),$$

*Proof.* The statement on the action of $L_o^{a,o}$ and $J_o^{ao'}$ by (quasi-)inverses is proved in [Lo79], Lemma 4.7 (the framework in loc. cit. is slightly different from ours, but the proof carries over by using Lemma 9.4; see also [BeNe04], Section 3 for another proof in a different setting). □

**Lemma 9.9.** *If $(-v,a)\in V^+\times V^-$ is quasi-invertible, then*

$$J_o^{ao'}(v)=L_o^{ao'}(-v)=(-v)^a=-B(-v,a)^{-1}\big(Q(v)a+v\big),$$

*and, with $v':=J_a^{vo}(o')=2a+Q(a)v$, we have the triple decomposition (3.7)*

$$J_a^{vo}=L_{o'}^{v,o}\circ(-\beta(v^{-a},a))\circ L_o^{o',v'}=L_{o'}^{v,o}\circ(-\beta(-v,a))^{-1}\circ L_o^{o',v'}\,.$$

*Proof.* To compute $J_a^{vo}$, note that

$$J_a^{v,o}=J_o^{ao'}J_{o'}^{J_o^{ao'}v,o}J_o^{ao'}=J_o^{ao'}J_{o'}^{w,o}J_o^{ao'}$$

with $w=J_o^{ao'}v=(-v)^a$. Now use the "commutation relation" from Lemma 3.2

$$J_o^{ao'}J_{o'}^{wo}=J_{o'}^{J_o^{o'a}(w),o}\,\beta(w,-a)\,J_o^{o',J_{o'}^{ow}(a)}$$

to get the triple decomposition

$$\begin{aligned}J_a^{v,o}=J_o^{ao'}J_{o'}^{w,o}J_o^{ao'}&=J_{o'}^{J_o^{o'a}(w),o}\,\beta(w,-a)\,J_o^{o',J_{o'}^{ow}(a)}J_o^{ao'}\\&=L_{o'}^{J_o^{o'a}(w),o}\big(-\beta(w,-a)\big)\,L_o^{J_{o'}^{ow}(a),a}\\&=L_{o'}^{v,o}\big(-\beta((-v)^a,-a)\big)\,L_o^{o',v'}\\&=L_{o'}^{v,o}\big(-\beta(-v,a)\big)^{-1}L_o^{o',v'}\end{aligned}$$

where $v'=a-J_{o'}^{ow}(a)$; note also that $\beta((-v)^a,-a))=\beta(v^{-a},a)=\beta(v,-a)^{-1}$ (identity JP 35 from [Lo75]). We give another expression for $v'$ by using the *symmetry principle* $x^y=x+Q(x)y^x$ ([Lo75], Prop. 3.3)

$$\begin{aligned}v'=a-J_{o'}^{ow}(a)&=a-(-a)^w=a+a^{-w}\\&=2a+Q(a)(-w)^a=2a+Q(a)v\,.\end{aligned}$$

This proves the triple decomposition for $J_a^{vo}$ given in the claim. $\square$

By uniqueness of the triple decomposition, it follows that

$$J_a^{v,o}(o') = 2a + Q(a)v. \tag{9.3}$$

The formula for $J_a^{vo}$ has also been given, in another framework, in [Be08], Th. 2.2.

**Theorem 9.10.** *For $x, y, z \in V^+$ and $a, b, c \in V^-$, the following holds: if $(x, -a)$ and $(z, Q(a)z)$ are quasi-invertible, then we have the triple decomposition (3.7)*

$$J_a^{xz} = L_{o'}^{vo} \circ h \circ L_o^{o'v'},$$

*where*

$$\begin{aligned}
v &= J_a^{xz}(o) = (xoz)_a = (x^{-a} + z^{-a})^a = x + B(x, -a)z^{Q(a)x}\,, \\
v' &= J_a^{xz}(o') = 2a + Q(a)x + Q(a)B(x, -a)z^{(Q(a)x)} \\
&= 2a + Q(a)x + B(a, -x)(Q(a)z))^x, \\
h &= D(J_a^{xz}) = -\beta\big((-v)^a, -a\big) = -\beta\big((-x)^a + (-z)^a, -a\big)\,.
\end{aligned}$$

*Using this notation, the action of $J_a^{xz}$ on $V^+$, resp. on $V^-$, is given by*

$$\begin{aligned}
J_a^{xz}(y) &= v - \beta(v^{-a}, a)y^{-v'} \\
&= x + B(x, -a)z^{Q(a)x} - B\big(x^{-a} + z^{-a}, -a\big)y^{(-2a - Q(a)x - B(a,-x)(Q(a)z)^x)} \\
J_a^{xz}(b) &= v' - \beta(v^{-a}, a)^{-1}b^{-v} = v' - \beta(v, a)b^{-v} = v' - B(a, v)^{-1}b^{-v} \\
&= 2a + Q(a)x + B(a, -x)(Q(a)z))^x - \\
&\qquad B\big(a, x + B(x, -a)z^{Q(a)z}\big) \cdot b^{(-2a - Q(a)x - B(a,-x)(Q(a)z))^x)}
\end{aligned}$$

*If, moreover, $(y, -a)$ is quasi-invertible, we have also*

$$J_a^{xz}(y) = (x^{-a} - y^{-a} + z^{-a})^a.$$

*Proof.* Using the "transplantation formula (2.2), $J_a^{xz} = J_a^{J_a^{xz}(o),o} = J_a^{v,o}$, we have to compute the value $v = J_a^{xz}(o)$, and then apply the preceding theorem to get the expressions from the claim. To compute $J_a^{xz}(o)$, start by observing that

$$J_{o'}^{xz}(y) = (xyz)_{o'} = x - y + z$$

whence, using that $L_o^{ao'}(o') = a$ and, by the preceding theorem, $L_o^{o'a}(y) = y^{-a}$,

$$J_a^{xz}(y) = J_{L_o^{ao'}(o')}^{xz}(y) = L_o^{ao'} J_{o'}^{L_o^{o'a}x, L_o^{o'a}z} L_o^{o'a}(y) = (x^{-a} - y^{-a} + z^{-a})^a\,,$$

proving the last formula from the claim, which for $y = o$ gives $J_a^{xz}(o) = (x^{-a} + z^{-a})^a$. By using [Lo75], Th. 3.7: $(x + z)^y = x^y + B(x, y)^{-1}.z^{(y^x)}$ and $x^{y+z} = (x^y)^z$, as well as (JP35) $B(x, y)^{-1} = B(x^y, -y)$ and the "symmetry principle" $x^y = x + Q(x)y^x$, we get the following Jordan theoretic formula

$$\begin{aligned}
v = (x^{-a} + z^{-a})^a &= (x^{-a})^a + B(x^{-a}, a)^{-1}(z^{-a})^{(a^{(x^{-a})})} \\
&= x + B(x, -a)z^{(a^{(x^{-a})} - a)} \\
&= x + B(x, -a)z^{(Q(a)x)}\,.
\end{aligned}$$

It follows that

$$\begin{aligned} J_a^{xz}(o') = v' &= 2a + Q(a)v \\ &= 2a + Q(a)x + Q(a)B(x,-a)z^{(Q(a)x)}. \end{aligned}$$

Now replace $v$ and $v'$ by these expressions in the triple decomposition from the preceding theorem, and writing its action on $V^+$ and (using that $(J_a^{xz} = (J_a^{xz})^{-1})$ on $V^-$, we get the explicit formulae from the claim. □

## 10. From Jordan pairs to Jordan geometries

The aim of this chapter is to construct a Jordan geometry starting from a Jordan pair $(V^+, V^-)$, or from a Jordan algebra:

**Theorem 10.1.** *For every Jordan pair $(V^+, V^-)$ over $\mathbb{K}$, there is a Jordan geometry, having $(V^+, V^-)$ as associated Jordan pair. More precisely, there is a functor from the category of Jordan pairs over $\mathbb{K}$ to Jordan geometries over $\mathbb{K}$ with base point. Under this functor, unital Jordan algebras correspond to Jordan geometries with a pairwise transversal triple.*

*Proof.* If 2 is invertible in $\mathbb{K}$, then, as shown in [Be02], Th. 10.1, there is a generalized projective geometry with base point having $(V^+, V^-)$ as associated Jordan pair; by Theorem 7.3, this geometry is a Jordan geometry, and thus the theorem is proved in this case.

If 2 is not invertible in $\mathbb{K}$, we cannot use midpoints in order to define the inversions $J_a^{xz}$, and hence we have to modify the construction: the set $\mathcal{X}$ and inversions of the type $J_a^{xx} = (-1)_x^a = (-1)_a^x$ are defined by the same methods as in [Be02], but inversions of the type $J_a^{xz}$ for $x \neq z$ have to be defined in a different way: we define first the translation operators $L_a^{xz}$, essentially by using a "Jordan version" of the exponential map for a Kantor-Koecher-Tits algebra, and then let

$$J_a^{xz} := L_a^{xz} J_a^{zz}. \tag{10.1}$$

To be more specific, recall from [Be02] or [BeNe04] that a transversal pair $(x, a) \in \mathcal{D}_2$ corresponds to an *Euler operator*, i.e., to a 3-grading of the associated "Kantor-Koecher-Tits algebra" $\mathfrak{g}$ of the Jordan pair. The base point $(o, o')$ corresponds to the 3-grading $\mathfrak{g} = V^+ \oplus \mathfrak{h} \oplus V^-$, coming directly with the construction of $\mathfrak{g}$. Thus, given a transversal pair $(x, a)$, we may assume without loss of generality that $(x, a) = (o, o')$ is the base point; then $U_a$ is naturally identified with $V^+$, and hence the condition $z \in U_a$ means that $z \in V^+$. Defining the "exponential" $\exp(z) \in \mathrm{Aut}(\mathfrak{g})$ as in [Lo95], we then let

$$L_a^{xz} := \exp(z) \tag{10.2}$$

(this depends on $(x, a)$ since $\exp(z)$ is defined with respect to a fixed 3-grading), and define $J_a^{xz}$ by (10.1). Now one has to prove that the Jordan structure map thus defined satisfies our axioms – this proof is quite lengthy, and essentially amounts to reverse the computations leading to the "explicit formulae" given in the preceding section; details are similar to the proof of [Be02], Th. 10.1, and will be omitted. □

## Appendix A. Symmetry actions

In this appendix, we recall the definition of some algebraic structures (torsors, reflection spaces, symmetric spaces), and we define their "actions" on a set. Since a group is defined by a binary law, there are just two kinds of actions (left and right actions); a torsor is defined by a ternary law, and therefore we have three kinds of actions: *left, right and symmetry torsor actions.*

### A.1. **Torsors.**

**Definition A.1.** *A* torsor *is a set $G$ with a map $G^3 \to G$, $(x,y,z) \mapsto (xyz)$ satisfying the following algebraic identities:*

(PA) para-associative identity*: $((xuv)wz) = (x(wuv)z) = (xu(vwz)))$.*
(IP) idempotency identity $(xxy) = y = (yxx)$.

*The* opposite torsor *is $G$ with $(xyz)^{opp} = (zyx)$, and a torsor is called* commutative *if $G = G^{opp}$, i.e., it satisfies the identity*

(C) $(xyz) = (zyx)$.

*Categorial notions are defined in the obvious way. In every torsor,* left-, right- and middle multiplication operators *are the maps $G \to G$ defined by equation (0.1).*

Every group $(G, e, \cdot)$ becomes a torsor by letting $(xyz) = xy^{-1}z$, and every torsor is obtained in this way: thus torsors are "groups with origin forgotten".

**Lemma A.2.** *In every torsor, the middle multiplication operators satisfy*

(SA) $m_{xy} \circ m_{uv} \circ m_{rs} = m_{m_{xr}(v), m_{sy}(u)}$,
(IP) $m_{xz}(x) = z$, $m_{xz}(z) = x$.

*Conversely, a set $G$ with a map $m : G \times G \to \mathrm{Bij}(G)$, $(x,z) \mapsto m_{xz}$ satisfying (SA) and (IP), becomes a torsor by letting $(xyz) := m_{xz}(y)$. The operator $m_{xz}$ is then invertible with inverse operator $m_{zx}$.*

*Proof.* Applied to an element $z$, (SA) reads: $(x(u(rzs)v)y) = ((xvr)z(suy))$. By direct check, it is seen that this holds in any torsor. Conversely, using (SA) and invoking (IP) twice, we get back (PA):
$(xy(uvw)) = m_{x, m_{uw}(v)}(y) = m_{m_{xy}(y), m_{uw}(v)}(y) = m_{xw} m_{vy} m_{yu}(y) = (x(vuy)w)$. □

Letting $u = y$ and $v = r$ in (SA), we get by (IP) the "Chasles relation"

(SA') $m_{xy} \circ m_{yv} \circ m_{vs} = m_{xs}$.

It can be shown that, conversely, (SA') and (IP) imply (SA).

### A.2. **Symmetry torsor actions.**

**Definition A.3.** *Let $(G, (- - -))$ be a torsor and $X$ a set. A* symmetry torsor action on $X$ *is a map of $G \times G$ into the set of bijections of $X$*

$$G \times G \to \mathrm{Bij}(X), \qquad (x,z) \mapsto M_{xz}$$

*such that the following identities hold*

(STA1) $M_{xz} \circ M_{zx} = \mathrm{id}_X$
(STA2) $M_{xz} \circ M_{uv} \circ M_{ab} = M_{(xva),(buz)}$

*The symmetry torsor action is called* commutative*, or:* an inversive action *if*

(CTA) $M_{xz} = M_{zx}$

*(equivalently, if all $M_{xz}$ are of order two). According to the preceding lemma, every torsor has a natural symmetry action on itself, given by $M_{xz} = m_{xz}$, which we call the* regular symmetry action (of $G$ on itself)*. Spaces with $G$-symmetry action form a category in the obvious way, and* subspaces *and* direct products *can be defined in this category.*

As above, letting $z = u$ and $v = a$ in (STA2), we get a *middle Chasles relation*

$$M_{xz} \circ M_{za} \circ M_{ab} = M_{xb}. \tag{A.1}$$

However, it is not true that (A.1) and (STA1) imply (STA2).

*Remark.* These axioms have the following categorial interpretation: with the usual torsor structure $(fgh) = fg^{-1}h$ on $\mathrm{Bij}(X)$, (STA2) can be rewritten in the form

$$\big(M_{xz} M_{vu} M_{ab}\big) = M_{(xva),(buz)}$$

which means that $M$ can be interpreted as a torsor homomorphism

$$M : G \times G^{opp} \to \mathrm{Bij}(X), \qquad (x, z) \mapsto M_{xz}\,.$$

*Remark.* It is not true that a symmetry action of a commutative torsor is always inversive. For instance, consider the following situation: if $H$ is a subgroup of a group $G$, then the regular action of $G$ on itself induces a symmetry action $H \times H \to \mathrm{Bij}(G)$ given by $M_{h,h'}(g) = hg^{-1}h'$. This action is in general not commutative, even if $H$ as a group is commutative. Indeed, $(M_{xz})^2(u) = xz^{-1}ux^{-1}z$ may be a non-trivial map on $G$, although it is trivial on $H$.

*Remark.* Assume, in the preceding situation, that $G$ is a compact Lie group and $H$ a maximal torus. Then the elements $M_{x,x^{-1}}$ with $x^2 \in Z(G)$ (in particular, those with $x^2 = e$) are involutive. On the other hand, when $x$ normalizes $H$, they stabilize $H$, and when $x$ centralizes $H$, they act trivially on $H$. Taken together, we get the following interpretation of the Weyl group, together with its set of generators of order two: it is the torsor of middle multiplications stabilizing $H$ and $e$, generated by its involutive elements.

A.3. **Left- and right torsor actions.**

**Lemma A.4** (Left- and right action)**.** *For a symmetry torsor action, the* left translation $L_{xv} \in \mathrm{Bij}(X)$ *defined by*

$$L_{xv} := M_{xz} \circ M_{zv}$$

*depends only on $x, v \in G$, but not on the choice of $z$. We have the identities*

(LTA1) $L_{xx} = \mathrm{id}_X$,

(LTA2) $L_{xv} L_{uw} = L_{(xvu),w} = L_{x,(wuv)}$.

*Similarly, the* right translation $R_{vx} := M_{zv} \circ M_{xz}$ *does not depend on $z$. Moreover, for any symmetry torsor action, left and right translations commute:*

$$L_{xv} \circ R_{yw} = R_{yw} \circ L_{xv},$$

*and if the symmetry action is inversive, then left- and right action agree: $L_{xv} = R_{xv}$.*

*Proof.* All claims are checked by direct computations. We show that $L_{xv}$ is well-defined: indeed, the equality $M_{xz}M_{zv} = M_{xw}M_{wv}$ is a direct consequence of (A.1) and (STA1). In order to show that $L_{xv}$ and $R_{wu}$ commute, we may first reduce to the case $x = w$, by observing that (LTA2) gives us the *left Chasles relation*

$$L_{xv} \circ L_{vw} = L_{xw}. \tag{A.2}$$

The relation $L_{xv}R_{xw} = R_{xw}L_{xv}$ is now proved by making appropriate choices when expressing the $L$- and $R$-operators by two $M$-operators. Finally, assuming that the action is inversive, we get $L_{xv} \circ R_{vx} = M_{xx}M_{xv}M_{xv}M_{xx} = \mathrm{id}_X$. □

**Corollary A.5** (Transplantation formula)**.** *Given a* commutative *symmetry torsor action, we have, for all $x, o, z \in G$,*

$$M_{xz} = M_{xo}M_{oo}M_{zo} = M_{(xoz),o} = M_{m_{xz}(o),o}.$$

*Proof.* $M_{xo}M_{oo}M_{zo} = L_{xo}M_{zo} = L_{xo}M_{oz} = M_{xz}$ □

There are some other algebraic identities valid for every symmetry torsor action, such as the *intertwining relation between left and right actions*

$$M_{xx} \circ L_{vx} \circ M_{xx} = R_{(xvx),x} \tag{A.3}$$

which can also be written, for $x = e$ and $M_{ee}(g) = j(g) = g^{-1}$, and $L_g := L_{g,e}$,

$$j \circ L_g = R_{g^{-1}} \circ j. \tag{A.4}$$

Note also that identities for $R$-operators correspond to identities for $L$-operators, with reversed composition in $\mathrm{Bij}(X)$ and reversed order of indices.

**Definition A.6.** *A* left torsor action *of a torsor $G$ on a set $X$ is a map*

$$G \times G \to \mathrm{Bij}(X), \qquad (x, y) \mapsto L_{xy}$$

*(if there is risk of confusion we write also $L_{x,y}$ instead of $L_{xy}$) such that the identities (LTA1) and (LTA2) from the preceding lemma hold.* Right actions *are defined similarly. The* regular left (right) action of $G$ on itself *is defined by the lemma.*

**Lemma A.7.** *Let $G$ be a group with neutral element $e$ and its usual torsor structure $(xyz) = xy^{-1}z$. Then we have an equivalence of categories between left group actions of $G$ and left torsor actions of $G$.*

*Proof.* Given a left group action $G \times X \to X$, $(g, x) \mapsto L_g(x)$, we let

$$L_{x,y} := L_x(L_y)^{-1}.$$

Conversely, given a left torsor action, let $L_g := L_{g,e}$, and the claim follows by a straightforward check of definitions. □

A.4. **On the structure of symmetry actions.** The preceding two lemmas say that left- and right actions of torsors are nothing new, compared to usual group actions, whereas symmetry actions are commutig left- and right actions together with some operator $j$ satisfying the intertwining relation (A.4). One may check that, conversely, if we have commuting left and right actions of a group $G$ on a set $X$, together with a map of order two $j : X \to X$ satisfying the intertwining relation, we can reconstruct a symmetry torsor action.

Motivated by this obervation, one will look at the behaviour of $G \times G$-orbits $\mathcal{O}$ under $j$. If $j(\mathcal{O}) \cap \mathcal{O}$ is empty, then $j$ is equivalent to the exchange map between two copies of this orbit, exchanging "left" and "right". In the other case, one will have to distinguish whether $j$ has a fixed point in $\mathcal{O}$, or not. If there is a fixed point $p$, the stabilizer $H$ of $p$ must be a normal subgroup, and we get a version of the regular symmetry action on the quotient group $G/H$. The remaining case, where $j$ has no fixed point in $\mathcal{O}$, seems to be more difficult to analyze.

A.5. **Reflection spaces and symmetric spaces.**

**Definition A.8.** *A* reflection space *is a set together with a map* $s : M \to \mathrm{Bij}(M)$*,* $x \mapsto s_x$ *such that the following identities hold:*

(R1) *(idempotency)* $s_x(x) = x$,
(R2) *(inversivity)* $s_x \circ s_x = \mathrm{id}_M$,
(R3) *(distributivity)* $s_x s_z s_x = s_{s_x(z)}$.

*Reflection spaces form a category. The subgroup* $G(M)$ *of* $\mathrm{Aut}(M, \mu)$ *generated by all* $s_x s_y$ *with* $(x, y) \in M^2$ *is called the* transvection group *of* $M$*. If, moreover,* $M$ *is a smooth manifold (in the usual sense, or in the general algebraic framework of Appendix B), then* $M$ *is called a* symmetric space *if*

(T) *for every* $x \in M$*, the tangent map* $T_x(s_s)$ *of* $s_x$ *at its fixed point* $x$ *is equal to* $-\mathrm{id}_{T_x M}$.

*Remark.* Every torsor with $s_x(y) := m_{xx}(y) = (xyx)$ becomes a reflection space, and every Lie torsor with this law becomes a symmetric space (Theorem B.2).

A.6. **Symmetry action of a reflection space.**

**Definition A.9.** *Let* $M$ *be a reflection space and* $X$ *a set. A* symmetry action of $M$ on $X$ *is a map* $M \to \mathrm{Bij}(X)$*,* $x \mapsto S_x$ *such that*

(S1) $S_x \circ S_x = \mathrm{id}_X$,
(S2) $S_x S_y S_x = S_{s_x(y)}$.

*For* $X = M$*, we have a symmetry action of* $M$ *on itself given by* $S_x = s_x$*, which we call the* regular symmetry action (of $M$ on itself)*. As above, categorial notions are defined.*

**Lemma A.10.** *If* $(x, z) \mapsto M_{xz}$ *is a symmetry action of a torsor* $G$*, then we get a symmetry action of* $G$*, seen as reflection space, by* $G \to \mathrm{Bij}(X)$*,* $x \mapsto S_x := M_{xx}$*.*

*Proof.* $S_x S_y S_x = M_{xx} M_{yy} M_{xx} = M_{(xyx),(xyx)} = M_{s_x(y), s_x(y)} = S_{s_x(y)}$ □

In general, left or right actions of $G$ do not give rise to symmetry actions of the symmetric space $G$; and in general, symmetry actions of reflection spaces do not give rise to actions of the group $G(M)$ (cf. remarks in [Be00]: already on the infinitesimal level this does not hold since the standard imbedding of a Lie triple system is in general not functorial).

**Definition A.11.** *Given a symmetry action* $M \to \mathrm{Bij}(X)$*,* $x \mapsto S_x$*, we define the* transvection operators *by* $Q_{xy} := S_x S_y \in \mathrm{Bij}(X)$.

These operators share some properties with the translation operators of left or right torsor actions: we have an analog of the Chasles relation (A.2) $Q_{xy}Q_{yz} = Q_{xz}$, and $Q_{xx} = \mathrm{id}_X$, whence $(Q_{xy})^{-1} = Q_{yx}$, but in contrast to left and right translations, the composition of two transvections is in general no longer a transvection. Instead, we have the *fundamental formula*

$$Q_{xy}Q_{zy}Q_{xy} = S_xS_yS_zS_yS_xS_y = S_{S_xS_y(z)}S_y = Q_{Q_{xy},y}\,. \tag{A.5}$$

## Appendix B. Differential geometry of smooth laws

In this appendix, a commutative base ring $\mathbb{K}$ with unit 1 is fixed. The following purely algebraic approach to differential geometry will be developed in full detail in [BeS13]. Methods and proof strategies follow the patterns developped in [Be08, BeS11], the difference being that here we do not assume that $\mathbb{K}$ carries a topology.

B.1. **Weil algebras.** A $\mathbb{K}$-*Weil algebra* is an associative and commutative algebra

$$\mathbb{A} = \mathbb{K} \oplus \mathring{\mathbb{A}} \tag{B.1}$$

where $\mathring{\mathbb{A}}$ is a nilpotent ideal of $\mathbb{A}$ which is free and finite-dimensional as a $\mathbb{K}$-module. Weil algebras form a category $\underline{\mathbb{K}\text{-Walg}}$, where morphisms are algebra homomorphisms preserving the decomposition. Note that projection $\mathbb{A} \to \mathbb{K}$ and injection $\mathbb{K} \to \mathbb{A}$ are morphisms. Main examples are the *k-th order jet rings*

$$J^k\mathbb{K} := \mathbb{K}[X]/(X^{k+1}) = \mathbb{K}[\delta], \qquad (\delta^{k+1} = 0), \tag{B.2}$$

and for $k = 1$ we get the *tangent ring of* $\mathbb{K}$ (or: *ring of dual numbers*)

$$T\mathbb{K} := J^1\mathbb{K} = \mathbb{K}[X]/(X^2) = \mathbb{K} \oplus \varepsilon\mathbb{K} \qquad (\varepsilon^2 = 0). \tag{B.3}$$

Given two Weil algebras $\mathbb{A} = \mathbb{K} \oplus \mathring{\mathbb{A}}$, $\mathbb{B} = \mathbb{K} \oplus \mathring{\mathbb{B}}$, their *tensor product* and *Whitney sum* are again Weil algebras:

$$\mathbb{A} \otimes_{\mathbb{K}} \mathbb{A} = \mathbb{K} \oplus (\mathring{\mathbb{A}} \oplus \mathring{\mathbb{B}} \oplus \mathring{\mathbb{A}} \otimes \mathring{\mathbb{B}}), \qquad \mathbb{A} \oplus_{\mathbb{K}} \mathbb{B} := \mathbb{K} \oplus (\mathring{\mathbb{A}} \oplus \mathring{\mathbb{B}}). \tag{B.4}$$

B.2. **Extended domains.** For any $\mathbb{K}$-module $V$ and Weil algebra $\mathbb{A}$, the scalar extension $V_{\mathbb{A}} = V \otimes_{\mathbb{K}} \mathbb{A}$ has a canonical decomposition $V_{\mathbb{A}} = V \otimes V_{\mathring{\mathbb{A}}}$, with projection $\pi : V_{\mathbb{A}} \to V$ and injection $V \to V_{\mathbb{A}}$. Therefore, for any set $U \subset V$, we may define the *extended domain*

$$T^{\mathbb{A}}U := U_{\mathbb{A}} := \pi^{-1}(U) = U \times V_{\mathring{\mathbb{A}}} \subset V_{\mathbb{A}}. \tag{B.5}$$

In this way every set $U \subset V$ gives rise to a functor

$$\underline{U} : \underline{\mathbb{K}\text{-Walg}} \to \underline{\text{set}}, \quad \mathbb{A} \mapsto \underline{U}(\mathbb{A}) := U_{\mathbb{A}}\,. \tag{B.6}$$

B.3. **Smooth laws.** A $\mathbb{K}$-*smooth law* between a set $U \subset V$ and $U' \subset W$ is a morphism (= natural transformation) of functors

$$\underline{f} : \underline{U} \to \underline{U'}.$$

In other words, for any $\mathbb{K}$-Weil algebra $\mathbb{A}$, we have a map

$$f_{\mathbb{A}} : U_{\mathbb{A}} \to U'_{\mathbb{A}},$$

varying functorially with $\mathbb{A}$, in the sense that, if $\phi : \mathbb{B} \to \mathbb{A}$ is a morphism of Weil algebras, then the diagram

$$\begin{array}{ccc} U_{\mathbb{A}} & \to & U'_{\mathbb{A}} \\ \uparrow & & \uparrow \\ U_{\mathbb{B}} & \to & U'_{\mathbb{B}} \end{array}$$

commutes. For instance, every polynomial defines a smooth law: if $f : V \to W$ is a polynomial, then its algebraic extensions $f_{\mathbb{A}} : V_{\mathbb{A}} \to W_{\mathbb{A}}$ (which exist for any $\mathbb{K}$-algebra, not only for Weil algebras) define a family satisfying the requirements above. If $f$ is $\mathbb{K}$-linear, then $f_{\mathbb{A}}$ is simply its $\mathbb{A}$-linear extension.

B.4. **$\mathbb{A}$-differentials.** Given a smooth law $\underline{f}$, by functoriality we get from the projection $\mathbb{A} \to \mathbb{K}$ that $f_{\mathbb{A}}$ is fibered over $f = f_{\mathbb{K}}$, that is, for all $(x, v) \in T^{\mathbb{A}}U$,

$$f_{\mathbb{A}}(x, v) = \big(f(x), d^{\mathbb{A}}f(x)v\big), \tag{B.7}$$

with a map $d^{\mathbb{A}}f(x) : V_{\mathring{\mathbb{A}}} \to V_{\mathring{\mathbb{A}}}$, called the $\mathbb{A}$-*differential of* $f$ *at* $x$. In general, this map is not linear; it is linear if $\mathbb{A} = T\mathbb{K}$, and in this case we just write $df(x)$, called "the" differential of $f$ at $x$ (see below, B.7).

B.5. **Smooth $\mathbb{K}$-manifolds and $\mathbb{A}$-tangent bundle.** Let $M$ be a set and $V$ a $\mathbb{K}$-module, called the *model space*. An *atlas of $M$, with model space $V$*, is a collection

$$\mathcal{A} = (\phi_i, U_i)_{i \in I},$$

where $I$ is an index set, $(U_i)_{i\in I}$ a covering of $M$ by subsets $U_i \subset M$, and $\phi_i : U_i \to V_i \subset V$ a bijection of $U_i$ onto some subset $V_i$ of $V$. The bijections

$$\phi_{i,j} := \phi_i \circ \phi_j^{-1} : V_{ji} \to V_{ij}$$

where $V_{ij} = \phi_i(U_j \cap U_i)$, are called *transition functions*. The data $\underline{M} := (M, \mathcal{A})$ are called a *smooth manifold over* $\mathbb{K}$, if all transition functions are smooth laws over $\mathbb{K}$. Morphisms are again called *smooth laws*, $\underline{f} : \underline{M} \to \underline{N}$, defined to be given locally, in each chart, by smooth laws in the above sense.

**Theorem B.1.** *Assume $(M, \mathcal{A})$ is a smooth manifold over $\mathbb{K}$ and modelled on $V$. Then the data $T^{\mathbb{A}}\mathcal{A} = ((\phi_i)_{\mathbb{A}}, (U_i)_{\mathbb{A}})_{i\in I}$ define an atlas, modelled on $V_{\mathbb{A}}$, of a set $M_{\mathbb{A}}$, such that $(M_{\mathbb{A}}, T^{\mathbb{A}}\mathcal{A})$ is a smooth manifold over $\mathbb{A}$. It depends functorially on $\mathbb{A}$.*

*Proof.* See [Be13], Theorem 3.2 and its proof (where only the formal properties are used, and the smooth setting serves to define the framework): the main point is that all data are local, and the set $M$ can be recovered from the local data as the quotient $M = S/\sim$ where $S = \{(i, x) \in I \times V \mid x \in V_i\}$ with equivalence relation $(i, x) \sim (j, y)$ iff $y = \phi_{ij}(x)$. Applying $T^{\mathbb{A}}$ to these local data gives local data of the same kind, but defined over $\mathbb{A}$. $\square$

We call $T^{\mathbb{A}}M := M_{\mathbb{A}}$ the $\mathbb{A}$-*tangent bundle of* $M$, and the functor $\underline{M} : \mathbb{A} \mapsto T^{\mathbb{A}}M$ a *smooth manifold law*. Smooth laws $\underline{f}$ than have well-defined *tangent maps* $T^{\mathbb{A}}f : T^{\mathbb{A}}M \to T^{\mathbb{A}}N$. The term "bundle" is justified by the fact that projection $\mathbb{A} \to \mathbb{K}$ and injection $\mathbb{K} \to \mathbb{A}$ induce maps

$$\pi : T^{\mathbb{A}}M \to M, \qquad z : M \to T^{\mathbb{A}}M, \tag{B.8}$$

and tangent maps are fiber preserving: $\pi \circ T^{\mathbb{A}}f = f \circ \pi$.

B.6. **Lie groups and symmetric spaces.** As usual, *Lie groups* are defined as groups in the category of smooth manifold laws, and *symmetric spaces* are smooth reflection space laws satisfying Axiom (T) from Definition A.8.

**Theorem B.2.** *Every Lie group $G$ becomes a symmetric space when equipped with the law $s_s(y) = xy^{-1}x$. The Weil bundle $T^{\mathbb{A}}G$ of a Lie group is a Lie group, and the Weil bundle $T^{\mathbb{A}}M$ of a symmetric space $M$ is again a symmetric space.*

*Proof.* One shows as usual that the tangent map of group multiplication at the origin $e$ is vector addition, and hence the tangent map of inversion at the origin is $-\mathrm{id}_{T_eG}$, and it follows that $G$ satisfies the axioms of a symmetric space. Since the axioms of a group or of a reflection space can be expressed by commutative diagrams involving only the structure maps and natural maps such as the diagonal imbedding (cf. [Lo69] for the case of symmetric spaces), applying the functor $T^{\mathbb{A}}$ yields structures of the same type, over $\mathbb{A}$. Therefore $T^{\mathbb{A}}G$ is a Lie group over $\mathbb{A}$, and we get in fact a Lie group law $\underline{G} : \mathbb{A} \mapsto T^{\mathbb{A}}G$; similarly for reflection spaces. The same argument applies also if we write the fourth axiom (T) of a symmetric space in the form of the "commutative diagram"

$$T_{\pi(u)}(s_{\pi(u)})u = (-1)_{TM}(u),$$

where $(-1)_{TM} : TM \to TM$ is action by the scalar $-1$ in tangent spaces; this map is induced by the automorphism $T\mathbb{K} \to T\mathbb{K}$, $a + \varepsilon b \mapsto a - \varepsilon b$. We then have a natural identification $T^{\mathbb{A}}(-1)_{TM} = (-1)_{T(T^{\mathbb{A}}M)}$, induced by the isomorphism $TT^{\mathbb{A}}\mathbb{K} = T\mathbb{K} \otimes_{\mathbb{K}} \mathbb{A} = \mathbb{A} \otimes_{\mathbb{K}} T\mathbb{K} = T^{\mathbb{A}}T\mathbb{K}$. (The proof given in [Be08], Proposition 5.6, is different, relying again on "midpoints", since in loc. cit. we had not yet the full functioriality of Weil functors at disposition.) □

The same principle applies to other kinds of "smooth geometries".

B.7. **Vector algebras and vector bundles.** A Weil algebra is called a *vector algebra* if $\mathring{\mathbb{A}}$ carries the zero product (as in the example of the tangent ring $T\mathbb{K} = \mathbb{K}[\varepsilon]$). A direct check shows that in this (and only in this) case the map

(B.9) $$\phi : \mathbb{A} \oplus_{\mathbb{K}} \mathbb{A} \to \mathbb{A}, \quad (x, a, b) \mapsto (x, a + b)$$

is an algebra homomomorphism. For every $\mathbb{K}$-module $V$, it induces a map $\Phi : V_{\mathbb{A}\oplus_{\mathbb{K}}\mathbb{A}} \to V_{\mathbb{A}}$, given by the same kind of formula. Note that $V_{\mathbb{A}\oplus_{\mathbb{K}}\mathbb{A}} = V \oplus V_{\mathring{\mathbb{A}}} \oplus V_{\mathring{\mathbb{A}}}$, and

$$U_{\mathbb{A}\oplus_{\mathbb{K}}\mathbb{A}} = U \times V_{\mathring{\mathbb{A}}} \times V_{\mathring{\mathbb{A}}}.$$

Any smooth law defined on $U \subset V$ is fiber preserving (see above, B.4), and from functoriality $\Phi \circ f_{\mathbb{A}\oplus_{\mathbb{K}}\mathbb{A}} = f_{\mathbb{A}} \circ \Phi$, we get additivity of the fiber map:

$$d^{\mathbb{A}}f(x)(v + w) = d^{\mathbb{A}}f(x)v + d^{\mathbb{A}}f(x)w.$$

In a similar way, using the algebra endomorphism $\mathbb{A} \to \mathbb{A}$, $(x, a) \mapsto (x, ra)$ for $r \in \mathbb{K}$, we see that the fiber map is fiberwise homogeneous of degree 1. Summing up, if $\mathbb{A}$ is a vector algebra, then $T^{\mathbb{A}}M$ carries a well-defined *vector bundle structure over* $M$, and the tangent map $T^{\mathbb{A}}f : T^{\mathbb{A}}U \to T^{\mathbb{A}}U'$ is fiberwise linear. Locally, in a bundle chart, the linear structure is given by the linear structure of $V_{\mathring{\mathbb{A}}}$. Moreover, these arguments imply that $T^{\mathbb{A}\oplus_{\mathbb{K}}\mathbb{A}}M$ is the Whitney sum of vector bundles $T^{\mathbb{A}}M \times_M T^{\mathbb{A}}M$. For $\mathbb{A} = T\mathbb{K} = \mathbb{K}[\varepsilon]$, we get the *tangent bundle*.

B.8. **Second and higher order tangent bundles.** The *iterated tangent rings* are defined by induction: $T^k\mathbb{K} = T(T^{k-1}\mathbb{K})$; they are Weil algebras, giving rise to the *iterated tangent bundles* $T^kM$. We use notation similar as in [Be08], in particular

$$TT\mathbb{K} = (\mathbb{K}[\varepsilon_1])[\varepsilon_2] = \mathbb{K} \oplus \varepsilon_1\mathbb{K} \oplus \varepsilon_2\mathbb{K} \oplus \varepsilon_1\varepsilon_2\mathbb{K} \ . \tag{B.10}$$

There is a natural action of the permutation group $S_k$ by automorphisms on $T^k\mathbb{K}$, and by functoriality also on $T^kM$. For $k = 2$, the action of the transposition (12) defines the *canonical flip* of $TTM$. The *vertical ideal* $\varepsilon_1\varepsilon_2\mathbb{K}$ is the kernel of the homomorphism $TT\mathbb{K} \to T\mathbb{K} \oplus_\mathbb{K} T\mathbb{K}$, and the *vertical bundle* is the corresponding subbundle of $TTM$ denoted by $\varepsilon_1\varepsilon_2TM$; it is canonically isomorphic to $TM$.

B.9. **Vector fields and infinitesimal automorphisms.** There is a one-to-one correspondence between $\mathbb{A}$-*vector fields on* $M$ (smooth laws $\underline{X} : \underline{M} \to \underline{T^\mathbb{A}M}$ such that $\underline{\pi} \circ \underline{X} = \underline{\mathrm{id}}$) and *infinitesimal automorphisms of* $M$ ($\mathbb{A}$-smooth laws $\underline{\xi} : \underline{T^\mathbb{A}M} \to \underline{T^\mathbb{A}M}$ such that $\underline{\pi} \circ \underline{\xi} = \underline{\pi}$). The proof is essentially the same as in [Be08], Chapter 28: given $\xi : T^\mathbb{A}M \to T^\mathbb{A}M$, we obtain a vector field $\xi \circ z : M \to T^\mathbb{A}M$, and conversely, given $X : M \to T^\mathbb{A}M$, consider $T^\mathbb{A}X : T^\mathbb{A}M \to T^\mathbb{A}(T^\mathbb{A}M) = T^{\mathbb{A}\otimes\mathbb{A}}M$; we compose with the canonical map $T^{\mathbb{A}\otimes\mathbb{A}}M \to T^\mathbb{A}M$ induced by the ring homomorphism

$$\mathbb{A} \otimes \mathbb{A} \to \mathbb{A}, \quad (x,u) \otimes (y,v) \mapsto (xy, xv + uy + uv)$$

to get a map $\tilde{X} : T^\mathbb{A}M \to T^\mathbb{A}M$. If $\mathbb{A} = T\mathbb{K}$, then this map is described in a chart domain by $(x,u) \mapsto (x, u + X(x))$ (in the general case an explicit description is much more complicated – see [Be08], Thm. 28.3 for the case $\mathbb{A} = T^k\mathbb{K}$), and in this case the group law of addition on the space $\mathfrak{X}(M)$ of vector fields (that is, pointwise addition) corresponds to composition of infinitesimal automorphisms.

**Theorem B.3** (Lie bracket of vector fields)**.** *There is a canonical structure of* $\mathbb{K}$*-Lie algebra on the space of vector fields* $\mathfrak{X}(M)$*; the Lie bracket* $[X,Y]_{alg}$ *of two vector fields* $X, Y \in \mathfrak{X}(M)$ *can be defined in two equivalent ways: in a conceptual way, it can be described as a group commutator* $[g,h]_{group} = ghg^{-1}h^{-1}$ *in the group of infinitesimal automorphisms of* $TTM$ *over* $M$*:*

$$[\widetilde{T\varepsilon_1 X}, \widetilde{T\varepsilon_2 Y}]_{group} = \varepsilon_1\varepsilon_2[X,Y]_{alg} \tag{B.11}$$

*Here, one extends* $X : M \to T_{\varepsilon_1}M$ *to an infinitesimal automorphism of* $T_{\varepsilon_1}M$ *and then take its tangent map with respect to* $\varepsilon_2$*; similarly for* $Y$*, but with exchanged rôles of* $\varepsilon_1, \varepsilon_2$*; then take their group commutator; this gives in infinitesimal automorphism of* $TTM$ *which stabilizes the vertical bundle* $\varepsilon_1\varepsilon_2TM$ *and acts there by translation by a vector field* $Z$ *which is the Lie bracket* $[X,Y]_{alg}$ *of* $X$ *and* $Y$*. Less conceptually, the bracket may be defined by the following chart description in any chart of* $M$*:*

$$[X,Y](x) = dX(x)Y(x) - dY(x)X(x). \tag{B.12}$$

*Proof.* [Be08], Thm. 14.4; see also Chapter 24 of [Be08]. □

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