# Jordan structures and non-associative geometry

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**Abstract.** We give an overview of constructions of geometries associated to Jordan structures (algebras, triple systems and pairs), featuring analogs of these constructions with the Lie functor on the one hand and with the approach of non-commutative geometry on the other hand.

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## Introduction

Let us compare two aspects of the vast mathematical topic "links between geometry and algebra": on the one hand, the *Lie functor* establishes a close relation between Lie groups (geometric side) and Lie algebras (algebraic side); this is generalized by a correspondence between *symmetric spaces* and *Lie triple systems* (see [Lo69]). On the other hand, the philosophy of *Non-Commutative Geometry* generalizes the relation between usual, geometric point-spaces $M$ (e.g., manifolds) and the commutative and associative algebra $\mathrm{Reg}(M, \mathbb{K})$ of "regular" (e.g., bounded, smooth, algebraic,..., according to the context) $\mathbb{K}$-valued functions on $M$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, by replacing the algebra $\mathrm{Reg}(M, \mathbb{K})$ by more general, possibly non-commutative algebras $\mathcal{A}$. The interaction between these two aspects seems to be rather weak; indeed, in some sense they are "orthogonal" to each other: first, the classical setting of Lie's third theorem is finite-dimensional, whereas the algebras $\mathcal{A}$ of commutative or non-commutative geometry are typically infinite dimensional; second, and more importantly, taking a commutative and associative algebra as input, we obtain as Lie bracket $[x, y] = xy - yx = 0$, and hence (if we forget the associative structure and retain only the Lie bracket) we are left with constructing a Lie group with zero Lie bracket. But this is not very interesting: it does not even capture the specific information encoded in "commutative geometries".[1]

This remark suggests that, if one looks for a link between the two aspects just mentioned, it should be interesting to ask for an analog of the Lie functor for a class of algebras that contains faithfully the class of commutative associative algebras, but rather sacrifices associativity than commutativity. At this point let us note that the algebras $\mathcal{A}$ of non-commutative geometry are of course always supposed to be *associative*, so that the term "associative geometry" might be more appropriate than "non-commutative geometry". Indeed, behind the garden of associative algebras starts the realm of general algebras, generically neither associative nor commutative, and, for the time being, nobody has an idea of what their "geometric interpretation" might be. Fortunately, two offsprings of associative algebras grow not too far away behind the garden walls: *Lie algebras* right on the other side (the branches reach over the wall so abundantly that some people even consider them as still belonging to the garden), and *Jordan algebras* a bit further. Let us just recall that the former are typically obtained by skew-symmetrizing an associative algebra, $[x, y] = xy - yx$, whereas the latter are typically obtained by symmetrizing them, $x \bullet y = \frac{1}{2}(xy + yx)$ (the factor $\frac{1}{2}$ being conventional, in order to obtain the same powers $x^k$ as in the associative algebra).

So let us look at *Jordan algebras* – their advantage being that the class of commutative associative algebras is faithfully embedded. In this survey paper we will explain their geometric interpretation via certain *generalized projective geometries*, emphasizing that this interpretation really combines both aspects mentioned above. Of course, it is then important to include the case of *infinite-dimensional* algebras, and to treat them in essentially the same way as finite-dimensional ones. This is best done in a purely algebraic framework, leaving aside all questions of topologizing our algebras and geometric spaces. Of course, such questions form an interesting topic for the further development of the theory: in a very general setting (topological algebraic structures over general topological fields or rings) basic results are given in [BeNe05], and certainly many results from the more specific setting of *Jordan operator algebras* (Banach-Jordan structures over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$; see [HOS84]) admit interesting geometric interpretations in our framework. Beyond the Banach-setting, it should be interesting to develop a theory of *locally convex topological Jordan structures* and their geometries, taking up the associative theory (cf. [Bil04]).

The general construction for Jordan algebras (and for other members of the family of Jordan algebraic structures, namely *Jordan pairs* and *Jordan triple systems*, which in some sense are easier to understand than Jordan algebras; cf. Section 1) being explained in the main text (Section 2 and Section 3), let us

[1] Of course, this does not exclude that *other* ways of associating Lie groups to function algebras are interesting, for example by looking at their derivations and automorphisms; but our point is precisely that such constructions need more input than the trivial Lie bracket on the algebra of functions.

here just look at the special case of "commutative geometry", i.e., the case of the commutative and associative algebra $\mathcal{A} = \mathrm{Reg}(M, \mathbb{K})$ of "regular $\mathbb{K}$-valued functions" on a geometric space $M$ (here, $\mathbb{K}$ is a commutative field with unit 1; in the main text we will allow also commutative rings with 1). Looking at $\mathcal{A}$ as a Jordan algebra, we associate to it the space $\mathcal{X} := \mathrm{Reg}(M, \mathbb{KP}^1)$ of "regular functions from $M$ into the projective line $\mathbb{KP}^1$". Of course, $\mathcal{X}$ is no longer an algebra or a vector space; however, we can recover all these structures if we want: we call two functions $f, g \in \mathcal{X}$ "transversal", and write $f \top g$, if $f(p) \neq g(p)$ for all points $p \in M$. Now choose three functions $f_0, f_1, f_\infty \in \mathcal{X}$ that are mutually transversal. For each point $p \in M$, the value $f_\infty(p)$ singles out an "affine gauge" (by taking $f_\infty(p)$ as point at infinity in the projective line $\mathbb{KP}^1$ and looking at the affine line $\mathbb{KP}^1 \setminus \{f_\infty(p)\}$); then $f_0(p)$ singles out a "linear gauge" (origin in the affine line) and $f_1(p)$ a "unit gauge" (unit element in the vector line). Obviously, any such choice of $f_0$, $f_1$, $f_\infty$ leads to an identification of the set $(f_\infty)^\top$ of all functions that are transversal to $f_\infty$ with the usual algebra $\mathcal{A}$ of regular functions on $M$. In other words, $\mathcal{A}$ and $(\mathcal{X}; f_0, f_1, f_\infty)$ carry the same information, and thus we see that the encoding via $\mathcal{A}$ depends on various choices, from which we are freed by looking instead at the "geometric space" $\mathcal{X}$. It thus becomes evident that $\mathcal{A}$ really is a sort of "tangent algebra" of the geometric ("non-flat") space $\mathcal{X}$ at the point $f_0$, in a similar way as the Lie algebra of a Lie group reflects its tangent structure at the origin.

This way of looking at ordinary "commutative geometry" leads to generalizations that are different from non-commutative geometry (in the sense of A. Connes) but still have much in common with it. For instance, both theories have concepts of "states", i.e., a notion that, in the commutative case, amounts to recover the point space $M$ from the algebra of regular functions. Whereas in case of non-commutative geometry this concept relies heavily on *positivity* (and hence on the ordered structure of the base field $\mathbb{R}$), the corresponding concept for generalized projective geometries is purely geometric and closely related to the notion of *inner ideals* in Jordan theory (Section 4). There are many interesting open problems related to these items, some of them mentioned in Section 4.

Summing up, it seems that the topic of geometrizing Jordan structures, incorporating both ideas from classical Lie theory and basic ideas from associative geometry, is well suited for opening the problem of "general non-associative geometry", i.e., the problem of finding geometries corresponding to more general non-associative algebraic structures.

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Notation: Throughout, $\mathbb{K}$ is a commutative base ring with unit 1 and such that 2 and 3 are invertible in $\mathbb{K}$.

# 1 Jordan pairs and graded Lie algebras

## 1.1 $\mathbb{Z}/(2)$-graded Lie algebras and Lie triple systems

Let $(\Gamma,+)$ be an abelian group. A Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is called *$\Gamma$-graded* if it is of the form

$$\mathfrak{g} = \bigoplus_{n\in\Gamma} \mathfrak{g}_n, \quad \text{with} \quad [\mathfrak{g}_m,\mathfrak{g}_n] \subset \mathfrak{g}_{n+m}.$$

Let us consider the example $\Gamma = \mathbb{Z}/(2)$. Here, $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$, with a subalgebra $\mathfrak{g}_{\overline{0}}$ and a subspace $\mathfrak{g}_{\overline{1}}$ which is stable under $\mathfrak{g}_{\overline{0}}$ and such that $[\mathfrak{g}_{\overline{1}},\mathfrak{g}_{\overline{1}}] \subset \mathfrak{g}_{\overline{0}}$. It is then easily seen that the linear map $\sigma : \mathfrak{g} \to \mathfrak{g}$ which is 1 on $\mathfrak{g}_{\overline{0}}$ and $-1$ on $\mathfrak{g}_{\overline{1}}$ is an involution of $\mathfrak{g}$ (automorphism of order 2). Conversely, every involution gives rise to $\mathbb{Z}/(2)$-grading of $\mathfrak{g}$, and hence such gradings correspond bijectively to *symmetric Lie algebras* $(\mathfrak{g},\sigma)$, i.e., to Lie algebras $\mathfrak{g}$ together with an automorphism $\sigma$ of order 2. Then $\mathfrak{g}_{\overline{1}}$ (the $-1$-eigenspace of $\sigma$) is stable under taking triple Lie brackets $[[x,y],z]$. This leads to the notion of *Lie triple system*:

**Definition 1.1.** *A* Lie triple system (LTS) *is a $\mathbb{K}$-module $\mathfrak{q}$ together with a trilinear map*

$$\mathfrak{q}\times\mathfrak{q}\times\mathfrak{q} \to \mathfrak{q}, \quad (X,Y,Z) \mapsto [X,Y,Z],$$

*or, equivalently, with a bilinear map*

$$R : \mathfrak{q}\times\mathfrak{q} \to \operatorname{End}(\mathfrak{q}), \quad (X,Y) \mapsto R(X,Y) =: [X,Y,\cdot],$$

*such that*

*(LT1)* $R(X,X) = 0$
*(LT2)* $R(X,Y)Z + R(Y,Z)X + R(Z,Z)Y = 0$ *(the Jacobi identity)*
*(LT3) the endomorphism $D := R(X,Y)$ is a* derivation *of the trilinear product, i.e.,*

$$D[U,V,W] = [DU,V,W] + [U,DV,W] + [U,V,DW].$$

If $\mathfrak{g}$ is a $\mathbb{Z}/(2)$-graded Lie algebra, then $\mathfrak{g}_{\overline{1}}$ with $[X,Y,Z] := [[X,Y],Z]$ becomes an LTS, and every LTS arises in this way since one may reconstruct a Lie algebra from an LTS via the *standard imbedding* (see [Lo69]): let $\mathfrak{h} \subset \operatorname{Der}(\mathfrak{q})$ the subalgebra of derivations of the LTS $\mathfrak{q}$ spanned by all operators $R(X,Y)$, $X,Y \in \mathfrak{q}$, and define a bracket on $\mathfrak{g} := \mathfrak{q} \oplus \mathfrak{h}$ by

$$[(X,D),(Y,E)] := \big(DY - EX, [D,E] - R(X,Y)\big).$$

One readily checks that $\mathfrak{g}$ is a $\mathbb{Z}/(2)$-graded Lie algebra. As a side remark, one may note that this construction (the *standard imbedding*) is in general not functorial – see [Sm05] for a discussion of this topic.

### 1.2 3-graded Lie algebras and Jordan pairs

A Lie algebra is called $2n+1$-*graded* if it is $\mathbb{Z}$-graded, with $\mathfrak{g}_j = 0$ if $j \notin \{-n, -n+1, \ldots, n\}$. The linear map $D : \mathfrak{g} \to \mathfrak{g}$ with $Dx = jx$ for $x \in \mathfrak{g}_j$ is then a derivation of $\mathfrak{g}$; if it is an inner derivation, $D = \mathrm{ad}(E)$, the element $E \in \mathfrak{g}$ is called an *Euler element*.

To any $\mathbb{Z}$-graded algebra one may associate a $\mathbb{Z}/(2)$-grading $\mathfrak{g} = \mathfrak{g}_{even} \oplus \mathfrak{g}_{odd}$, by putting together all homogeneous parts with even, resp. odd index. In the sequel we are mainly interested in 3-graded and 5-graded Lie algebras; in both cases, $\mathfrak{g}_{odd} = \mathfrak{g}_1 \oplus \mathfrak{g}_{-1}$. We then let $V^\pm := \mathfrak{g}_{\pm 1}$ and define trilinear maps by

$$T^\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm, \quad (x,y,z) \mapsto [[x,y],z].$$

The maps $T_\pm$ satisfy the identity

$$\text{(LJP2)}\ T^\pm(u,v,T^\pm(x,y,z)) = T^\pm(T^\pm(u,v,x),y,z) - T^\pm(x,T^\mp(v,u,y),z) + T^\pm(x,y,T^\pm(u,v,z))$$

Indeed, this is just another version of the identity (LT3), reflecting the fact that $\mathrm{ad}[u,v]$ is a derivation of $\mathfrak{g}$ respecting the grading since $[u,v] \in \mathfrak{g}_0$. In the 3-graded case, we have moreover, for all $x, z \in V^\pm$, $y \in V^\mp$:

$$\text{(LJP1)}\ T^\pm(x,y,z) = T^\pm(z,y,x)$$

This follows from the fact that $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are abelian, and so by the Jacobi identity $[[x,y],z] - [[z,y],x] = [[z,x],y] = 0$.

**Definition 1.2.** *A pair of $\mathbb{K}$-modules $(V^+, V^-)$ together with trilinear maps $T^\pm : V^\pm \times V^\mp \times V^\pm \to V^\pm$ is called a* (linear) Jordan pair *if (LJP1) and (LJP2) hold. (The term "linear" refers to the fact that the identities (LJP1) and (LJP2) are linear in each of their variables, whereas the definition given in [Lo75], which is valid also in case that* 2 *or* 3 *are not invertible in $\mathbb{K}$, is based on identities that are* quadratic *in some of their variables.)*

Every linear Jordan pair arises by the construction just described: if $(V^+, V^-)$ is a linear Jordan pair, let $\mathfrak{q} := V^+ \oplus V^-$ and define a trilinear map $\tilde{T} : \mathfrak{q}^3 \to \mathfrak{q}$ via

$$\tilde{T}\big((x,x'),(y,y'),(z,z')\big) := \big(T^+(x,y',z), T^-(x',y,z')\big).$$

Then the map $\tilde{T}$ satisfies the same identities as $(T^+, T^-)$, with $T^+$ and $T^-$ both replaced by $\tilde{T}$ (in other words, $(\mathfrak{q}, \tilde{T})$ is a *Jordan triple system*, see below). Now we define a trilinear bracket $\mathfrak{q}^3 \to \mathfrak{q}$ by

$$\big[(x,x'),(y,y'),(z,z')\big] := \tilde{T}((x,x'),(y,y'),(z,z')) - \tilde{T}((y,y'),(x,x'),(z,z')).$$

By a direct calculation (cf. Lemma 1.4 below), one checks that $\mathfrak{q}$ with this bracket is a Lie triple system, and its standard imbedding $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$ with $\mathfrak{g}_{\pm 1} = V^{\pm}$ and $\mathfrak{g}_0 = \mathfrak{h}$, is a 3-graded Lie algebra. Without loss of generality we may assume that $\mathfrak{g}$ contains an Euler element $E$: in fact, the endomorphism $E : \mathfrak{q} \to \mathfrak{q}$ which is 1 on $V^+$ and $-1$ on $V^-$, is a derivation of $\mathfrak{q}$ commuting with $\mathfrak{h}$, and hence we may replace $\mathfrak{h}$ by $\mathfrak{h} + \mathbb{K}E$ in the construction of the standard embedding.

For 5-graded Lie algebras, the identity (LJP1) has to be replaced by another, more complicated identity, which leads to the notion of *Kantor pair*, see [AF99]. As for Jordan pairs, Kantor pairs give rise to Lie triple systems of the form $\mathfrak{q} = V^+ \oplus V^-$, where now the standard imbedding leads back to a 5-graded Lie algebra.

### 1.3 Involutive $\mathbb{Z}$-graded Lie algebras

An *involution of a $\mathbb{Z}$-graded Lie algebra* is an automorphism $\theta$ of order 2 such that $\theta(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ for $j \in \mathbb{Z}$. If $\mathfrak{g}$ is 3- or 5-graded, we let $V := \mathfrak{g}_1$ and define

$$T : V \times V \times V \to V, \quad (X, Y, Z) \mapsto [[X, \theta Y], Z].$$

Then $T$ satisfies the identity (LJT2) obtained from (LJP2) by omitting the indices $\pm$, and if $\mathfrak{g}$ is 3-graded, we have moreover the analog of (LJP1).

**Definition 1.3.** *A $\mathbb{K}$-module together $V$ with a trilinear map $T : V^3 \to V$ satisfying the identities (JP1) and (JP2) obtained from (LJP1) and (LJP2) by omitting indices is called a (linear) Jordan triple system (JTS).*

Every linear Jordan triple system arises by the construction just described: just let $V^+ := V^- := V$ and $T^{\pm} := T$; then $(V^+, V^-)$ is a Jordan pair carrying an involution (=isomorphism $\tau$ from $(V^+, V^-)$ onto the Jordan pair $(V^-, V^+)$, namely $\tau(x, x') = (x', x)$). Let $\mathfrak{g}$ be the 3-graded Lie algebra associated to this Jordan pair; then the involution $\tau$ of the Jordan pair induces an involution $\theta$ of $\mathfrak{g}$. By the way, these arguments show that Jordan triple systems are nothing but Jordan pairs with involution (cf. [Lo75]). For any JTS $(V, T)$, the Jordan pair $(V^+, V^-) = (V, V)$ with $T^{\pm} = T$ is called the *underlying Jordan pair.*

The Lie algebra $\mathfrak{g}$ now carries *two* automorphisms of order 2, namely $\theta$ and the automorphism $\sigma$ corresponding to the $\mathbb{Z}/(2)$-grading $\mathfrak{g} = \mathfrak{g}_{even} \oplus \mathfrak{g}_{odd}$. These two automorphisms commute. In particular, $\sigma$ restricts to the $\theta$-fixed subalgebra $\mathfrak{g}^{\theta}$, with $-1$-eigenspace being $\{X + \theta(X) | \, X \in \mathfrak{g}_1\}$. The LTS-structure of this space is described by the following lemma.

**Lemma 1.4.** (The Jordan-Lie functor.) *If $T$ is a JTS on $V$, then*

$$[X, Y, Z] = T(X, Y, Z) - T(Y, X, Z)$$

*defines a LTS on $V$.*

*Proof.* The proof of (LT1) is trivial, and (LT2) follows easily from the symmetry of $T$ in the outer variables. In order to prove (LT3), we define the endomorphism $R(X,Y) := T(X,Y,\cdot) - T(Y,X,\cdot)$ of $V$. Then $R(X,Y)$ is a derivation of the trilinear map $T$, as follows from the second defining identity (JT2). But every derivation of $T$ is also a derivation of $R$, since $R$ is simply defined by skew-symmetrization of $T$ in the first two variables.

The lemma defines a functor from the category of JTS to the category of LTS over $\mathbb{K}$, which we call the *Jordan-Lie functor*. In general, it is neither injective nor surjective; all the more surprising is the fact that, in the real, simple and finite-dimensional case, the Jordan-Lie functor is not too far from setting up a one-to-one correspondence (classification results due to E. Neher, cf. tables given in [Be00]).

### 1.4 The link with Jordan algebras

Whereas all results from the preceding sections are naturally and easily understood from the point of view of Lie theory, the results presented next are "genuinely Jordan theoretic" – by this we mean that proofs by direct calculation in 3-graded Lie algebras are much less straightforward; the reader may try to do so, in order to better appreciate the examples to be given in the next section.

In the following, $\mathfrak{g}$ is a 3-graded Lie algebra, with associated Jordan pair $(V^+,V^-) = (\mathfrak{g}_1,\mathfrak{g}_{-1})$. For $x \in V^-$, we define a $\mathbb{K}$-linear map

$$Q^-(x) : V^+ \to V^-, \quad y \mapsto Q^-(x)y := \frac{1}{2}T^-(x,y,x) = -\frac{1}{2}[x,[x,y]].$$

In the same way $Q^+(y)$ for $y \in V^+$ is defined. Then the following hold:

**(1) The Fundamental Formula.** For all $x \in V^-$, $y \in V^+$,

$$Q^-(Q^-(x)y) = Q^-(x)Q^+(y)Q^-(x).$$

**(2) Meyberg's Theorem.** Fix $a \in V^-$. Then $V_a := V^+$ with product

$$x \bullet_a y := \frac{1}{2}T^+(x,a,y)$$

is a Jordan algebra (called the *a-homotope algebra*). Recall (see, e.g., [McC04]), that a *(linear) Jordan algebra* is a $\mathbb{K}$-module $J$ with a bilinear and commutative product $x \bullet y$ such that the identity

(J2) $x \bullet (x^2 \bullet y) = x^2 \bullet (x \bullet y)$

holds.

**(3) Invertible elements and unital Jordan algebras.** An element $a \in V^-$ is called *invertible in* $(V^+,V^-)$ if the the operator $Q^-(a) : V^+ \to V^-$ is invertible. The element $a \in V^-$ is invertible in $(V^+, V^-)$ if, and only if, the Jordan algebra $V_a$ described in the preceding point admits a unit element; this unit element is then $a^\sharp := (Q(a))^{-1}(a)$. Moreover, the Jordan pair $(V^+, V^-)$ can then be recovered from the Jordan algebra $(V, \bullet) = (V_a, \bullet_a)$ as follows: let $V^+ := V^- := V$ as $\mathbb{K}$-modules and let $T : V^3 \to V$,

$$T(x, y, z) := (x \bullet y) \bullet z - y \bullet (x \bullet z) + x \bullet (y \bullet z).$$

Then $(V, T)$ is a Jordan triple system, and its underlying Jordan pair $(V^+, V^-)$ is the one we started with.

### 1.5 Some examples

**(1) Jordan pairs of rectangular matrices.** Let $A$ and $B$ two $\mathbb{K}$-modules and $W := A \oplus B$. Let $I : W \to W$ be the linear map that is 1 on $A$ and $-1$ on $B$. Then the Lie algebra $\mathfrak{g} := \mathfrak{gl}_{\mathbb{K}}(W)$ is 3-graded, with, for $i = 1, 0, -1$, $\mathfrak{g}_j = \{X \in \mathfrak{g} | \, [I, X] = 2jX\}$. In an obvious matrix notation, this corresponds to describing $\mathfrak{g}$ by $2 \times 2$-matrices, $\mathfrak{g}_0$ as diagonal and $\mathfrak{g}_1$ as upper and $\mathfrak{g}_{-1}$ as lower triangular matrices:

$$\begin{pmatrix} \mathfrak{g}_0 & V^+ \\ V^- & \mathfrak{g}_0 \end{pmatrix}$$

Therefore

$$(V^+, V^-) = (\mathrm{Hom}_{\mathbb{K}}(A, B), \mathrm{Hom}_{\mathbb{K}}(B, A))$$

carries the structure of a linear Jordan pair. The Jordan pair structure is given by

$$T^\pm(X, Y, Z) = XYZ + ZYX.$$

In fact, this is proved by the following calculation (using matrix notation for elements of $\mathfrak{g}$):

$$\Big[ \Big[ \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} \Big], \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \Big] = \Big[ \begin{pmatrix} XY & 0 \\ 0 & -YX \end{pmatrix}, \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \Big] = \begin{pmatrix} 0 & XYZ + ZYX \\ 0 & 0 \end{pmatrix}.$$

The proof of the Fundamental Formula for this Jordan pair is easy since we may calculate in the associative algebra $\mathrm{Hom}_{\mathbb{K}}(W, W)$, where $Q(x) = \ell_x \circ r_x$ is simply composition of left and of right translation by $x$. Similarly, the proof of Meyberg's Theorem becomes an easy exercise: for any $a \in \mathrm{Hom}_{\mathbb{K}}(W, W)$, the product $(x, y) \mapsto xay$ is again an associative product, and by symmetrizing and restricting to $V^+$, if $a \in V^-$, we get the Jordan algebra $\bullet_a$.

In general, the Jordan pair $(V^+, V^-)$ does not contain invertible elements. For instance, if $\mathbb{K}$ is a field and $A$ and $B$ are not isomorphic as vector spaces, then it is easily seen that $Q^-(x)$ is never bijective.

Finally, it is clear that, if $A$ and $B$ are isomorphic as $\mathbb{K}$-modules, and fixing such an isomorphism in order to identify $A$ and $B$, an element $a \in V^+ = A =$

$V^-$ is invertible if it is invertible in the usual sense in $\mathrm{Hom}(A, A)$, and then $\bullet_a$ has $a^{-1}$ as unit element. Note that, in particular, this happens for the Lie algebra $\mathfrak{gl}(2, \mathbb{K})$, with $V^+ \cong V^- \cong \mathbb{K}$, where $x \bullet_a y = xay = yax$.

**(2) Function spaces.** As for any algebraic structure defined by identities, spaces of functions with values in such a structure, equipped with the "pointwise product", form again a structure of the given type. In our case, if $M$ is a set and $(V^+, V^-)$ a Jordan pair, then $(\mathrm{Fun}(M, V^+), \mathrm{Fun}(M, V^-))$ is a again a Jordan pair, and similarly for Jordan triple systems and Jordan algebras. Moreover, if $(V^+, V^-)$ corresponds to a 3-graded Lie algebra $\mathfrak{g}$, then the function space corresponds to the 3-graded Lie algebra $\mathrm{Fun}(M, \mathfrak{g})$. In particular, pairs of scalar functions $(\mathrm{Fun}(M, \mathbb{K}), \mathrm{Fun}(M, \mathbb{K}))$ form a Jordan pair with pointwise product

$$(T(f, g, h))(p) = 2f(p)\, g(p)\, h(p).$$

Under suitable assumptions, "regular" functions (continuous, smooth, algebraic, ...) will form subpairs.

**(3) The classical examples.** Besides rectangular matrices, these include:

- full asociative algebras $A$. Here $(V^+, V^-) = (A, A)$, and $\mathfrak{g} \subset \mathfrak{gl}(2, A)$ is the subalgebra generated by the strict upper triangular and strict lower triangular matrices;
- Hermitian elements of an involutive associative algebra $(A, *)$. Here, $(V^+, V^-) = (A_h, A_h)$ is the fixed space of $*$, and $\mathfrak{g}$ is the symplectic algebra of $(A, *)$ (cf. [BeNe04], Section 8.2). Note that symmetric and Hermitian matrices are a special case;
- skew-Hermitian elements of an involutive associative algebra $(A, *)$: similar as above, replacing $*$ by its negative. As above, Jordan pairs of skew-symmetric or skew-Hermitian matrices are a special case;
- conformal geometries (or "spin factors"). Here, $V^+ = V^- = V$ is a $\mathbb{K}$-module with a non-degenerate symmetric bilinear form $(\cdot|\cdot) : V \times V \to \mathbb{K}$ and trilinear map

  $$T(x, y, z) = (x|z)y - (x|y)z - (z|y)x\,.$$

  Then $\mathfrak{g}$ is the orthogonal Lie algebra of the quadratic form on $\mathbb{K} \oplus V \oplus \mathbb{K}$ given by

  $$\beta((r, v, s), (r, v, s)) = rs + (v|v)\,.$$

There are also exceptional Jordan systems, namely octonionic $1 \times 2$-matrices, resp. the Hermitian octonionic $3 \times 3$-matrices, corresponding to 3-gradings of the exceptional Lie algebras of type $E_6$, resp. $E_7$.

## 2 The generalized projective geometry of a Jordan pair

### 2.1 The construction

The following construction of a pair $(\mathcal{X}^+, \mathcal{X}^-)$ of homogeneous spaces associated to a Jordan pair $(V^+, V^-)$ is due to J.R. Faulkner and O. Loos (see [Lo95]): starting with a (linear) Jordan pair $(V^+, V^-)$, let $\mathfrak{g}$ be the 3-graded Lie algebra $\mathfrak{g}$ constructed in Section 1.2; as explained there, we may assume that $\mathfrak{g}$ contains an *Euler operator* $E$, i.e., an element such that $[E, X] = iX$ for $X \in \mathfrak{g}_i$.

Next we define two abelian groups $U^\pm = \exp(\mathfrak{g}_{\pm 1})$ by observing that, for $X \in \mathfrak{g}_{\pm 1}$, the operator $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is 3-step nilpotent, and that

$$\exp(X) := e^{\mathrm{ad}(X)} = \mathrm{id} + \mathrm{ad}(X) + \frac{1}{2}\mathrm{ad}(X)^2$$

is an automorphism of $\mathfrak{g}$ (for this we need that 2 and 3 are invertible in $\mathbb{K}$). Since $\mathfrak{g}_{\pm 1}$ are abelian, $U^\pm$ are abelian subgroups of $\mathrm{Aut}(\mathfrak{g})$. The maps $\exp_\pm : \mathfrak{g}_\pm \to \mathrm{Aut}(\mathfrak{g})$ are injective (here we use our Euler operator: $\exp_\pm(X) = \mathrm{id}$ implies $\exp_\pm(X)E = E \pm X = E$, whence $X = 0$). Thus $U^\pm$ is isomorphic to $(V^\pm, +)$. The *elementary projective group* associated to the Jordan pair $(V^+, V^-)$ is the subgroup

$$G := \mathrm{PE}(V^+, V^-) := \langle U^+, U^- \rangle$$

of $\mathrm{Aut}(\mathfrak{g})$ generated by $U^+$ and $U^-$. Let $H$ be the subgroup of $G$ stabilizing the grading (i.e., commuting with $\mathrm{ad}(E)$); then $U^+$ and $U^-$ are normalized by $H$, and the groups $P^\pm$ generated by $H$ and $U^\pm$ are semidirect products:

$$P^\pm := \langle H, U^\pm \rangle \cong H \rtimes V^\pm.$$

Finally, we define two homogeneous spaces $\mathcal{X}^\pm := G/P^\mp$ with base points $o^\pm = e/P^\mp$ ($e$ being the unit element of $G$).

The pair of "geometric spaces" $(\mathcal{X}^+, \mathcal{X}^-)$ is called the *generalized projective geometry associated to* $(V^+, V^-)$. It is indeed the geometric object associated to the Jordan pair $(V^+, V^-)$, in a similar way as a Lie group is the geometric object associated to a Lie algebra. Let us explain this briefly.

### 2.2 Generalized projective geometries

The spaces $\mathcal{X}^\pm$ being defined as above, the direct product $\mathcal{X}^+ \times \mathcal{X}^-$ is equipped with a *transversality relation*: call $(x, \alpha) \in \mathcal{X}^+ \times \mathcal{X}^-$ *transversal*, and write $x \top \alpha$, if they are conjugate under $G$ to the base point $(o^+, o^-)$. It is easy to describe the set $\alpha^\top := \{x \in \mathcal{X}^+ | \, x \top \alpha\}$ : if $\alpha = o^-$ is the base point (whose stabilizer is $P^+$), then this is the $P^+$-orbit $P^+.o^+$ which is isomorphic to $V^+ \cong U^+.o^+$. Note here that the set $\alpha^\top$ carries a canonical structure of an affine space over $\mathbb{K}$, since the vector group $U^+$ acts on it simply transitively. By homogenity, the same statements hold for any $\alpha \in \mathcal{X}^-$. This observation leads to the following definition:

**Definition 2.1.** *A* pair geometry *is a pair of sets* $\mathcal{X} = (\mathcal{X}^+, \mathcal{X}^-)$ *together with a binary relation* $\top \subset \mathcal{X}^+ \times \mathcal{X}^-$ *(called* transversality*, and we write* $x \top \alpha$ *for* $(x, \alpha) \in \top$*) such that* $\mathcal{X}^\pm$ *is covered by subsets of the form*

$$\alpha^\top = \{x \in \mathcal{X}^\pm |\, x \top \alpha\}, \quad \alpha \in \mathcal{X}^\mp.$$

*A* linear pair geometry (over $\mathbb{K}$) *is a pair geometry* $(\mathcal{X}^+, \mathcal{X}^-, \top)$ *such that, for any transversal pair* $x \top \alpha$*, the set* $\alpha^\top$ *is equipped with a structure of linear space over* $\mathbb{K}$ *(i.e., a* $\mathbb{K}$*-module), with origin* $x$*, and the same property holds by exchanging the rôles of* $\mathcal{X}^+$ *and* $\mathcal{X}^-$*.*

*An* affine pair geometry *is a linear pair geometry such that the underlying affine structure of the linear space* $(\alpha^\top, x)$ *does not depend on* $x$*, for all transversal pairs* $(x, \alpha)$*, and dually. In other words, for all* $\alpha \in \mathcal{X}^-$*, the set* $\alpha^\top$ *carries the structure of an affine space over* $\mathbb{K}$*, and dually.*

The discussion above shows that, to any Jordan pair $(V^+, V^-)$ over $\mathbb{K}$, we can associate an affine pair geometry $(\mathcal{X}^+, \mathcal{X}^-, \top)$. The link with triple products is given by introducting the *structure maps of a linear pair geometry*: if $x \top \alpha$, $y \top \alpha$, $z \top \alpha$ and $r \in \mathbb{K}$, then let $r_{x,\alpha}(y) := ry$ denote the product $r \cdot y$, and $y +_{x,\alpha} z$ the sum of $y$ and $z$ in the $\mathbb{K}$-module $\alpha^\top$ with zero vector $x$. In other words, we define maps of three (resp. four) arguments by

$$\begin{aligned} \mathrm{P}_r : \quad (\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+)^\top \to \mathcal{X}^+, \quad (x, \alpha, y) \mapsto \mathrm{P}_r(x, \alpha, y) := r_{x,\alpha}(y), \\ \mathrm{S} : (\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ \times \mathcal{X}^+)^\top \to \mathcal{X}^+, (x, \alpha, y, z) \mapsto \mathrm{S}(x, \alpha, y, z) := y +_{x,\alpha} z, \end{aligned}$$

where the domain of $\mathrm{P}_r$ is the "space of transversal triples",

$$(\mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+)^\top = \{(x, \alpha, y) \in \mathcal{X}^+ \times \mathcal{X}^- \times \mathcal{X}^+ |\, x \top \alpha,\, y \top \alpha\},$$

and the domain of S is the similarly defined space of generic quadruples. Of course, we should rather write $\mathrm{P}_r^+$ instead of $\mathrm{P}_r$ and $\mathrm{S}^+$ instead of S; dually, the maps $\mathrm{P}_r^-$ and $\mathrm{S}^-$ are then also defined. The structure maps encode all the information of a linear pair geometry: by fixing the pair $(x, \alpha)$, the structure maps describe the linear structure of $(\alpha^\top, x)$, resp. of $(x^\top, \alpha)$. In this way, linear pair geometries can be regarded as objects whose structure is defined by (one or several) "multiplication maps", just like groups, rings, modules, symmetric spaces . . .

This point of view naturally leads to the question whether there are more specific "identities" satisfied by the structure maps. If $(\mathcal{X}^+, \mathcal{X}^-; \top)$ is the geometry associated to a Jordan pair, then this is indeed the case: there are two such identities, denoted by (PG1) and (PG2) in [Be02]; the first identity (PG1) can be seen as an "integrated version" of the defining identity (LJP2) of a Jordan pair; it implies the existence of a "big" automorphism group, whereas (PG2) is rather a global version of the Fundamental Formula and implies the existence of "structural maps" exchanging the two partners $\mathcal{X}^+$ and $\mathcal{X}^-$ (we refer the reader to [Be02] for details). The important point about these identities is that we can also go the other way round: by "deriving" them in a suitable way, one can show that any affine pair geometry satisfying (PG1) and (PG2) in all scalar extensions gives rise to a Jordan pair $(V^+, V^-)$ as "tangent geometry" with respect to a fixed base point $(o^+, o^-)$. This construction clearly parallels the correspondence between Lie groups and Lie algebras (and even more closely the one between symmetric spaces and Lie triple systems, [Lo69]; see also [Be06]); however, in contrast to Lie theory, the constructions are algebraic in nature and therefore work much more generally in arbitrary dimension and over general base fields and rings. They define a bijection (even an equivalence of categories) between Jordan pairs and connected generalized projective geometries with base point – and, in the same way, between Jordan triple systems, (resp. unital Jordan algebras), and *generalized polar (resp. null) geometries*, i.e., generalized projective geometries with the additional structure of a certain kind of involution, cf. [Be02], [Be03]. Thus, in principle, all Jordan theoretic notions can be translated into geometric ones; in general, it is not at all obvious what the correct translation should be, but once it has been found, it often sheds new light onto the algebraic notion. In Chapter 4 we illustrate this at the example of the notion of *inner ideals* and their *complementation relation.*

### 2.3 The geometric Jordan-Lie functor

*Symmetric spaces* are well-known examples of "non-associative geometries" (cf. [Lo69]): they are manifolds $M$ together with a point reflection $\sigma_x$ attached to each point $x \in M$ such that the *multiplication map* $\mu : M \times M \to M$, $(x, y) \mapsto \sigma_x(y)$ satisfies the properties

- (M1) $\mu(x, x) = x$,
- (M2) $\mu(x, \mu(x, y)) = y$,
- (M3) $\mu(x, \mu(y, z)) = \mu(\mu(x, y), \mu(x, z))$,
- (M4) the tangent map $T_x(\sigma_x)$ is the negative of the identity of the tangent space $T_x M$.

Recall from Lemma 1.4 the (algebraic) Jordan-Lie functor, associating a Lie triple system to a Jordan triple system. The geometric analog of this functor associates to each generalized polar geometry $(\mathcal{X}^+, \mathcal{X}^-; p)$ the *symmetric space* $M^{(p)}$ of its non-isotropic points. The geometric construction is very simple: essentially, the multiplication map $\mu$ is obtained from the structure maps $\mathrm{P}_r$ by taking the scalar $r = -1$. More precisely, given a polarity (i.e., a pair of bijections $(p^+, p^-) : (\mathcal{X}^+, \mathcal{X}^-) \to (\mathcal{X}^-, \mathcal{X}^+)$ that are inverses of each other, compatible with the structure maps and such that there exists a *non-isotropic point* $x \in \mathcal{X}^+$, which means that $x \top p^+(x)$), for each non-isotropic point $x$ we define the point-reflection by

$$\sigma_x(y) := \mathrm{P}_{-1}\big(x, p^+(x), y\big).$$

The above mentioned identity (PG1) then implies in a straightforward way that the set $M^{(p)}$ of non-isotropic points of $p$ becomes a symmetric space (cf. [Be02] for the purely algebraic setting and [BeNe05] for the smooth setting). Moreover, the Lie triple system of the symmetric space $M^{(p)}$ is precisely the one defined by Lemma 1.4. The well-known construction of finite and infinite dimensional *bounded symmetric domains* (cf. [Up85]) arises as a special case of the geometric Jordan-Lie functor. The remarks following Lemma 1.4 imply that in fact "most" symmetric spaces (in the finite dimensional real case) are obtained by the preceding construction (essentially, all classical finite-dimensional real ones and about half of the exceptional ones, cf. tables in [Be00]).

### 2.4 Examples revisited

**(1) Rectangular matrices correspond to Grassmannians.** Let $W = A \oplus B$ be as in Example (1) of Section 1.5 and let $\mathrm{Gras}_A^B(W)$ be the set of all submodules $V \subset W$ that are isomorphic to $A$ ("type") and admit a complement $U \subset W$ isomorphic to $B$ ("cotype"). Then

$$(\mathcal{X}^+, \mathcal{X}^-) = (\mathrm{Gras}_A^B(W), \mathrm{Gras}_B^A(W)),$$

with transversality of $(V, U) \in \mathcal{X}^+ \times \mathcal{X}^-$ being usual complementarity of subspaces ($W = U \oplus V$), is an affine pair geometry over $\mathbb{K}$: it is a standard exercise in linear algebra to show that the set $U^\top$ of complements of $U$ carries a natural structure of affine space. Note that by our definition of Grassmannians as *complemented* Grassmannians this affine space is never empty; if $\mathbb{K}$ is a field this condition is automatic. (However, if $\mathbb{K}$ is a *topological* field or ring and $W$ a topological $\mathbb{K}$-module, then one will prefer modified definitions of "restricted Grassmannians" by imposing conditions of closedness, of boundedness, of Fredholm type or other, cf., e.g., [PS86], Chapter 7. All such conditions lead to subgeometries of the algebraically defined geometries considered here.)

Fixing the decomposition $W = A \oplus B$ as base point $(o^+, o^-)$ in $(\mathcal{X}^+, \mathcal{X}^-; \top)$, by elementary linear algebra one may identify the pair of linear spaces $((o^-)^\top, (o^+)^\top)$ with the pair $(V^+, V^-) = (\mathrm{Hom}(B, A), \mathrm{Hom}(A, B))$. By still elementary (though less standard) linear algebra, one can now give explicit expressions for the structure maps introduced in Section 2.2 and check their fundamental identities, thus describing a direct link between the geometry and its associated Jordan pair (see [Be04]): namely, elements of $\mathcal{X}^+$ are realized as images of injective maps $f : A \to W$, modulo equivalence under the general linear group $\mathrm{Gl}_{\mathbb{K}}(A)$ ($f \sim f'$ iff $\exists g \in \mathrm{Gl}_{\mathbb{K}}(A)$: $f' = f \circ g$), and similarly elements of $\mathcal{X}^-$ are realized as kernels of surjective maps $\phi : W \to A$, again modulo equivalence under $\mathrm{Gl}_{\mathbb{K}}(A)$. Two such elements are transversal, $[f] \top [\phi]$, if and only if $\phi \circ f : A \to A$ is a bijection, and the structure map $\mathrm{P}_r$ is now given by

$$\mathrm{P}_r([f],[\phi],[h]) = \big[(1-r)f\circ(\phi\circ f)^{-1} + rh\circ(\phi\circ h)^{-1}\big].$$

As in ordinary projective geometry, an affinization is given by writing $f: A \to W = B \oplus A$ as "column vector" and normalizing the second component to be $\mathbf{1}_A$, the identity map of $A$, and similarly for the "row vector" $\phi: B\oplus A \to A$ (see [Be04]). To get a feeling for the kind of non-linear formulas that appear in such contexts, the reader may re-write the preceding formula for $\mathrm{P}_r$ by replacing $f, \phi$ and $h$ by such column-, resp. row-vectors, and then re-normalize the right-hand side, in order to get the formula for the multiplication map in the affine picture. The special case $r = \frac{1}{2}$ ("midpoint map") is particularly important from a Jordan-theoretic point of view.

One may as well also consider the *total* Grassmannian geometry of all complemented subspaces of $W$,

$$\mathrm{Gras}(W) := \{V \subset W \,|\, (V \,\mathrm{submodule}),\, \exists V' : W = V \oplus V' (V' \,\mathrm{submodule})\}.$$

Then the pair $(\mathrm{Gras}(W), \mathrm{Gras}(W))$ still is a generalized projective geometry, but it is not *connected* in general (there is a natural notion of *connectedness* for any affine pair geometry, see [Be02]). In infinite dimension, the geometries $(\mathcal{X}^+, \mathcal{X}^-)$ introduced above need not be connected either, but in finite dimension over a field they are; in fact, they then reduce to the usual Grassmannians $\mathrm{Gras}_k(\mathbb{K}^n) \cong \mathrm{Gl}(n,\mathbb{K})/(\mathrm{Gl}(k,\mathbb{K}) \times \mathrm{Gl}(n-k,\mathbb{K}))$ of $k$-spaces in $\mathbb{K}^n$.

The case $n = 2k$, or, more generally, $A \cong B$, deserves special attention: in this case, $\mathcal{X}^+$ and $\mathcal{X}^-$ really are the same sets; the identity map $\mathcal{X}^+ \to \mathcal{X}^-$ is a canonical antiautomorphism of the geometry. Its properties are similar to the well-known *null-systems* from classical projective geometry. In particular, in the case of the projective line $(\mathrm{Gras}_1(\mathbb{K}^2), \mathrm{Gras}_1(\mathbb{K}^2))$, this really is the canonical null-system coming from the (up to a scalar) unique symplectic form on $\mathbb{K}^2$.

**(2) Geometries of maps and functions.** We fix some generalized projective geometry $\mathcal{X} = (\mathcal{X}^+, \mathcal{X}^-)$, and let $M$ be any set. Then the *geometry of maps*

$$(\mathrm{Fun}(M, \mathcal{X}^+), \mathrm{Fun}(M, \mathcal{X}^-))$$

with "pointwise transversality and structure maps", is again a generalized projective geometry. By arguments similar as above, we see that it corresponds to the Jordan pair of functions introduced in Example (2) of Section 1.5. As explained there, the case of scalar valued functions corresponds to $V^+ = V^- = \mathrm{Fun}(M,\mathbb{K})$ with pointwise triple product $(T(f,g,h))(p) = 2f(p)\,g(p)\,h(p)$. On the geometric side, then $\mathcal{X}^+ = \mathcal{X}^- = \mathbb{KP}^1$ is the projective line, and $\mathrm{Fun}(M,\mathcal{X}^+) = \mathrm{Fun}(M,\mathcal{X}^-)$ is the space of functions from $M$ into the projective line. Note that these definitions parallel the usual ones for groups (cf. [PS86]); in particular, taking for $M$ the unit circle, we would get "loop geometries".

**(3) The classical examples.** We briefly characterize the geometries $(\mathcal{X}^+, \mathcal{X}^-)$ corresponding to the examples of item (3) of Section 1.5:

- full asociative algebras $A$ correspond to the *projective line* $(A\mathbb{P}^1, A\mathbb{P}^1)$ *over* $A$ (cf. [BeNe04]);
- Hermitian elements of an involutive associative algebra $(A, *)$ correspond to the $*$-Hermitian projective line over $A$ (cf. [BeNe04], Section 8.3), and similarly for skew-Hermitian elements. *Lagrangian geometries* are a special case of this construction, corresponding to (skew-) Hermitian or (skew-) symmetric matrices or operators
- conformal geometries (or “spin factors”): here, $\mathcal{X}^+ \cong \mathcal{X}^-$ is the projective quadric of $\mathbb{K} \oplus V \oplus \mathbb{K}$ with the same quadratic form as in Section 1.5.

As to the exceptional Jordan systems, their geometries are among those constructed by Tits; to our knowledge, their structure of generalized projective geometry has not yet been fully investigated.

## 3 The universal model

We have given above the construction of the generalized projective geometry $(\mathcal{X}^+, \mathcal{X}^-)$ associated to a 3-graded Lie algebra $\mathfrak{g}$ via homogeneous spaces $(G/P^-, G/P^+)$. For a more detailed study, a geometric realization of these spaces is useful, which in some sense generalizes the example of the Grassmannian geometries. This “universal model” has been introduced in [BeNe04] and used in [BeNe05] to define (under suitable topological assumptions) a manifold structure on $(\mathcal{X}^+, \mathcal{X}^-)$.

### 3.1 Ordinary flag geometries

We fix some $\mathbb{K}$-module $\mathfrak{g}$ and denote by $\mathcal{F}_k$ the set of all flags

$$\mathfrak{f} = \big(0 = \mathfrak{f}_0 \subset \mathfrak{f}_1 \subset \ldots \subset \mathfrak{f}_k = \mathfrak{g}\big)$$

of length $k$ in $E$ (i.e., the $\mathfrak{f}_i$ are linear subspaces of $E$, and all inclusions are supposed to be strict). We say that two flags $\mathfrak{e}, \mathfrak{f} \in \mathcal{F}$ are *transversal* if they are “crosswise complementary” in the sense that

$$\forall i = 1, \ldots, k : \qquad \mathfrak{g} = \mathfrak{f}_i \oplus \mathfrak{e}_{k-i}.$$

It is a nice exercise in linear algebra to show that $\mathfrak{e}$ and $\mathfrak{f}$ are transversal if, and only if, they come from a *grading of* $\mathfrak{g}$ *of length* $k$: given a grading $g = (\mathfrak{g}_1, \ldots, \mathfrak{g}_k)$, i.e., a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$, we define two flags $\mathfrak{f}^+(g)$ and $\mathfrak{f}^-(g)$ via

$$\mathfrak{f}_i^+(g) := \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_i, \qquad \mathfrak{f}_{k-i}^-(g) := \mathfrak{g}_{i+1} \oplus \ldots \oplus \mathfrak{g}_k.$$

Then $\mathfrak{f}^+(g)$ and $\mathfrak{f}^-(g)$ are obviously transversal, but the converse is also true: every pair of transversal flags can be obtained in this way! From this exercise, one deduces easily that we get a linear pair geometry $(\mathcal{X}^+, \mathcal{X}^-; \top)$ by taking

$$\mathcal{X}^+ := \mathcal{X}^- := \mathcal{X} := \{\mathfrak{f} \in \mathcal{F}_k \mid \exists \mathfrak{e} \in \mathcal{F}_k : \mathfrak{e} \top \mathfrak{f}\},$$

the space of all flags that admit at least one transversal flag (cf. [BL06]). For $k = 2$, this is the total Grassmannian geometry, and similarly as in that case, one may for general $k$ single out connected components by taking *flags of fixed type and cotype*. In general, these geometries will be *linear, but not affine* pair geometries, and we do not know what kind of "laws" (in a sense generalizing the laws of generalized projective geometries) can be used to describe them.

### 3.2 Filtrations and gradings of Lie algebras

Let us assume now that $\mathfrak{g}$ is a *Lie algebra* with Lie bracket denoted by $[\cdot,\cdot]$, and that all our gradings and filtrations are compatible with the Lie bracket in the sense that

$$[\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j}, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

We will assume that our Lie algebra is $2k+1$-*graded*, i.e., as index set we take $I_k = \{-k, -k+1, \ldots, k\}$, and we say that a $2k+1$-grading is *inner* if it can be defined by an *Euler element*, i.e., by an element $E \in \mathfrak{g}$ such that $[E, X] = iX$ for all $X \in \mathfrak{g}_i$, $i \in I_k$. We denote by $\mathcal{G}_k$ the set of all inner $2k+1$-gradings of $\mathfrak{g}$ (it is harmless to assume that the center of $\mathfrak{g}$ is reduced to zero, and so $\mathcal{G}_k$ can be identified with set of all Euler elements). An *inner* $2k+1$-*filtration of* $\mathfrak{g}$ is a flag

$$\mathfrak{f} = (0 = \mathfrak{f}_{k+1} \subset \mathfrak{f}_k \subset \ldots \subset \mathfrak{f}_{-k} = \mathfrak{g})$$

such that there is some inner $2k+1$-grading with $\mathfrak{f}_i = \mathfrak{g}_i \oplus \mathfrak{g}_{i+1} \oplus \ldots \oplus \mathfrak{g}_k$, for all $i \in I_k$; we write $\mathcal{F}_k$ for the set of all such filtrations. Then the following holds ([BeNe04] for the case of 3-gradings ($k = 1$), and [Ch07] for the general case). We have to impose some mild restrictions on the characteristic of $\mathbb{K}$ since we want to use, for $X \in \mathfrak{f}_1$, the operator $\exp(X) = e^{\mathrm{ad}(X)} = \sum_{j=0}^{\infty} \frac{1}{j!} \mathrm{ad}(X)^j$, the sum being finite since $[\mathfrak{f}_1, \mathfrak{f}_i] \subset \mathfrak{f}_{i+1}$, so that $\mathrm{ad}(X)$ is nilpotent.

**Theorem.** *Assume that $\mathbb{K}$ is a commutative ring such that the integers $2, 3, \ldots, 2k+1$ are invertible in $\mathbb{K}$.*

1. *Two inner $2k+1$-filtrations $\mathfrak{f}$ and $\mathfrak{e}$ are transversal if, and only if, there exists a $2k+1$-grading $g = (\mathfrak{g}_{-k}, \ldots, \mathfrak{g}_k)$ such that $\mathfrak{f} = \mathfrak{f}^+(g)$ and $\mathfrak{e} = \mathfrak{f}^-(g)$.*
2. *The geometry of inner $2k+1$-filtrations $(\mathcal{F}_k, \mathcal{F}_k; \top)$ is a linear pair geometry. More precisely, for any $\mathfrak{f} \in \mathcal{F}_k$, the nilpotent Lie algebra $\mathfrak{f}_1$ is in bijection with $\mathfrak{f}^\top$ via $X \mapsto \exp(X).\mathfrak{e}$, for an arbitrary choice of $\mathfrak{e} \in \mathfrak{f}^\top$.*

Now assume that $k = 1$, i.e., we are in the 3-graded case. Then $[\mathfrak{f}_1, \mathfrak{f}_1] \subset \mathfrak{f}_2 = 0$ (i.e., $\mathfrak{f}_1$ is an abelian subalgebra), whence part (2) of the theorem says that we have a simply transitive action of the vector group $\mathfrak{f}_1 \cong \exp(\mathfrak{f}_1)$ on $\mathfrak{f}^\top$, and thus we see that the geometry of inner 3-filtrations is an *affine* pair geometry. In fact, fixing some inner 3-grading as base point in the space of all inner 3-gradings, it now easily follows that the geometry $(G/P^-, G/P^+)$ constructed in the preceding chapter is imbedded into the geometry from the theorem simply by taking the $G$-orbit of the base point.

As mentioned above, this "universal geometric realization" turns out to be useful for giving a precise description of the intersections of the "chart domains" $\alpha^\top$ and $\beta^\top$ for $\alpha, \beta \in \mathcal{X}^-$ and for calculating the corresponding "transition functions" (see [BeNe04]); calculations become similar to the case of Grassmannian geometries (Example (1) in Section 2.3 above), the difference being that the usual fractional linear transformations have to be replaced by certain *fractional quadratic* transformations (this difference corresponds to the fact that we are now working with true flags instead of single subspaces). Using these purely algebraic results, one can now give necessary and sufficient conditions for defining on $(\mathcal{X}^+, \mathcal{X}^-)$ the structure of a smooth manifold ([BeNe05]). The theory works nicely over general topological base fields and rings (cf. [Be06] for basic differential geometry and Lie theory in this general framework); complex or real Banach, Fréchet or locally convex manifolds are special cases of it. For higher gradings, similar results can be expected (J. Chenal, work in progress, [Ch07]).

## 4 The geometry of states

Let us come back to the analogies, mentioned in the introduction, of the preceding constructions with methods of commutative and non-commutative geometry. In the language of classical physics, one wants to recover the pure states of a system (the point space $M$) from its observables (the function algebra $\mathcal{A} = \mathrm{Reg}(M, \mathbb{R})$). The usual procedure is to look at the (maximal) ideals $I_p = \{f \in \mathcal{A} | \, f(p) = 0\}$ corresponding to points $p \in M$. Left or right ideals in associative algebras are generalized by *inner ideals* in Jordan theory. Therefore we shall interprete *states* in generalized projective geometries as the geometric objects corresponding to inner ideals.

### 4.1 Intrinsic subspaces

Assume $(\mathcal{X}^+, \mathcal{X}^-; \top)$ is a linear pair geometry over the commutative ring $\mathbb{K}$.

**Definition 4.1.** *A pair* $(\mathcal{Y}^+, \mathcal{Y}^-)$ *of subsets* $\mathcal{Y}^\pm \subset \mathcal{X}^\pm$ *is called a* subspace of $(\mathcal{X}^+, \mathcal{X}^-)$*, if for all* $x \in \mathcal{Y}^\pm$ *there exists an element* $\alpha \in \mathcal{Y}^\mp$ *such that* $x \top \alpha$*, and if for every such pair* $(x, \alpha)$ *the set* $\mathcal{Y}_\alpha := \mathcal{Y} \cap \alpha^\top$ *is a* linear *subspace of* $(\alpha^\top, x)$ *and* $\mathcal{Y}'_x := \mathcal{Y}' \cap x^\top$ *is a* linear *subspace of* $(x^\top, \alpha)$.

*A subset* $\mathcal{I} \subset \mathcal{X}^+$ *is called a* state *or* intrinsic subspace (in $\mathcal{X}^+$)*, if* $\mathcal{I}$ *"appears linearly to all possible observers": for all* $\alpha \in \mathcal{X}^-$ *with* $\alpha^\top \cap \mathcal{I} \neq \emptyset$ *and for all* $x \in \alpha^\top \cap \mathcal{I}$*, the set* $(\alpha^\top \cap \mathcal{I}, x)$ *is a linear subspace of* $(\alpha^\top, x)$.

For instance, if $(\mathcal{X}^+, \mathcal{X}^-) = (\mathrm{Gras}_1(\mathbb{K}^{n+1}), \mathrm{Gras}_n(\mathbb{K}^{n+1}))$ is an ordinary projective geometry, then all projective subspaces of $\mathcal{X}^+$ are intrinsic subspaces, and every affine subspace of an affinization of $\mathcal{X}^+$ is obtained in this way. In contrast, for Grassmannian geometries of higher rank the situation becomes more complicated: only rather specific linear subspaces of the affinization $M(p,q;\mathbb{K})$ of $\mathrm{Gras}_p(\mathbb{K}^{p+q})$ are obtained in this way, namely the so-called *inner ideals*. This observation generalizes to all geometries $(\mathcal{X}^+,\mathcal{X}^-)$ associated to a Jordan pairs $(V^+,V^-)$: subspaces containing the base point $(o^+,o^-)$ correspond to subpairs of $(V^+,V^-)$, and intrinsic subspaces of $\mathcal{X}^+$ containing $o^+$ to *inner ideals* $I \subset V^+$, i.e., submodules such that $T^+(I,V^-,I) \subset I$ (see [BL06]). The notions of *minimal, maximal, principal,...* inner ideals may also be suitably translated into a geometric language (minimal intrinsic subspaces will also be called *intrinsic lines* and maximal ones *intrinsic hyperplanes*; readers coming from axiomatic geometry or from Jordan theory will consider the former as more fundamental and call them accordingly *pure states*, whereas readers coming from algebraic geometry or $C^*$-algebra theory rather would reserve this term for the latter ones). The collection of states associated to a linear pair geometry can again be turned into a pair geometry by introducing the following transversality relation:

**Definition 4.2.** *Let $\mathcal{I}$ be an intrinsic subspace in $\mathcal{X}^+$ and $\mathcal{J}$ one in $\mathcal{X}^-$. We say that $\mathcal{I}$ and $\mathcal{J}$ are* transversal*, and we write again $\mathcal{I}\top\mathcal{J}$, if*

*(1) the pair $(\mathcal{I},\mathcal{J})$ is a subspace,*
*(2) the linear pair geometry $(\mathcal{I},\mathcal{J})$ is* faithful *in the following sense:*

*A linear pair geometry $(\mathcal{Y}^+,\mathcal{Y}^-)$ is called* faithful *if $\mathcal{Y}^-$ is faithfully represented by its effect of linearizing $\mathcal{Y}^+$, and vice versa: whenever $\alpha^\top = \beta^\top$ as sets* and *as linear spaces (with respect to some origin $o$), then $\alpha = \beta$, and the dual property holds.*

Let us give some motivation for this definition (which does not appear in [BL06]). First of all, in general, a linear pair geometry need not be faithful (take, for instance, a pair of $\mathbb{K}$-modules with trivial structure maps), but ordinary projective geometries over a field are. (In case of geometries corresponding to Jordan pairs, faithfulness corresponds to *non-degeneracy* in the Jordan-theoretic sense.) Next note that, even if the geometry $(\mathcal{X}^+,\mathcal{X}^-)$ was faithful, the geometry $(\mathcal{I},\mathcal{X}^-)$ will in general not be faithful: there is a *kernel*, i.e., the equivalence relation "$\alpha \sim_\mathcal{I} \beta$ iff $\alpha$ and $\beta$ induce the *same* linear structure on $\mathcal{I}$" will be non-trivial on $\mathcal{X}^-$. The condition $\mathcal{I}\top\mathcal{J}$ means, then, that the space $\mathcal{J} \subset \mathcal{X}^-$ is transversal to the fibers of this equivalence relation, in the usual sense. (This is the geometric translation of the notion of *complementation of inner ideals* introduced in [LoNe95]: the fiber of the equivalence relation $\sim_\mathcal{I}$ corresponds to the *kernel of an inner ideal* introduced in [LoNe95]). For instance, if $\mathcal{I}$ is a projective line in an ordinary projective space $\mathcal{X}^+ = \mathrm{Gras}_1(\mathbb{K}^{n+1})$, corresponding to a 2-dimensional subspace $I$ of $\mathbb{K}^{n+1}$, then $\mathcal{J}$ should be a projective subspace of the dual projective space $\mathcal{X}^- = \mathrm{Gras}_n(\mathbb{K}^{n+1})$ such that different elements of $\mathcal{J}$ define different affine lines in $\mathcal{I}$. If $\mathcal{J}$ is the set of hyperplanes lying over some $k$-dimensional subspace of $\mathbb{K}^{n+1}$, then this means that $I$ should contain some complement of $J$.

Vice versa, $J$ then also should contain some complement of $I$, and hence $I$ and $J$ have to be complements of each other in $\mathbb{K}^{n+1}$. Thus $(\mathcal{I}, \mathcal{J})$ is a projective line which is faithfully imbedded in the projective geometry. Moreover, in this example we clearly recover the Grassmannian geometry of 2-spaces as the geometry of intrinsic lines in the 1-Grassmannian.

Parts of the observations from the example generalize: let $\mathcal{S}^{\pm}$ be the set of all intrinsic subspaces $\mathcal{I}$ in $\mathcal{X}^{\pm}$ that admit a transversal intrinsic subspace $\mathcal{J}$ in $\mathcal{X}^{\mp}$. It is a natural question to ask whether $(\mathcal{S}^+, \mathcal{S}^-)$ is again a linear pair geometry, or under which conditions on $(\mathcal{X}^+, \mathcal{X}^-)$ and on the states we obtain one. For the time being, we have no general answers, but results from [BL06] point into the direction that, indeed, for the most interesting cases we get geometries $(\mathcal{S}^+, \mathcal{S}^-)$ that are again linear pair geometries. More precisely, if $(\mathcal{X}^+, \mathcal{X}^-)$ corresponds to a Jordan pair $(V^+, V^-)$ and we consider only states that are associated, via a Peirce-decomposition, to idempotents, we obtain geometries coming from 5-graded Lie algebras ([BL06], Theorem 5.8), and these are indeed linear pair geometries, according to Theorem 3.2.

### 4.2 Examples

**(1) Classical states and observables revisited.** Let $M$ be a set and $\mathcal{A} = \mathrm{Fun}(M, \mathbb{K})$ the Jordan algebra of all functions from $M$ to $\mathbb{K}$. It corresponds to the geometry $(\mathcal{X}, \mathcal{X})$ with $\mathcal{X} = \mathrm{Fun}(M, \mathbb{KP}^1)$. We claim that (if $\mathbb{K}$ is a field) the set $M$ can be recovered from the geometry $\mathcal{X}$ as the geometry of all pure states (intrinsic lines) running through the "zero function" $f_0 \equiv o$, by associating to a point $p \in M$ the intrinsic line

$$L_p = \{f : M \to \mathbb{KP}^1 |\, f(x) = o \text{ if } x \neq p\} \subset \mathrm{Fun}(M, \mathbb{KP}^1).$$

Indeed, note first that $L_p$ is indeed a minimal intrinsic subspace since $L_p \cong \mathbb{KP}^1$ and the geometry $(\mathbb{KP}^1, \mathbb{KP}^1)$ itself does not admit any proper intrinsic subspaces that are not points. Now assume $f : M \to \mathbb{KP}^1$ is a function taking non-zero values at two different points $p, q \in M$. Then a look at the direct product geometry $(\mathbb{KP}^1, \mathbb{KP}^1) \times (\mathbb{KP}^1, \mathbb{KP}^1)$ shows that the intrinsic subspace generated by $f$ contains at least all functions obtained from $f$ by altering the values at $p$ and $q$ in an arbitrary way, hence contains a homomorphic image of $\mathbb{KP}^1 \times \mathbb{KP}^1$ and thus is not minimal, proving our claim. Of course, we may as well work with the *maximal* intrinsic subspaces

$$H_p = \{f : M \to \mathbb{KP}^1 |\, f(p) = o\} \subset \mathrm{Fun}(M, \mathbb{KP}^1).$$

This is certainly closer to the philosophy of algebraic geometry and to the usual imbedding of "classical" into "non-commative" geometry, when $M$ is a smooth or algebraic manifold over $\mathbb{R}$ or $\mathbb{C}$ and we look at geometries of smooth or algebraic functions $f : M \to \mathbb{KP}^1$. Both points of view are dual to each other in the sense that one really should look at the geometry $(\mathcal{S}^+, \mathcal{S}^-)$, where one of the $\mathcal{S}^\pm$ is the collection of minimal, and the other of maximal intrinsic subspaces, each having a complement (in the sense of Definition 4.2) of the other type.

**(2) Grassmann geometries $\mathcal{X}^+ = \mathrm{Gras}_A^B(W)$.** Let us fix a flag in $W$ of length two, $\mathfrak{f} : 0 \subset \mathfrak{f}_1 \subset \mathfrak{f}_2 \subset W$. Then the set of all subspaces of $W$ that are "squeezed" by this flag,

$$\mathcal{I}_\mathfrak{f} := \{E \in \mathcal{X}^+ | \, \mathfrak{f}_1 \subset E \subset \mathfrak{f}_2\},$$

is an intrinsic subspace of $\mathcal{X}$, and if $\mathbb{K}$ is a field and $W$ is finite-dimensional over $\mathbb{K}$, then all intrinsc subspaces are of this form for a suitable flag $\mathfrak{f}$ (see [BL06], Theorem 3.11). Moreover, one can show that the intrinsic subspaces $\mathcal{I}_\mathfrak{f}$ and $\mathcal{I}_\mathfrak{e}$ are transversal in the sense defined above if, and only if, the flags $\mathfrak{e}$ and $\mathfrak{f}$ are transversal in the sense defined in Section 3.1 above. Therefore the geometry of intrinsic subspaces of the Grassmannian geometry is a geometry of flags of length two and hence is a linear pair geometry. The case of ordinary projective geometry (i.e., $A \cong \mathbb{K}$) is somewhat degenerate: in this case there is not much choice for the first component $\mathfrak{f}_1$ of the flag $\mathfrak{f}$, and therefore the intrinsic subspace $\mathcal{I}_\mathfrak{f}$ is already determined by $\mathfrak{f}_2$ alone; we get back the usual projective subspaces of a projective space. In all other cases, the geometry of intrinsic subspaces is a true flag geometry corresponding to 5-gradings and hence is a linear, but no longer an affine pair geometry.

**(3) Quantum states.** In the language of quantum mechanics (widely used also in non-commutative geometry), *states* are defined as positive normalized linear functionals $\phi : \mathcal{A} \to \mathbb{C}$ on a $C^*$-algebra $(\mathcal{A}, *)$. They form a convex set, whose extremal points are interpreted as *pure states.* These, in turn, admit a Jordan theoretic interpretation via *(primitive) idempotents* – see [FK94] for the theory of finite-dimensional *symmetric cones* and their geometry; there is a vast literature on infinite dimensional generalizations, cf., e.g., [HOS84], [ER92], [Up85]. Whereas the notion of state just mentioned depends highly on the ordered structure of the base field $\mathbb{R}$ (via *positivity* and *convexity*), this is not the case for the notion of inner ideal and intrinsic subspace, which hence seem to be more general and more geometric. Accordingly, it has been advocated to use inner ideals as a basic ingredient for an approach to quantum mechanics (see [F80]).

## References

[AF99] Allison, B. N. and J.R. Faulkner, "Elementary groups and invertibility for Kantor pairs" Comm. Algebra **27** (1999), no. 2, 519–556.

[Be00] Bertram, W., *The geometry of Jordan- and Lie structures*, Springer Lecture Notes in Mathematics **1754**, Berlin 2000

[Be02] Bertram, W., "Generalized projective geometries: general theory and equivalence with Jordan structures", Adv.

Geom. **2** (2002), 329–369 (electronic version: preprint 90 at `http://www.uibk.ac.at/mathematik/loos/jordan/index.html`)

[Be03] Bertram, W., "The geometry of null systems, Jordan algebras and von Staudt's Theorem", Ann. Inst. Fourier **53** (2003) fasc. 1, 193–225

[Be04] Bertram, W., "From Linear Algebra via Affine Algebra to Projective Algebra", Linear Algebra and its Applications **378** (2004), 109–134

[Be06] Bertram, W., *Differential Geometry, Lie Groups and Symmetric Spaces over General Base Fields and Rings.* Mem. AMS, to appear; math.DG/ 0502168

[BL06] Bertram, W. and H. Löwe, "Inner Ideals and Intrinsic Subspaces", Adv. Geom. (to appear); math.RA/0606448

[BeNe04] Bertram, W. and K.-H. Neeb, "Projective completions of Jordan pairs. I: The generalized projective geometry of a Lie algebra" J. Algebra **277**(2) (2004), 193 – 225; math.RA/0306272

[BeNe05] Bertram, W. and K.-H. Neeb, "Projective completions of Jordan pairs. II: Manifold structures and symmetric spaces" Geom. Dedicata **112** (2005), 73 – 113; math.GR/0401236

[Bil04] Biller, H., *Continuous Inverse Algebras and Infinite-Dimensional Linear Lie Groups*, Habilitationsschrift, TU Darmstadt, 2004

[Bue95] Buekenhout, F. (editor), *Handbook of Incidence Geometry*, Elsevier, Amsterdam 1995

[Ch07] Chenal, J. (work in progress)

[ER92] Edwards, C. M., and G.T. Rüttimann, "A characterization of inner ideals in $JB^*$-triples", Proc. Amer. Math. Soc. **116** (1992), no. 4, 1049–1057.

[FK94] Faraut, J. and A. Koranyi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford 1994

[F80] Faulkner, J.R., "Incidence Geometry of Lie Groups in Quantum Theory", Group theoretical methods in physics (Proc. Eigth Internat. Colloq., Kiryat Anavim, 1979) Ann. Israel Phys. Soc., **3**, 73 – 79, Hilger, Bristol, 1980

[HOS84] Hanche-Olsen, H. and E. Størmer, *Jordan Operator Algebras*, Pitman, Boston 1984

[Lo69] Loos, O., *Symmetric Spaces I*, Benjamin, New York, 1969

[Lo75] Loos, O., *Jordan Pairs*, Springer LN 460, New York 1975

[Lo95] Loos, O., "Elementary groups and stability for Jordan pairs." $K$-Theory **9** (1995), no. 1, 77–116

[LoNe95] Loos, O. and E. Neher, "Complementation of Inner Ideals in Jordan Pairs", J. Alg. **166** (1994), 255 – 295

[McC04] Mc. Crimmon, K. *A Taste of Jordan Algebras.* Springer, New York (1994)

[PS86] Pressley, A. and G. Segal, *Loop Groups*, Clarendon Press, Oxford 1986

[Sm05] Smirnov, O.N., "Imbedding of Lie triple systems into Lie algebras", preprint, 2005

[Up85] Upmeier, H., *Symmetric Banach Manifolds and Jordan $C^*$-algebras,* North Holland Mathematics Studies, 1985