RAMANUJAN's Master Theorem and Duality of Symmetric Spaces

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Abstract. — We interprete RAMANUJAN's master theorem (see [B85])

$$(R) \qquad \int_0^\infty x^{-s-1} \sum_{k=0}^{\infty} ((-1)^k a(k) x^k)\, dx = -\frac{\pi}{\sin(\pi s)} a(s)$$

as a relation between the Fourier transforms of an analytic function f with respect to the real forms $U(1)$ (compact) and $\mathbb{R}^+$ (non-compact) of the multiplicative group of non-zero complex numbers, and we ask for a similar relation between the spherical Fourier transforms of an analytical function with respect to a compact real form and the non-compact dual real form of a complex symmetric space. We obtained results in the case of symmetric cones ([Be93]) and in the rank-one case (announced in [Be94a]). Here we present the latter case in detail, describing features which will be also important for the general rank case.

0. Introduction: "RAMANUJAN's transform" for symmetric spaces

0.1 The geometrical setting: duality of riemannian symmetric spaces. — We consider a compact symmetric space $X_U = U/K$, its non-compact dual $X_G = G/K$, both realized as real forms of their common complexifcation $X_\mathbb{C} = U_\mathbb{C}/K_\mathbb{C}$. The simplest example of this situation is given by the circle and the positive real half line, both real forms of the multiplicative group of non-zero complex numbers. So let U be a compact Lie group, σ a non-trivial involution of U and K an open subgroup of the fixpoint group U^σ of σ. Then $X_U = U/K$ is a compact symmetric space, and its complexification is by definition the complex symmetric space $X_\mathbb{C} = U_\mathbb{C}/K_\mathbb{C}$, where, for a compact Lie group L, $L_\mathbb{C}$ denotes its *universal complexification.* This is a complex Lie group having as Lie algebra the complexification $\mathfrak{l}_\mathbb{C} = \mathfrak{l} \oplus i\mathfrak{l}$ of the Lie algebra $\mathfrak{l}$ of L and admitting a *Cartan decomposition* $L_\mathbb{C} = L \exp(i\mathfrak{l})$ (see [BtD85] III.(8.3)). This decomposition implies that, in the above case, $K_\mathbb{C} \cap U = K$, so

$$X_U \to X_\mathbb{C}, \quad uK \mapsto uK_\mathbb{C}$$

is injective, and we may and will consider X_U as the U-orbit of the base point in $X_\mathbb{C}$.

The *non-compact dual of* X_U is the riemannian symmetric space $X_G := G/K_0$, where K_0 denotes the identity component of K and G is the analytic subgroup of $U_{\mathbb{C}}$ having as Lie algebra $\mathfrak{g} := \mathfrak{k} \oplus i\mathfrak{q}$, where $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$ is the decomposition in ± 1-eigenspaces of $\mathfrak{u}$ with respect to the differential $\dot{\sigma}$ of σ at the origin. More conceptually: if τ ist the antiholomorphic involution of $U_{\mathbb{C}}$ having U as fixed point group, and the holomorphic continuation of σ onto $U_{\mathbb{C}}$ is also denoted by σ, then $\theta := \sigma \circ \tau = \tau \circ \sigma$ is an antiholomorphic involution of $U_{\mathbb{C}}$ having G as the connected component of the identity of its fixed point group. In terms of the Cartan decomposition: $\theta(u \exp(iX)) = \sigma(u) \exp(-i\dot{\sigma}X)$; so G has Cartan decomposition $G = K_0 \exp(i\mathfrak{q})$. From this we see that $G \cap U_{\mathbb{C}} = K_0$, so

$$X_G \to X_{\mathbb{C}}, \quad gK_0 \mapsto gK_{\mathbb{C}}$$

is injective, and we will consider X_G as the G-orbit of the base point in $X_{\mathbb{C}}$. We remark that the three involutions σ, τ and θ induce involutions on $X_{\mathbb{C}}$ such that σ is the symmetry with respect to the origin of the symmmetric space $X_{\mathbb{C}}$, X_U is open in the fixed point set of τ and X_G is open in the fixed point set of θ.

Besides the above (abelian) example $X_U = U(1)$, let us mention two other important (non-abelian) examples: firstly, the complexification of the n-sphere $X_U = S^n$ which is just the complex quadric with the same equation, and the non-compact dual X_G is the real hyperbolic space H^n; secondly, the *group case*: K is a compact connected Lie group, $X_U = K \times K/dia(K \times K) \cong K$, $X_{\mathbb{C}} \cong K_{\mathbb{C}}$, $X_G \cong K_{\mathbb{C}}/K$, the latter imbedded into $K_{\mathbb{C}}$ as the set $\{z\tau(z)^{-1} \,|\, z \in K_{\mathbb{C}}\}$ where τ is the conjugation of $K_{\mathbb{C}}$ w.r. to K.

0.2 The analytical setting: compact and non-compact spherical Fourier transform. — We are interested in analyzing complex-valued functions f which are holomorphic on $X_{\mathbb{C}}$ or at least on parts of $X_{\mathbb{C}}$ which contain the orbits X_U and X_G, and we suppose that f is $K_{\mathbb{C}}$-invariant. Then we define the *non-compact spherical Fourier transform* $\tilde{f}$ of f by

$$\tilde{f}(\lambda) = \int_{X_G} f(x)\phi_{-\lambda}(x)\, dx, \tag{0.1}$$

where dx is a G-invariant measure on X_G (unique up to a constant) and

$$\phi_\lambda(x) = \int_K e^{(\rho+\lambda)H(kg)}\, dk$$

is Harish chandra's formula for the spherical functions of X_G, with respect to an Iwasawa decomposition $G = NAK$ of G, A being a vector group having a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} = i\mathfrak{q}$ as Lie algebra and

$H : G \to \mathfrak{a}$, $nak \mapsto \log a$ being the Iwasawa projection. The parameter λ runs through $\mathfrak{a}^*_{\mathbb{C}}$, the dual space of the complexification of $\mathfrak{a}$, and $\rho \in \mathfrak{a}^*_{\mathbb{C}}$ is the half sum of the positive roots in the restricted root system $\Sigma := \Sigma(\mathfrak{a}, \mathfrak{g})$. We say that the spherical Fourier transform of f is defined at λ if (0.1) converges absolutely. It is known that $\tilde{f}$ is invariant under the action of the *Weyl group* $W := W(\Sigma)$ of Σ.

We will define the *compact spherical Fourier transform* of f in a somewhat different manner than usual in order to be able to compare it with the non-compact transform defined above. The key will be the following lemma (see [He94] III.9):

0.2.1 Lemma. — *The spherical function ϕ_λ admits a holomorphic continuation onto $X_{\mathbb{C}}$ if and only if λ belongs to the W-orbit of the set $\Lambda + \rho$, where Λ is the set of the* dominant spherical weights, *i.e. the highest weights of K-spherical irreducible representations of U which can also be described as the set of $\lambda \in \mathfrak{a}^*$ such that (see [La78] thm.B and references there)*

$$\forall \mu \in \Sigma^+ : \ \frac{(\lambda|\mu)}{(\mu|\mu)} \in \mathbb{Z}^+, \qquad \lambda(\Gamma_{U/K}) \subset 2\pi i \mathbb{Z},$$

where $\Gamma_{U/K} = \{X \in i\mathfrak{a} |\ \exp X \in K\}$. Then the restriction of ϕ_λ to X_U is a spherical function of X_U, and all spherical functions of X_U arise in this manner.

$\square$

The lemma permits us to define, by a formula very similar to (0.1), the *compact spherical Fourier transform* of f for $\lambda \in \Lambda + \rho$ by

$$\hat{f}(\lambda) := \int_{X_U} f(x)\phi_{-\lambda}(x)\, dx = \int_{X_U} f(x)\overline{\phi_\lambda(x)}\, dx; \tag{2}$$

here dx is a Haar measure on X_U, and the last equality is due to the fact that $\phi_{-\lambda}(x) = \phi_\lambda(x^{-1})$ (where ϕ_λ is considered as bi-invariant function on the group) and that the spherical functions of X_U are of *positive type.*

0.3 "Ramanujan's transform". — If f is a "good" analytic and $K_{\mathbb{C}}$-invariant function on $X_{\mathbb{C}}$, then, as the spherical Fourier transform is invertible, it will be determined by its non-compact transform $\tilde{f}$ as well as by its compact transform $\hat{f}$, which in turn means that these transforms determine each other. We express this by the diagram

$$\begin{array}{ccc} & f & \\ f|_{X_U} & & f|_{X_G} \\ \downarrow & & \downarrow \\ \hat{f} & \overset{(R)}{\longleftrightarrow} & \tilde{f}\,, \end{array}$$

where we would like to call the bottom line the "RAMANUJAN transform" because of the fact that it was RAMANUJAN who discovered in 1913 a relation, nowadays called his "master theorem" (see [B85] p.298), of the kind we are looking for. It concerns the abelian case $X_U = U(1)$, $X_{\mathbb{C}} = \mathbb{C}^*$, $X_G = \mathbb{R}^+$, and we state it in the way RAMANUJAN did:

$$(R) \qquad \int_0^\infty x^{-s-1}\Big(\sum_{k=0}^\infty (-1)^k a(k) x^k\Big)\, dx = -\frac{\pi}{\sin(\pi s)} a(s).$$

If we let $f(x) := \sum_{k=0}^\infty (-1)^k a(k) x^k$, and we remark that x^s is the spherical function ϕ_s for the multiplicative group $\mathbb{R}^+$ and x^m, $m \in \mathbb{Z}$ are the spherical functions of $U(1)$, then the formula takes the form

$$(R) \qquad \tilde{f}(s) = -\frac{\pi}{\sin(\pi s)} a(s), \qquad \hat{f}(m) = (-1)^m a(m).$$

Here RAMANUJAN implicitely assumes that the series, not converging on the whole of $[0, \infty[$ in general, admits an analytic continuation, still denoted by the same series symbol. For example, if $a(s) = 1$, then the series $f(x)$ converges on the unit disk and has an analytic continuation, given by $f(x) = \frac{1}{1+x}$, and $\int_0^\infty \frac{x^{-s-1}}{1+x} dx = -\Gamma(s)\Gamma(1-s) = -\frac{\pi}{\sin(\pi s)}$, as stated by (R). Let us give one precise formulation of the "master theorem" (there are several formulations possible; see [B85] for a discussion): *Assume that $a(s)$ is holomorphic on the half-plane $\{\Re s > -1\}$ and decreasing in the sense that there is an integer $N > 1$ and a constant $M < \infty$ such that $|a(s)| < M(1+|s|)^{-N}$, then $f(x) = \sum_{m=0}^\infty a(m)(-x)^m$ converges on the unit disc and admits a holomorphic continuation onto a neighbourhood of $\mathbb{R}^+$ such that (R) is valid for all s with $-1 < \Re s < 0$.* We can read (R) as an interpolation formula for the coefficients $a(m)$: there is a (in some sense unique) holomorphic interpolation of the coefficients $a(m)$, which is related to the non-compact Fourier transform of f by the formula (R). Under certain other assumptions on f, we can also read this formula in the other direction: given the non-compact transform $\tilde{f}(s)$ of f, we may find the coefficients of its compact transform as the values of the holomorphic function $a(s) = -\frac{1}{\pi}\sin(\pi s)\tilde{f}(s)$ at $s = m \in \mathbb{N}$, multiplied by $(-1)^m$. We see that we need these operations: "analytic continuation", "multiplying by a particular function" and "evaluating at integer points" in order to describe RAMANUJAN's transform, and so will we in other cases than the abelian one.

The expression for the transform (R) in the general case of a Riemannian symmetric space is not known. We have obtained results in two special cases: firstly, in the case of symmetric cones ([Be93], see also

[DGR93]), and secondly, in the case of rank-one symmetric spaces, announced in [Be94a] and which will be the main result of the present work. This result, stated in a precise form below (theorems 1.5 and 2.5.1), can be written "à la Ramanujan" as

(R_1)

$$\int_{X_G} \left(\sum_{m=0}^{\infty} (-1)^m d_m a(m+\rho)\phi_{m+\rho}(x)\right)\phi_s(x)\, dx = \mathbf{b}(s)a(s) + \mathbf{b}(-s)a(-s),$$

where $\mathbf{b}(s)$ is a meromorphic function depending only on the structure of the space and for which we give an explicit formula, namely

$$\mathbf{b}(s) = \frac{1}{\sin \pi(s + \frac{m_{\beta/2}}{4})},$$

where β is the *bigger* (or *unmultipliable*) root in the positive system Σ^+, and the multiplicity $m_{\beta/2}$ of its half may be zero (case of the spheres). In all our formulas concerning the rank-one case we identify $\mathfrak{a}^*$ and $\mathbb{R}$ by identifying β and 1. Note that the right-hand side of (R_1) has two terms instead of a single one as was the case in Ramanujan's formula; this is due to the non-commutativity of the groups entering in the harmonic analysis in this case and can in fact be interpreted as a sum over the Weyl group. We will give two versions of (R_1): the first one (chapter 1) is more conceptual and many of its ingredients can be generalized to the general rank case; the second one (chapter 2) describes an interesting family of functions for which (R_1) holds. It could be seen as an application of the first version if the assumptions needed for the first approach held for this family of functions; but this is not the case, and thus the second version is completely independent of the first. However, the result is formally the same, given by (R_1), and it seems to be difficult, just as in the classical case, to describe the set of "all" functions for which (R_1) holds.

0.4 First version using the residue theorem. — Hardy proved Ramanujan's Master Theorem by using the residue theorem (see [Ha20], [Ha37], [DGR93]). Adapted to the rank-one case, this approach can be presented as follows: the idea is to relate the Fourier-inversion formulas

$$f(x) = \sum_{m=0}^{\infty} (-1)^m d_m a(m+\rho)\phi_{m+\rho}(x)$$

with $(-1)^m a(m+\rho) = \hat{f}(m+\rho)$ (compact) and

$$f(x) = \int_{-i\infty}^{i\infty} \tilde{f}(s)\phi_s(x) \frac{ds}{\mathbf{c}(s)\mathbf{c}(-s)}$$

(non-compact) by the residue theorem. Suppose that there is a function $\mathbf{b}(s)$ such that $\tilde{f}(s) = a(s)\mathbf{b}(s) + a(-s)\mathbf{b}(-s)$ holds for all "good" a (which define f by the series expression). Substituting this into the integral and using that $\phi_s = \phi_{-s}$ we get

$$\sum_{m=0}^{\infty} (-1)^m d_m a(m+\rho)\phi_{\rho+m}(x) = 2\int_{-i\infty}^{i\infty} a(s)\phi_s(x)\frac{\mathbf{b}(s)}{\mathbf{c}(s)\mathbf{c}(-s)}ds.$$

One should expect this to be a consequence of the residue theorem, so $\mathbf{b}(s)$ should satisfy the requirement

$$\mathrm{res}_{s=m+\rho}\frac{\mathbf{b}(s)}{\mathbf{c}(s)\mathbf{c}(-s)} = (-1)^m d_m.$$

Now, going the inverse way and defining $\mathbf{b}(s)$ by the formula given in the previous section, one shows that this is indeed the case, and under suitable assumptions on $a(s)$ (theorem 1.5) the residue theorem indeed implies that the integral and the series representation define the *same* analytic function f, the Fourier transforms of which then satisfy (R_1). In the course of the proof we get a very useful interpolation formula for the dimensions d_m (section 1.3) and find a simple relation of this interpolation with the Plancherel measure of G/K (section 1.4) which holds also in the general rank case.

0.5 Second version using the generalized Mehler **and** Heine **formulae**. — Our original strategy to find relations of the form (R) was different from the above: we were looking for a sufficiently large set of functions $f_j, j \in J$, where J is some index set, of which we are able to calculate both the compact transform $\hat{f}_j$ and the non-compact transform $\tilde{f}_j$. Then we find a relation (R) satisfied by these transforms, which will then be satisfied also for all "superpositions" $f = \int_J f_j dj$, dj some measure on J. In all cases we treated we had $J = X_{\mathbb{C}}$, that is, we were looking at *kernel functions* $k(x,y)$.

Let us describe the kernel function for the rank-one case: we normalize the Riemannian metric on the compact space $X_U = U/K$ such that the diameter (the maximal distance $d(x,y)$ between points $x, y \in X_U$) is π. Then (if we suppose that K is connected) the function $z(x) := \cos(d(x,x_0))$ (x_0 being the base point in X_U) is K-invariant on X_U, admits a holomorphic and $K_{\mathbb{C}}$-invariant continuation onto $X_{\mathbb{C}}$, and every holomorphic and $K_{\mathbb{C}}$-invariant function on $X_{\mathbb{C}}$ factors through z. The kernel function is then defined by

$$k(u,w) := (z(u) + z(w))^{-2\rho},$$

using the identification of $\mathbb{R}$ and $\mathfrak{a}^*$ explained in section 0.3. The calculation of the compact and non-compact transforms of $k_u(x) := k(u,x)$ is of some interest in its own right. In the special case where X_U is the 2-sphere S^2 the results are classical; the compact transform is given by NEUMANN's formula ([Er53] 3.6 (29))

$$(N) \qquad \int_{-1}^{1} P_m(x)(x+y)^{-1}dx = (-1)^m Q_m(y), \quad m \in \mathbb{N}$$

(which by Fourier inversion is equivalent to HEINE's formula [Er53] 3.10 (10)), and the non-compact transform is given by MEHLER's formula ([Er53] 3.14 (6))

$$(M) \qquad \int_{1}^{\infty} P_{s-1/2}(x)(x+y)^{-1}dx = \frac{\pi}{\cos(\pi s)} P_{s-1/2}(y), \quad |\Re s| < 1/2,$$

where P_s and Q_s are *Legendre functions of the first*, resp. *second kind.* Using these classical formulae, together with the classical formula [Er53] 3.3.1 (3)

$$(HC') \qquad \pi \tan(\pi s) P_{s-1/2}(y) = (Q_{-s-1/2}(y) - Q_{s-1/2}(y)),$$

E.Stein and M.Wainger have treated the case of the sphere S^2 in [SW63], which has been of interest long before in the context of the so-called *Watson-Sommerfeld transform* in quantum mechanics. Our results are a generalization of theirs. In fact, the rank-one case could be treated as a problem in the analysis of JACOBI functions, but we tried to replace this by group-theoretic arguments whenever possible, so giving some new proofs to classical results, in order to prepare the understanding of the general case of RAMANUJAN's transform. We show first that the kernel k is a solution of the *generalized Darboux equation* $L_x u(x,y) = L_y u(x,y)$ (L being the Laplacian, acting with respect to first, resp. second variable) and deduce then that the *integral transformations associated to* k commute with the action of the Laplacian. This implies in a rather straightforward way the fact that the integrals in (M) and (N) "reproduce" the Legendre functions (which correspond to the spherical functions). A more detailed analysis is needed to calculate the proportionality factor as a function of s resp. m appearing at the right hand side of (M) and (N). Finally, we deduce (R_1) by putting the results together with *Harish-Chandra's expansion of the spherical functions* (see [He84] p.430) which in the rank-one case reads

$$(HC) \qquad \phi_s = \mathbf{c}(s) f_s + \mathbf{c}(-s) f_{-s}, \qquad -2s \notin \mathbb{N},$$

where f_s is an eigenfunction of the Laplacian for the same eigenvalue as ϕ_s, behaving as $e^{(\rho-s)r}$ as r goes to infinity. The formula (HC') is just a special case of (HC) expressed in the language of Legendre functions.

0.6 Some problems and remarks on the general case. — 1. Recall that the right-hand side of (R_1) can be interpreted as a sum over the Weyl group. This leads to the conjecture that in the general case (R) takes the form

$$\tilde{f}(s) = \sum_{w \in W} \mathbf{b}(w \cdot s) a(w \cdot s), \qquad \forall m \in \Lambda + \rho : \; a(m) = \epsilon(m) \hat{f}(m),$$

where $\epsilon(m)$ is some ± 1-valued function on $\rho + \Lambda$ and $\mathbf{b}$ a function depending only on the structure of the space. Then an interesting question would be how to interprete this function, and in particular to examine its relation with Harish-chandra's c-function. The arguments developped in section 0.4 apply to this general situation and show that $\mathbf{b}$ should satisfy some requirements on the residues at the points of $\Lambda + \rho$.

2. When calculating the compact and non-compact transforms of the kernel function in the rank-one case by comparing asymptotic behaviours at infinity, we also get information about the asymptotic behaviour of the spherical functions at their singularities situated at the *antipodal set of the base point* x_0 *in* X_U, and similarly for the singularity of f_s situated at x_0. The function $\mathbf{b}$ we determine in the rank-one case is (though rather implicitely) related to these asymptotic behaviours. Therefore it would be interesting to have information about the behaviour of the spherical functions at the antipodal set of x_0 in X_U for the general case, and to investigate how this is related to Ramanujan's transform.

3. The original strategy we used to obtain relations of the form (R) depended on the knowledge of functions of which we can calculate the compact and non-compact spherical Fourier transforms. Are there good candidates for such functions in the general rank case? It would be helpful to have a geometric interpretation of the kernel-function we use in the rank-one case in order to investigate a possible generalization.

4. As holomorphic continuation is compatible with the algebra structure of function spaces given by ordinary function multiplication, the top line of the diagram on section 0.3 is a homomorphism of usual function algebras. Thus Ramanujan's transform (the bottom line) will be a homomorphism of the algebra structures obtained from the function-algebras by forward transport by the Fourier transform. In an abstract language this can be expressed by saying that (R) should be a homomorphism of the *dual hypergroups associated to* X_U *and* X_G. See [He94] for some results on the dual hypergroups (p. 333 f. for X_U, p.365 f. for X_G).

5. In a similar way, we remark that (R) commutes with multiplication by W-invariant polynomials. This is due to the fact that the algebras of invariant differential operators $\mathbb{D}(X_G)$ and $\mathbb{D}(X_U)$ can both be identyfied with the algebra of holomorphic invariant differential operators on $X_{\mathbb{C}}$, and on both sides of the diagram in section 0.3 this action comes down by spherical Fourier transform to multiplication by the same W-invariant polynomial.

6. Finally, one can interprete Ramanujan's master theorem in various ways and generalize or extend our attempt in different directions. F.ex. one may consider vector-valued functions or functions transforming by certain K-types, or one may interprete the formula as a relation between the "compact" and "non-compact" spectral resolutions of a certain differential operator (not necessarily related to group theory). A particularly interesting viewpoint might be to investigate the relation to the harmonic analysis on a *non-riemannian* real form of the complex symmetric space $X_{\mathbb{C}}$. For the case where X_U is a sphere and X_G a real hyperbolic space one can consider the non-riemannian real form given by a one-sheeted hyperboloid, as was done by Bros and Viano [BV].

This work grew out of the author's thesis [Be94b]. I would like to thank Jacques Faraut for suggesting me the problem and encouraging my work on it all the way long, and the Mittag-Leffler Institute for hospitality when I worked out this version, as well as M. Flensted-Jensen and S. Helgason for helpful discussions on this subject during the program "Harmonic Analysis on Lie Groups".

1. The rank-one case: approach by the residue theorem

1.1 Geometric preliminaries. — Let $X_U = U/K$ be a compact rank-one symmetric space. Then U will be simple, except for the one-dimensional spaces $X_U = S^1$ and $X_U = \mathbb{PR}^2$. The classification of the rank-one spaces contains three series and one exceptional space. They are characterized by the number of connected components of K and by the root system Σ of their non-compact dual $X_G = G/K$ which contains one or two positive roots, depending on the case. General notations being as in the introduction, we will write β for the *bigger* root in Σ^+, m_β for its multiplicity and $m_{\beta/2}$ for the multiplicity of $\beta/2$ which may be zero. We give the classification and the characterizing data following [He84] p.167. In all cases except the second K is connected; in the remainig case K has two connected components.

1. The spheres: $U/K = S^n = SO(n+1)/SO(n)$, $m_{\beta/2} = 0$, $m_\beta = n-1$.
2. The real projective space: $U/K = S^n/\pm 1 = O(n+1)/O(n)$,

$m_{\beta/2} = 0$, $m_\beta = n-1$.

3. The complex projective space: $U/K = SU(n+1)/S(U(n)\times U(1))$, $m_\beta = 1$, $m_{\beta/2} = 2n-2$.

4. The quaternionic projective space: $U/K = Sp(n+1)/S(Sp(n)\times Sp(1))$, $m_\beta = 3$, $m_{\beta/2} = 2n-2$.

5. The Cayley projective plane: $U/K = F_4/Spin(9)$, $m_\beta = 7$, $m_{\beta/2} = 8$.

In the sequel we will assume that K is connected; i.e. we exclude case 2. (In fact, the final result for case 2 may be obtained by applying the result of case 1 to functions which are invariant by the group $\{\pm 1\}$.) We then normalize the Riemannian metric on X_U such that the diameter of X_U is π (see section 0.4). This metric is a multiple of the one induced by the Killing form; from [He84]p.169 we deduce the following formulas which characterize the metric: let $H \in \mathfrak{a}$ be the element such that $\beta(H) = 1$. Then the scalar product $(\cdot|\cdot)$ on the tangent space $\mathfrak{q} = i\mathfrak{p}$ of the origin is such that $(iH|iH) = 1$, which in turn implies that $d(\exp(irH), x_0) = r$. This metric identifies $\mathfrak{a}$ and $\mathfrak{a}^*$ with $\mathbb{R}$ (and their complexifations with $\mathbb{C}$) by identifying H and β with 1. From the conditions stated in Lemma 0.1.2 it is clear then that the set of *dominant spherical weights* of U is generated by β, so we will identify it with $\mathbb{N}$. If we introduce the parameters

$$a := \frac{m_\beta + m_{\beta/2} - 1}{2}, \quad b := \frac{m_\beta - 1}{2}, \tag{1.1}$$

then $\rho = \frac{a+b+1}{2} \in \frac{\mathbb{N}}{4}$. More precisely, we have in

case 1 and 2: $a = b = \frac{n-2}{2}$, $\rho = \frac{n-1}{2}$;

case 3: $a = n-1$, $b = 0$, $\rho = \frac{n}{2}$;

case 4: $a = n$, $b = 1$, $\rho = \frac{n+2}{2}$;

case 5: $a = 7$, $b = 3$, $\rho = \frac{11}{2}$.

We remark that in all cases ρ turns out to be an integer or half-integer.

1.2 Relation of spherical functions to Jacobi-functions. — As said in the introduction, we define the function z on X_U by

$$z(x) := \cos(d(x, x_0)), \tag{1.2}$$

$d(x, x_0)$ being the riemannian distance of a point x from the base point x_0 in X_U. It is clear that this function is invariant by the isotropy subgroup K of x_0. Furthermore, every K-invariant function f on X_U factors through z: there is a function F defined on the interval $[-1, 1]$ such that $f = F \circ z$. This is a consequence of the fact that f, in the rank-one case, is a function of the distance $d(x, x_0)$, and that $\cos : [-\pi, \pi] \to [-1, 1]$ is bijective. We shall always use lower-case letters for K-invariant functions defined on X_U and and upper-case letters for the corresponding function on $[-1, 1]$.

For example, $\phi_m = \Phi_m \circ z$, ϕ_m being a spherical function on X_U. The following proposition is classical (see [La78]).

1.2.1 Proposition. — *The function z admits a holomorphic continuation onto $X_{\mathbb{C}}$, and every spherical function of X_U is a polynomial in z. More precisely, there are constants r, s, $r+s=1$, such that $rz+s$ is a spherical function of X_U, and $\Phi_{m+\rho} = P_m^{(a,b)}$, where $P_m^{(a,b)}$ is a normalized Jacobi polynomial (see definition below) with a and b given by* (1.1).

Proof. — If we write L for the Laplace-Beltrami operator of X_U, let L_0 be the second degree ordinary differential operator defined by $Lf = L_0 F \circ z$ for a K-invariant function f. Using the expression of L in polar coordinates ([He84] p.169) and a change of variables we get

$$L_0 F(z) = (1-z^2)\frac{d^2F}{dz^2} - ((a-b)+(a+b+2)z)\frac{dF}{dz}. \tag{1.3}$$

This is just the *Jacobi operator* ([Er53] 10.8.(14)). Knowing that the spherical function $\phi_{m+\rho}$ is caracterized by the equation $L\phi_{m+\rho} = (m^2 - \rho^2)\phi_{m+\rho}$, we conclude that $\Phi_{m+\rho}$ is a solution of the *hypergeometric equation* $L_0F = (s^2-\rho^2)F$. The solution which is regular at the point 1 may be expressed using the *hypergeometric function* $F(a,b;c;z) = \sum_m \frac{(a)_m(b)_m}{(c)_m m!} z^m$, where $(a)_m := \frac{\Gamma(a+m)}{\Gamma(a)}$, see [Er53] 2.9 and 10.8.(16):

$$\Phi_s(z) = F(\rho+s, \rho-s; a+1; \frac{1-z}{2}). \tag{1.4}$$

It is known that this function is regular on the interval $[-1,1]$ if and only if it is polynomial, and this is the case if and only if $s = \rho+m$ with $m \in \mathbb{N}$. The function

$$P_m^{(a,b)}(z) := \Phi_{s+\rho}(z) = F(m+a+b+1, -m; a+1; \frac{1-z}{2})$$

is a Jacobi polynomial of degree m, normalized by $P_m^{(a,b)}(1) = 1$. We have shown that $\phi_{m+\rho} = P_m^{(a,b)} \circ z$. In particular, as $P_1^{(a,b)}$ is a polynomial of degree one, $P_1(x) = rx+s$, i.e. $rz+s = \phi_{1+\rho}$ is spherical. We know that every spherical function of X_U admits a holomorphic continuation onto $X_{\mathbb{C}}$ (lemma 0.2.1); so this is also the case for z. $\square$

1.2.2 Remark. Let us indicate a more geometrical proof of the proposition. The above proof shows that z is a linear combination of the trivial spherical function and the first non-trivial spherical function of X_U, in particular it is a K-invariant matrix coefficient of some finite-dimensional spherical representation of U. Let us describe this representation. In the case of the spheres, this representation of $SO(n+1)$ is the natural one and z is in fact the associated spherical function (the JACOBI-polynomials are called GEGENBAUER-polynomials in this case, and we have $P_1^{(a,a)}(z) = z$). For the general case, we recall a result of U.Hirzebruch (see [FK94], exercise IV.5 and V.5 and references there) stating that every compact rank-one symmetric space can be realized as an orbit of a linear group action in a *euclidean Jordan-algebra* V, namely as the set $\mathcal{J}(V)$ of *primitive idempotents* of this algebra, the compact group U acting transitively on this space being the *automorphism group* of the algebra. We choose a base point $c \in \mathcal{J}(V)$ and let K be its stabilizer in U. Using results of Hirzebruch (see [FK94] IV, exercise 4) one shows that for $x \in \mathcal{J}(V)$,

$$z(x) = \langle x|2c - e\rangle,$$

where e is the unit element of the algebra V. In other terms, writing z as a K-biinvariant matrix coefficient of U,

$$z(u) = \langle u \cdot c, 2c - e\rangle = \frac{1}{2}\langle u \cdot (2c - e)|2c - e\rangle + \langle e|c\rangle - \frac{1}{2}$$

for $u \in U$ (where we have used that $U \cdot e = e$). Now one can check, using dimension arguments, that the lowest-dimensional non-trivial and irreducible K-spherical respresentation of U is realized on the orthocomplement of the unit element e of V. If $2c - e = \lambda e + f$ is the corresponding orthogonal decomposition, then $z(u) = \langle u \cdot f|f\rangle + \lambda^2 + \langle e|c\rangle - \frac{1}{2}$ is the decomposition of z as a linear combination of the trivial and the first non-trivial spherical function of U/K. □

The holomorphic continuation of z, restricted to the non-compact dual X_G of X_U, is (using holomorphy of the exponential map) given by the formula

$$z(\exp(rH)) = \operatorname{ch}(r).$$

Every K-invariant function f on X_G factors through z; as in the compact case we write $f = F \circ z$, where F is now defined on $[1, \infty[$. As a particular case, $\phi_s = \Phi_s \circ z$, where Φ_s is a Jacobi function, given by equation (1.4). We can rewrite *Harish-Chandra's spherical function expansion* (see introduction, formula (HC)) in terms of the hypergeometric function. As Φ_s is given by formula (1.3), we can rewrite [Er53] 2.10 (2) as

$$\Phi_s(z) = \mathbf{c}(s)F_s(z) + \mathbf{c}(-s)F_{-s}(z)$$

with

$$F_s(z) = (2z - 2)^{s-\rho}F(\rho - s, b + 1 - s; 1 - 2s; \frac{2}{1 - z}), \quad -2s \notin \mathbb{N},$$

(this function behaving as $(2z)^{s-\rho}$ as z (real) goes to infinity) and

$$\mathbf{c}(s) = 2^{2\rho-2s}\frac{\Gamma(a+1)\Gamma(2s)}{\Gamma(s+\rho)\Gamma(s+\frac{a-b+1}{2})} = 2^{-2\rho}\frac{\Gamma(a+1)\Gamma(2s)\Gamma(s+\frac{a-b}{2})}{\Gamma(s+\rho)\Gamma(2s+a-b)}.$$

We remind the reader that all our formulas are written with respect to the *bigger* root β (that is, if we replace s by $\frac{(s|\beta)}{(\beta|\beta)}$, we can interprete s as a parameter in $\mathfrak{a}_{\mathbb{C}}^*$); for this reason our formula for the $\mathbf{c}$-function looks slightly different from the usual ones taking the *smaller* root as reference (f.ex. [He84] p.437). (In our note [Be94a] we followed this usual choice which made our formulas unnecessarily complicated.)

1.3 Interpolation of the Plancherel measure of X_U. —

1.3.1 Lemma. — *The dimension d_m of the irreducible U-representation having highest weight $m\beta$ is given by the formula*

$$d_m = c \cdot (2m+a+b+1)\frac{\Gamma(m+a+1)\Gamma(m+a+b+1)}{\Gamma(m+b+1)\Gamma(m+1)}$$

with the constant $c = \frac{\Gamma(b+1)}{\Gamma(a+1)\Gamma(a+b+2)}$.

Proof. — This formula may be obtained as a consequence of [Er53] 10.8 (4) and (3). One may also specialise the coefficients a and b to the values listed in section 1.1 and obtains then the table given in [She90] p.90. We shall give here a purely group-theoretic proof which generalizes to the higher rank case. Recall the following relation (due to Vretare) between the Plancherel measures of X_G and X_U (see [He94] thm. 9.10):

$$d_m = \lim_{s\to\rho}\frac{\mathbf{c}(s)\mathbf{c}(-s)}{\mathbf{c}(s+m)\mathbf{c}(-s-m)}.$$

Using the (first) explicit formula for the c-function given above we obtain

$$d_m = \lim_{s\to\rho}\Big(\frac{\Gamma(2s)\Gamma(-2s)}{\Gamma(2s+2m)\Gamma(-2s-2m)}\cdot$$

$$\frac{\Gamma(s+\rho+m)\Gamma(\rho-s-m)}{\Gamma(s+\rho)\Gamma(\rho-s)}\frac{\Gamma(s+m+\frac{a-b+1}{2})\Gamma(-s-m+\frac{a-b+1}{2})}{\Gamma(s+\frac{a-b+1}{2})\Gamma(-s+\frac{a-b+1}{2})}\Big).$$

We then transform each of the three terms using the identity

$$\frac{\Gamma(z+w+m)\Gamma(w-z-m)}{\Gamma(z+w)\Gamma(w-z)} = (-1)^{m-1}\frac{(z+w)(z+w+1)...(z+w+m-1)}{(z-w+1)(z-w+2)...(z-w+m)}$$

for $m \in \mathbb{Z}_+$, $z - w \neq 1, ..., m$ (which is an easy consequence of the functional equation of the Gamma-function). Passing to the limit $s \to \rho$ gives (modulo constants which are later determined by the requirement $d_0 = 1$) for the fist term: $2\rho + 2m$, for the second term: $\frac{(m+2\rho-1)!}{m!}$, and for the third term: $\frac{(m+a)!}{(m+b)!}$ (in the case where a and b are integers (case 3,4 and 5); in case 1 this term cancels), which altogether gives the announced formula for d_m. ∎

We define the polynomial δ by

$$\delta(s) := s \cdot \frac{\Gamma(s+\rho)\Gamma(s+\frac{a-b+1}{2})}{\Gamma(s-\rho+1)\Gamma(s+\frac{b-a+1}{2})}$$

$$= s \cdot (s-\rho+1)(s-\rho+2)...(s+\rho-1)\cdot$$

$$(s - \frac{m_{\beta/2}}{4} + \frac{1}{2})(s - \frac{m_{\beta/2}}{4} + \frac{3}{2})...(s + \frac{m_{\beta/2}}{4} - \frac{1}{2})$$

(if $m_{\beta/2} = 0$, then the last term does not appear, and if $\rho = \frac{1}{2}$, then $\delta(s) = s$), then $d_m = \frac{\delta(m+\rho)}{\delta(\rho)}$, and so δ gives a polynomial interpolation of the coefficients d_m. Its zeroes are the points 0 and $-\rho+1, -\rho+2, ..., \rho-1$ in the case of the spheres, and in the other cases there are in addition also zeroes at the points $b-\rho+1, b-\rho+2, ..., \rho-b-1$ (and b is then an integer). The order of the zero at the point 0 decides whether δ is an odd or an even function: if X_U is not an odd-dimensional sphere, the zero at the point 0 is of order one or three, corresponding to ρ being a half-integer resp. integer, and δ is an odd function. In the case of the spheres S^n, the zero at the point 0 is of order one or two, correspondig to ρ being a half-integer (n even) resp. integer (n odd), and only in the case of the odd-dimensional spheres δ is an even function.

1.4 The functions $\delta(s)\mathbf{c}(s)\mathbf{c}(-s)$ and $\mathbf{b}(s)$. — Using the explicit formulae for the polynomial δ and the $\mathbf{c}$-function from the previous section, as well as the relations $\Gamma(z)\Gamma(1-z) = -\frac{\pi}{\sin(\pi z)}$ and $\Gamma(1/2+z)\Gamma(1/2-z) = \frac{\pi}{\cos(\pi z)}$, we obtain

$$\delta(s)\mathbf{c}(s)\mathbf{c}(-s) = \frac{\sin\pi(s-\rho)}{\sin(2\pi s)}\cos\pi(s - \frac{a-b}{2})$$

where $\frac{a-b}{2} = \frac{m_{\beta/2}}{4} = \rho - \frac{m_\beta}{2}$. We use the formula $\sin(2\pi s) = 2\sin(\pi s)\cos(\pi s)$ and observe that, because ρ is either integer or half-integer (see 1.1), we can replace s by $s-\rho$ in the term $\sin(2\pi s)$; we obtain (neglecting constants)

$$\delta(s)\mathbf{c}(s)\mathbf{c}(-s) = \frac{\cos\pi(s-\rho+\frac{m_\beta}{2})}{\cos\pi(s-\rho)}. \tag{1.5}$$

If m_β is even (this happens only in the case of the odd-dimensional spheres), then this expression equals ± 1, and if m_β is odd (all other cases), then it equals $\pm\tan\pi(s-\rho)$. Let us define the meromorphic function $\mathbf{b}(s)$ by

$$\mathbf{b}(s) := \frac{\delta(s)\mathbf{c}(s)\mathbf{c}(-s)}{\sin\pi(s-\rho)} \tag{1.6}$$

$$= \frac{\cos\pi(s-\rho+\frac{m_\beta}{2})}{\cos\pi(s-\rho)\sin\pi(s-\rho)} = \frac{\pm 1}{\sin\pi(s-\rho+\frac{m_\beta}{2})} = \frac{\pm 1}{\sin\pi(s-\frac{m_{\beta/2}}{4})}$$

which equals either $\frac{\pm 1}{\sin(\pi s)}$ (cases 1,5 and 3,4 with n even) or $\frac{\pm 1}{\cos(\pi s)}$ (cases 3 and 4 with n odd).

1.4.1 Proposition. — *The following relation between the Plancherel measures of the rank-one spaces X_G and X_U holds:*

$$\frac{1}{\mathbf{c}(s)\mathbf{c}(-s)} = \delta(s)(\cot\pi(s-\rho))^\epsilon$$

where $\delta(s)$ is the polynomial defined in the previous section and $\epsilon = 0$ in the case of the odd-dimensional spheres and $\epsilon = 1$ in all other cases. In all cases, $\frac{1}{\mathbf{c}(is)\mathbf{c}(-is)} \sim \pm\delta(is)$ $(s \to \infty)$. If X_U is not an odd-dimensional sphere, the function $\frac{1}{\mathbf{c}(s)\mathbf{c}(-s)}$ has simple poles at the points $\rho + m$, $m = 0, 1, ...$ with residues d_m (up to a common factor); there are no poles between ρ and $-\rho$. The function

$$\frac{\mathbf{b}(s)}{\mathbf{c}(s)\mathbf{c}(-s)} = \frac{\delta(s)}{\sin\pi(s-\rho)}$$

has (in all cases) simple poles situated at $s = \rho + m$, $m = 0, 1, ...$ with residues $(-1)^m d_m$ (up to a common factor); there are no poles between ρ and $-\rho$.

Proof. — The statements about the asymptotic behaviour, the poles and residues are immediate consequences of the given formulae and of the location of the zeroes of $\delta(s)$, see 1.3. $\square$

We remark that the above lemma is also true for $X_U = S^1$ (which is an odd-dimensional sphere). The generalization of the formulas for $\delta(s)$ and $\delta(s)\mathbf{c}(s)\mathbf{c}(-s)$ (but not for $\mathbf{b}(s)$!) to general-rank spaces is immediate, because the interpolation formula for the dimensions has the same product structure as the product formula of Gindikin and Karpelevic for the $\mathbf{c}$-function ([He84] p.447) - this follows just by putting the product formula for the $\mathbf{c}$-function into the formula of Vretare we used in 1.3.1 to derive our formula for $\delta(s)$. As a special case, the group case will give us a product of several factors, each of them of the type corresponding to $X_U = SU(2) = S^3$, and we get WEYL's dimension formula.

1.5 RAMANUJAN**'s master theorem for rank-one spaces (first version).** — *Let $a(s)$ be a holomorphic function on the half-plane $\{\Re s > -\rho\}$ satisfying the decrease condition $\exists A < \pi, C < \infty, \varepsilon > 0$: $|a(s)| < Ce^{A|\Im s|}e^{-\varepsilon \Re s}$. Then the series*

$$f(x) := \sum_{m=0}^{\infty} (-1)^m d_m a(m+\rho)\phi_{m+\rho}(x)$$

converges on a neighbourhood of X_U, defining there a holomorphic and K-invariant function, and admits a holomorphic continuation onto a neighbourhood of X_G, given by the formula

$$f(x) = \int_{-i\infty}^{i\infty} \phi_s(x)(\mathbf{b}(s)a(s) + \mathbf{b}(-s)a(-s)) \frac{ds}{|\mathbf{c}(s)|^2}.$$

The non-compact spherical Fourier transform $\tilde{f}$ exists for $|\Re s| < \rho$ and there satisfies the relation

$$(R_1) \qquad \tilde{f}(s) = \mathbf{b}(s)a(s) + \mathbf{b}(-s)a(-s)$$

with $\mathbf{b}(s) = \frac{1}{\sin \pi(s - \frac{m_{\beta/2}}{4})}$ (see formula (1.6)).

Proof. — It was shown by Lassalle ([La78]; there the case of arbitrary rank is treated) that the decrease condition $|a(m+\rho)| < Ce^{-\varepsilon(m+\rho)}$ assures the normal convergence of the series $f(x)$ on a neighbourhood (depending on ε and which can be described explicitely) of X_U.

Let us show that the integral given in the theorem defines indeed an analytic continuation of f. We first use the fact that $|\mathbf{c}(s)|^2 = \mathbf{c}(s)\mathbf{c}(-s)$ for imaginary s and that $\phi_s = \phi_{-s}$ to write the given integral as

$$I(x) := \int_{-i\infty}^{i\infty} \phi_s(x)a(s) \frac{\mathbf{b}(s)}{\mathbf{c}(s)\mathbf{c}(-s)}\, ds$$

$$(1.8) \qquad = \int_{-i\infty}^{i\infty} \phi_s(x)a(s) \frac{\delta(s)}{\sin \pi(s-\rho)}\, ds$$

(by (1.6)). As remarked in proposition 1.4.1, the poles of the integrand on the half-plane $\Re s > -\rho$ are situated at the points $\rho + m$, $m = 0, 1, \ldots$.

In particular, there are no poles on the imaginary axis. Furthermore, by the assumption on a, $|a(s)\frac{\delta(s)}{\sin\pi(s-\rho)}| < C\delta(s)e^{(A-\pi)|s|}$ with $A-\pi<0$, and thus by proposition A.1 (appendix) $I(x)$ defines a holomorphic function on a neighbourhood of X_G.

We now show, using the residue theorem, that $f(x) = I(x)$ for $x \in K\exp([0,\varepsilon[H)\cdot x_0$. We define the integration path γ_r joining the vertices $(-ir, ir)$, $(ir, ir+r)$, $(ir+r, -ir+r)$, $(-ir+r, -ir)$, and recall that the poles of the integrand are situated at the points $\rho+m$, $m = 0, 1, ...$ with residues $(-1)^m a(m+\rho)\delta(m+\rho)\phi_{\rho+m}(x)$ (see prop. 1.4.1). Thus we obtain by the residue theorem

$$\int_{\gamma_r} a(s)\phi_s(x)\frac{\delta(s)}{\sin\pi(s-\rho)}ds = \sum_{m=0}^{[r-m]}(-1)^m d_m a(m+\rho)\phi_{m+\rho}(x).$$

To get the desired expression for the limit $r = \rho + n + \frac{1}{2} \to \infty$ ($n \in \mathbb{N}$), we need the following estimation on $\phi_s(x)$ as a function of the parameter s (see [Ko84] p.53 eqn. (6.3)): there is $M < \infty$ such that for all $t > 0$ and $s \in \mathbb{C}$,

$$|\phi_s(\exp(tH)\cdot x_0)| < M(1+t)e^{t(|\Re s|-\rho)}.$$

We now fix $t \in [0, \varepsilon[$ and consider the above integral for $x := \exp(tH)\cdot x_0$. Using the quoted estimation on $\phi_s(x)$ and the decrease assumption on $a(s)$, we get $|\phi_s(x)a(s)\frac{\delta(s)}{\sin\pi(s-\rho)}| < C'\delta(s)e^{(t-\varepsilon)\Re s}e^{(A-\pi)|\Im s|}$ where $t-\varepsilon<0$ and $A-\pi<0$. This and the fact that $\delta(s)$ is polynomial imply that in the limit $r = \rho+n+\frac{1}{2} \to \infty$ only the integral over the imaginary axis contributes, and so

$$I(x) = \int_{-i\infty}^{i\infty} a(s)\phi_s(x)\frac{\delta(s)}{\sin\pi(s-\rho)}\,ds$$

$$= \sum_{m=0}^{\infty}(-1)^m d_m a(m+\rho)\phi_{m+\rho}(x) = f(x)$$

as was to be shown.

Let us show that the function $h(s) := \mathbf{b}(s)a(s) + \mathbf{b}(-s)a(-s)$ is square integrable over the imaginary axis w.r.t. the measure $\frac{ds}{|\mathbf{c}(s)|^2}$. In fact, using the definition (1.6) of $\mathbf{b}(s)$ we easily get for pure imaginary s, $\frac{|h(s)|^2}{|\mathbf{c}(s)|^2} \leq \frac{|\delta(s)|^2\mathbf{c}(s)\mathbf{c}(-s)}{|\sin\pi(s-\rho)|^2}4Ce^{A|s|} \leq 4C\delta(s)e^{(2A-2\pi)|s|}$, which has a finite integral over the imaginary axis. Now the inversion theorem of the spherical Fourier transform (see [He84]) assures that $h(s)$ is the spherical Fourier transform of f in the L^2-sense, that is

$$\tilde{f}(s) = \mathbf{b}(s)a(s) + \mathbf{b}(-s)a(-s)$$

for almost all $s \in i\mathbb{R}$. But as f is bounded on X_G (which is an easy consequence of Prop. A 1), this is an identity between holomorphic functions on $\{|\Re s| < \rho\}$. $\square$

1.6 Remark. — Recall from proposition 1.4.1 that, if X_U is *not* an odd-dimensional sphere, then $\mathrm{res}_{s=\rho+m} \frac{1}{\mathbf{c}(s)\mathbf{c}(-s)} = d_m$. One may wonder whether then the series $\sum_m d_m a(m+\rho)\phi_{m+\rho}(x)$ can be linked to the integral $\int_{i\mathbb{R}}(a(s)+a(-s))\phi_s(x)\frac{ds}{\mathbf{c}(s)\mathbf{c}(-s)}$ by the residue theorem. This would give a "simple" version of the previous theorem, where $\mathbf{b}(s)$ is replaced by 1 and the factors $(-1)^m$ do not appear. However, the following arguments strongly suggest that no such version exists: let us try to find a decrease condition on $a(s)$ required in order to adapt the previous proof to this situation. We claim that the condition on $a(s)$ stated in the theorem is essentially necessary in order to conclude as in the proof. In fact, for the convergence of the series $f(x)$ on a neighbourhood of X_U this is shown in [La78] and for the convergence of the integral on a neighbourhood of X_G this can be shown in similar way. The estimation on $\phi_s(x)$ we used in the proof is also quite sharp, making it very unlikely to find a less restrictive assumption on $a(s)$. One should remark that the growth of $a(s)$ we admit in imaginary direction is thus only due to the strong decay of $\mathbf{b}(s)$ in this direction. But, $\mathbf{b}(s)$ replaced by 1, we would have to make on $a(s)$ the much stronger requirement $|a(s)| < Ce^{-\varepsilon|s|}$ (for some $\varepsilon > 0$) on a half-plane. Then $|a(s)\sin(\pi s)| < 2Ce^{(\pi-\varepsilon)|s|}$, so Carlson's uniqueness theorem (see [Ti39], [Ha20]) applied to the function $a(s)\sin(\pi s)$ yields $a(s) = 0$. Thus it seems that there are no non-trivial examples for the "simple" version of Ramanujan's transform. We may conclude that a non-trivial function $\mathbf{b}(s)$ should appear in the formula for (at least) two reasons: firstly, the function $\mathbf{b}(s)$ has to produce residues also in the cases where $\frac{1}{\mathbf{c}(s)\mathbf{c}(-s)}$ has no poles (the odd-dimensional spheres), and secondly, the strong decay of the $\mathbf{b}$-function in the imaginary direction allows the function $a(s)$ to be non-trivial.

1.7 An example. — Let $X_U = S^n = SO(n+1)/SO(n)$. We define a K-invariant and holomorphic function on $X_{\mathbb{C}}$ by $f(x) := e^{-rz(x)}$ with $r > 0$. The compact and non-compact spherical Fourier transforms of f are known: by [FH87] p.30 we have (up to a constant not depending on r) the following expansion of a plaine wave in spherical harmonics:

$$e^{-r\cos\theta} = \sum_{m=0}^{\infty} d_m(-1)^m r^{-\rho} I_{\rho+m}(r)\Phi_{m+\rho}(\cos\theta),$$

where $I_\nu(r) = r^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)}(\frac{r}{2})^{2k}$ is the modified Bessel function.

In other terms

$$\hat{f}(m+\rho) = r^{-\rho}(-1)^m I_{\rho+m}(r), \quad m \in \mathbb{N}_0.$$

So $a(s) := r^{-\rho} I_s(r)$ is an analytyic interpolation of the coefficients $(-1)^m \hat{f}(\rho+m)$. By [FH87] p.45 we have for all complex numbers s

$$\tilde{f}(s) = r^{-\rho} \frac{I_{-s}(r) - I_s(r)}{\sin(\pi s)}.$$

So the relation (R_1) (case of the spheres) $\tilde{f}(s) = \frac{1}{\sin(\pi s)}(a(s) - a(-s))$ is satisfied for all $s \in \mathbb{C}$. We do not know whether $a(s)$ satisfies the assumptions of the previous theorem.

2. The rank one case by the generalized Mehler and Heine formulae

2.1 The kernel function. — We continue to use all notations and conventions introduced in the previous chapter for the rank one case. We define the *kernel function* k by the formula

$$k(u,w) := (z(u) + z(w))^{-2\rho},$$

where 2ρ is the integer $a+b+1$ (see 1.1) and z the holomorphic function introduced in 1.2. The kernel k is meromorphic on $X_{\mathbb{C}} \times X_{\mathbb{C}}$ and well-defined and analytic on the whole of $X_G \times X_G$.

2.1.1 Proposition. — *The kernel k is a solution of the* generalized Darboux equation

$$L_u k(u,w) = L_w k(u,w),$$

where L_u, resp. L_w, denotes the Laplacian acting with respect to the first, resp. second, variable. (The Laplacians of X_U and of X_G act in a natural way on holomorphic functions defined on domains of $X_{\mathbb{C}}$, and these actions agree; so we need not specify which of these we mean.)

Proof. — Let us calculate $L_0 K_y$, where $K_y(x) := K(x,y) := (x+y)^{-(a+b+1)}$, and L_0 is the operator given by (1.2) acting w.r. to the variable x:

$$L_0 K_y(x) = 2\rho\,(x+y)^{-\frac{a+b+1}{2}-2}\left(\frac{a+b+3}{2}xy + (a-b)(x+y)\right).$$

This function is symmetric in x and y. $\square$

The *generalized Darboux equation* is studied in [He84] p. 288.

2.2 The integral transforms associated to the kernel function. The kernel k being as introduced above, let us define the *compact and non-compact integral transforms associated to* k by

$$(\mathcal{R}_U f)(x) := \int_{X_U} f(y)k(x,y)dy$$

and

$$(\mathcal{R}_G f)(x) := \int_{X_G} f(y)k(x,y)dy.$$

Taking for f a spherical function will give us later the spherical Fourier transforms of the functions $k_y(x) := k(x,y)$.

2.2.1 Proposition. — *(i) If f is a K-invariant differentiable function on X_G which is integrable or satisfies the following decrease condition:*

$$(*) \qquad \exists \varepsilon > 0, M < \infty : \quad |f(\exp(rH)x_0)| \leq Me^{-\varepsilon r},$$

then $\mathcal{R}_G f$ is a differentiable and K-invariant function on X_G and a holomorphic function on the domain of $X_{\mathbb{C}}$ defined by $z(x) \notin]-\infty,-1]$. If f is in addition square-integrable, then

$$L(\mathcal{R}_G f) = \mathcal{R}_G(Lf).$$

(ii) If f is a K-invariant and differentiable function on X_U, then $\mathcal{R}_U f$ is a differentiable and K-invariant function on $X_G - \{x_0\}$ which tends to zero as the distance r to the base point tends to infinity, and it is a holomorphic function on the domain of $X_{\mathbb{C}}$ defined by $z(x) \notin]-\infty, 1]$. The transform $\mathcal{R}_U$ commutes with the action of the Laplacian:

$$L(\mathcal{R}_U f) = \mathcal{R}_U(Lf).$$

Proof. — The K-invariance of the transforms is clear from the invariance of k. (We even could have dropped the invariance assumption on f; but then $\mathcal{R}(f \circ k) = \mathcal{R}f$ for all $k \in K$, so the assumption means no restriction.) Using the integral formulas for K-invariant functions f in polar coordinates ([He84] p.186 and 190) we get, after a change of variable,

$$\int_{X_G} f(g)dg = \int_1^\infty F(x)(x-1)^a(x+1)^b dx,$$

$$\int_{X_U} f(u)du = \int_{-1}^1 F(x)(1-x)^a(1+x)^b dx.$$

Substituting in these expressions $F(x)K(x,y) = F(x)(x+y)^{-(a+b+1)}$ in the place of $F(x)$ we easily see that the transforms $\mathcal{R}_U f$ and $\mathcal{R}_G f$ are defined and holomorphic on the domains stated, as well as the decrease stated in (ii). (We remark, however, that the base point will be a singular point for $\mathcal{R}_U f$ in general.) If f is square-integrable, using Prop. 2.1.1, the selfadjointness of the Laplacian and that k_x is square-integrable, we get

$$\begin{aligned}(L(\mathcal{R}_G f))(x) &= \int_{X_G} f(y) L_x k(x,y) dy \\ &= \int_{X_G} f(y) L_y k(x,y) dy \\ &= \int_{X_G} L_y f(y) k(x,y) dy \\ &= (\mathcal{R}_G(Lf))(x),\end{aligned}$$

The same argument holds for $\mathcal{R}_U$. ∎

For later use we state the asymptotic behaviour of the transforms $\mathcal{R}_G f$, resp. $\mathcal{R}_U f$ at their singularities possibly situated at the antipodal set of the origin in X_U, resp. at the origin. The following asymptotic formulas hold all up to a constant; so we suppose that Haar measures are suitably normalized.

2.2.2 Proposition. — *(i) If f satisfies the assumptions of 2.1.1 (i) and if $b > 0$, then*

$$(\mathcal{R}_G f)(x) \sim f(x_0)(z(x)+1)^{-b} \quad (x \in X_U, z(x) \to -1).$$

If $b = 0$ (i.e. if $X_U = S^2$ or $X_U = \mathbb{P}^n(\mathbb{C})$), then

$$(\mathcal{R}_G f)(x) \sim f(x_0)(\log(z(x)+1) \quad (x \in X_U, z(x) \to -1).$$

(ii) Let $A(x_0)$ be the antipodal orbit of x_0 in X_U. Then for every continous and K-invariant function f defined on X_U,

$$(\mathcal{R}_U f)(x) \sim f(A(x_0))(z(x)-1)^{-a} \quad (x \in X_G, x \to x_0),$$

if $a \neq 0$. If $a = 0$ (i.e. $X_U = S^2$),

$$(\mathcal{R}_U f)(x) \sim f(A(x_0)) \log(z(x)-1) \quad (y \in X_G, y \to x_0).$$

Proof. — (i) Let $b > 0$. We show that

$$(x+1)^b(\mathcal{R}_G F)(x) = \int_1^\infty F(y)(x+y)^{-(a+b+1)}(x+1)^b(y-1)^a(y+1)^b \, dy$$

tends (up to a constant) to $F(1)$ as x tends to -1. We choose $\delta > 0$ and write the integral as a sum of two integrals I_1 and I_2 over the intervals $[1, 1+\delta]$ and $[1+\delta, \infty[$. The second one is easily seen to tend to zero as x tends to -1. For the first one, we introduce the new variable $z := \frac{y-1}{x+1}$ and obtain

$$I_1 = \int_0^{\frac{\delta}{x+1}} F(z(x+1)+1)(z+1)^{-(a+b+1)} z^a ((x+1)z+2)^b \, dz$$

$$\to F(1)2^b \int_0^\infty (z+1)^{-(a+b+1)} z^a \, dz = F(1)2^b \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} \quad (x \to -1)$$

using Lebesgue's theorem on dominated convergence. In the case $b = 0$ we use similar arguments, but we have to use L'Hospital's rule before passing to the limit in I_1.

(ii) Let $a > 0$. We show that

$$I := (x-1)^a (\mathcal{R}_G F)(x) = \int_{-1}^1 F(y)(x-1)^a (x+y)^{-(a+b+1)} (1-y)^a (1+y)^b \, dy$$

tends (up to a constant) to $F(-1)$ as x tends to 1. We introduce the new variable $z := \frac{y+1}{x-1}$ and obtain

$$I = \int_0^{\frac{2}{x-1}} F(z(x-1)-1)(z+1)^{-(a+b+1)} (2-(x-1)z)^a z^b dz$$

$$\to F(-1)2^a \int_0^\infty (z+1)^{-(a+b+1)} z^b \, dz = F(-1)2^a \frac{\Gamma(b+1)\Gamma(a)}{\Gamma(a+b+1)} \quad (x \to 1).$$

In the case $a = 0$ we use the same argument as in *(i)*, case $b = 0$. $\square$

2.3 Non-compact spherical Fourier transform of the kernel and asymptotic behaviour of the spherical functions at the singularity. —

2.3.1 Proposition (Mehler**'s formula). —** *If $|\Re s| < \rho$, then, for all x such that $z(x) \notin]-\infty, -1]$,*

$$\int_{X_G} \phi_s(x) k(x, y) dx = \mathbf{d}(s) \phi_s(y),$$

with

$$\mathbf{d}(s) = \frac{\Gamma(\rho+s)\Gamma(\rho-s)}{\Gamma(2\rho)}.$$

Proof. — Using the asymptotic behaviour of the spherical functions at infinity,

$$\phi_s(\exp(rH)x_0) \sim \mathbf{c}(s)e^{r(s-\rho)H} \quad (r \to \infty)$$

for $\Re s > 0$ (see [He84] p.435), where $\mathbf{c}$ is Harish-Chandra's c-function, we see that $f = \phi_s$ for $|\Re s| < \rho$ satisfies the assumptions of prop.2.2.1 (i). We will show that $L(\mathcal{R}_G\phi_s) = \mathcal{R}_G(L\phi_s) = (s^2 - \rho^2)\mathcal{R}_G\phi_s$ and conclude that $\mathcal{R}_G\phi_s$, being a K-invariant eigenfunction of L for the same eigenvalue as ϕ_s, defined on the whole of X_G, must be a multiple of the spherical function ϕ_s. Because ϕ_s is not square integrable we cannot apply directly the last statement of 2.2.1 (i). We use the formula (see [Fa81] p.415)

$$\int_t^s (gLf - fLg)(r)\Delta(r)dr = [f,g](s) - [f,g](t), \tag{2.1}$$

for differentiable functions f and g on $]0,\infty[$ (here r is the distance from the origin) and $0 < t < s$, whith

$$[f,g](r) = \Delta(r)(\frac{df(r)}{dr}g(r) - f(r)\frac{dg(r)}{dr})$$

and $\Delta(r) = (\operatorname{sh}\frac{r}{2})^{m_\beta/2}(\operatorname{sh} r)^{m_\beta}$, and we remark that $[\phi_s, k_y](0) = 0$ and $[\phi_s, k_y](r) \to 0 (r \to \infty)$, which permits to conclude as in the proof of the last statement of 2.2.1 (i). As said above, this implies that $\mathcal{R}_G(\phi_s)(y) = \mathbf{d}(s)\phi_s(y)$ where $\mathbf{d}(s)$ is a complex number which we are going to determine now by comparing the asymptotic behaviours of both sides as y tends to infinity. Using the asymptotic behaviour of the spherical functions, $\Phi_s(x) \sim \mathbf{c}(s)(2x)^{s-\rho} \quad (x \to \infty)$ (see above), we can, for $0 < |\Re s| < \rho$, calculate

$$\begin{aligned}
\mathbf{d}(s) &= \lim_{y\to\infty} \frac{\mathcal{R}_g\Phi_s}{\mathbf{c}(s)(2y)^{s-\rho}} \\
&= \frac{1}{\mathbf{c}(s)} \lim_{y\to\infty} \int_1^\infty \frac{\Phi_s(x)}{(2y)^{s-\rho}}(x+y)^{-(a+b+1)}(x-1)^a(x+1)^b dx \\
&= \frac{1}{\mathbf{c}(s)} \lim_{y\to\infty} \int_{\frac{1}{y}}^\infty \frac{\Phi_s(yz)}{(2yz)^{s-\rho}} z^{s-\rho}(z+1)^{-(a+b+1)}(z-\frac{1}{y})^a(z+\frac{1}{y})^b dz,
\end{aligned}$$

where we introduced the new variable $z := \frac{x}{y}$. As the function $\frac{\Phi_s(x)}{(2x)^{s-\rho}}$ is continous on $[1,\infty[$ and tends to a constant for $x \to \infty$, it is bounded by a constant $C_1 < \infty$ (depending on s). So the integrand is dominated by

$$C_1 z^{s-\rho}(z+1)^{-(a+b+1)} z^a (z+1)^b < C_1 (z+1)^{s-\rho-1},$$

which is integrable for $|\Re s| < \rho$. As the integrand tends to

$$\mathbf{c}(s) z^{s-\rho} z^{2\rho-1} (1+z)^{-(a+b+1)} = \mathbf{c}(s)(1+z)^{-2\rho} z^{\rho+s-1}$$

as y tends to infinity, we have by Lebesgue's theorem

$$\mathbf{d}(s) = \int_0^\infty z^{s+\rho-1}(z+1)^{-2\rho} dz = \frac{\Gamma(\rho+s)\Gamma(\rho-s)}{\Gamma(2\rho)}.$$

$\square$

We have the following interpretation of the function $\mathbf{d}(s)$: it describes the asymptotic behaviour of the spherical function ϕ_s at its singularity situated at the antipodal orbit $A(x_0)$ of the base point x_0:

2.3.2 Proposition. — *The function Φ_s, $|\Re s| < \rho$, has the following asymptotic behaviour at the point -1: If $b > 0$,*

$$\Phi_s(x) \sim \frac{1}{\mathbf{d}(s)}(x+1)^{-b} \quad (x \to -1),$$

and if $b = 0$ (i.e. case 3 listed in section 1.1),

$$\Phi_s(x) \sim \frac{1}{\mathbf{d}(s)} \log(x+1) \quad (x \to -1).$$

Proof. — Applying MEHLER's formula and 2.2.2 ($f = \phi_s$ for $|\Re s| < \rho$ satisfies the assumptions of 2.1.2), we get in the case $b > 0$

$$\mathbf{d}(s)\Phi_s(x) = (\mathcal{R}_G \Phi_s)(x) \sim \Phi_s(1)(x+1)^{-b} \quad (x > -1, x \to -1),$$

For the case $b = 0$ we use the same argument. $\square$

Remark. In the case of the sphere $X_U = S^2$, $\Phi_s = P_{s-1/2}$, where P_s is a Legendre function of the first kind, and the classical MEHLER formula (M) (see introduction) appears as a special case of 2.3.1.

2.4 Compact transform of the kernel and asymptotic behaviour of the functions of the second kind. — As for the noncompact transform, prop. 2.1.1 implies that $\mathcal{R}_U \phi_s$ for $s \in \rho + \Lambda$ is an eigenfunction of the Laplacian for the same eigenvalue $s^2 - \rho^2$ as ϕ_s. But now $\mathcal{R}_U \phi_s$ is not defined on the whole of X_G; it is singular at the base point x_0 and goes to zero at infinity. We conclude that $\mathcal{R}_U \phi_s$ is a multiple of the eigenfunction f_{-s} of the Laplacian defined by *Harish-Chandra's expansion of the spherical function* (HC) (see section 1.2): there is a complex number $\mu(s)$ such that

$$\mathcal{R}_U \varphi_s = \mu(s) f_{-s}, \quad s \in \Lambda + \rho.$$

An explicit formula for $\mu(s)$ is given by

2.4.1 Proposition (NEUMANN**'s formula).** — *For $x \in X_{\mathbb{C}}$ such that $z(x) \notin]-\infty, 1]$,*

$$\int_U k(x,y)\phi_s(y)\,dy = (-1)^m \mathbf{e}(s) f_{-s}(x), \quad s = m+\rho, m \in \mathbb{N},$$

where f_{-s} is the function defined by (HC) and $\mathbf{e}$ is the meromorphic function defined by one of the following equivalent formulas:

$$\mathbf{e}(s) = \frac{1}{\delta(s)\mathbf{c}(s)} \frac{\Gamma(s+\rho)}{\Gamma(s-\rho+1)},$$

$$\mathbf{e}(s) = \frac{1}{s\,\mathbf{c}(s)} \frac{\Gamma(s+\frac{b-a+1}{2})}{\Gamma(s+\frac{a-b+1}{2})},$$

$$\mathbf{e}(s) = \frac{\Gamma(s+\rho)\Gamma(2s+b-a)}{\Gamma(2s+1)\Gamma(s+\frac{b-a}{2})}.$$

Proof. — For $s \in \rho + \mathbb{N}$, we will calculate $\mu(s)$ by determining the asymptotic behaviour of the function

$$\mathcal{R}_U \Phi_s(x) = \int_{-1}^{1} (x+y)^{-(a+b+1)} \Phi_s(y)(y-1)^a (y+1)^b \,dy,$$

as x goes to infinity. First, using the binominal expansion for $K_x(y) = (x+y)^{-(a+b+1)}$, we get

$$\mathcal{R}_U \Phi_{m+\rho}(x) = x^{-(a+b+1)}.$$

$$\int_{-1}^{1} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(a+b+1+k)}{\Gamma(a+b+1)\Gamma(k+1)} (\frac{y}{x})^k \Phi_{m+\rho}(y)(y-1)^a (y+1)^b \,dy.$$

Because of the Schur orthogonality relations, the scalar product in $L^2(X_U)$ of $\phi_{m+\rho}$ with a polynomial in z of lower degree is zero, so that we can replace the summation from zero to infinity by summation from m to infinity. When taking the limit $x \to \infty$, the terms of higher order can be neglected with respect to the term of order m. We end up with

$$\mathcal{R}_U \Phi_{m+\rho}(x) \sim (2x)^{-(a+b+1)-m}$$

$$\cdot (-1)^m \frac{2^{a+b+1+m}\Gamma(a+b+1+m)}{\Gamma(a+b+1)\Gamma(m+1)} \int_{-1}^{1} y^m \Phi_{\rho+m}(y)(y-1)^a (y+1)^b \,dy$$

as x goes to infinity. This last integral is the scalar product of $\varphi_{\rho+m}$ and z^m in $L^2(X_U)$. If we write

$$\phi_{m+\rho} = \sum_{j=1}^{m} c_j z^j,$$

then, once more because of Schur's orthogonality relations,

$$\int_U z^m(u)\phi_{\rho+m}(u)\,du = \frac{1}{c_m}\int_U (\phi_{\rho+m}(u))^2\,du = \frac{1}{c_m\,d_m},$$

where d_m is the dimension of the representation associated to $\phi_{m+\rho}$. As $\Phi_{\rho+m}(x)$ has asymptotic behaviour $c_m(2x)^m$ as x goes to infinity, we deduce

$$c_m = 2^m\mathbf{c}(m+\rho).$$

Hence $\mathcal{R}_U\Phi_s(x) \sim (-1)^m\mu(s)(2x)^{-2\rho-m}$ $(x\to\infty)$ with

$$\mu(s) = \frac{(-1)^m}{d_m}\frac{\Gamma(a+b+1+m)}{\mathbf{c}(\rho+m)\Gamma(a+b+1)\Gamma(m+1)}, \quad s = m+\rho, m\in\mathbb{N}.$$

This gives the first formula stated for $\mathbf{e}(s)$. Using the formula $d_m = \frac{\delta(m+\rho)}{\delta(\rho)}$ and the explicit formula for $\delta(s)$ (lemma 1.3.1), we get the second formula stated for $\mathbf{e}$. We now observe, using the first explicit formula for the $\mathbf{c}$-function given in section 1.2, that the term $\mathbf{c}(s)\frac{\Gamma(s+\frac{a-b+1}{2})}{\Gamma(s+\frac{b-a+1}{2})}$ gives the formula for the $\mathbf{c}$-function with *permutet* parameters a and b. Now the third formula stated for $\mathbf{e}$ is just a consequence of the functional equation of the Gamma function. ∎

The proposition states that, for $x \in X_G - \{x_0\}$, the function $\{\Re s > \frac{m_{\beta/2}}{4}\} \to \mathbb{C}, s \mapsto \mathbf{e}(s)f_{-s}(x)$ is a holomorphic interpolation of the coefficients $(-1)^m\widehat{k_x}(\rho+m)$. One can show that this interpolation tends to zero as $\Re s$ tends to infinity, and Carlson's uniqueness theorem (see [Ti39] 5.8) implies that it is the unique interpolation having this property. Let us now show that the function $\mathbf{e}$ is related to the asymptotic behaviour of the function f_{-s} at its singularity at the origin x_0 of X_G:

2.4.2 Proposition. — *For $s\in\Lambda+\rho$, the function f_{-s} has the following asymptotic behaviour at x_0: if $a>0$,*

$$F_{-s}(x) \sim s\cdot\mathbf{c}(s)(x-1)^{-a} \quad (x\to 1),$$

and if $a=0$,

$$F_{-s}(x) \sim s\cdot\mathbf{c}(s)\log(x-1) \quad (x\to 1).$$

Proof. — By NEUMANN's formula and 2.2.2, in the case $a > 0$,

$$F_{-s}(x) = (-1)^m \mathbf{e}(s)^{-1}(\mathcal{R}_U \Phi_s)(x) \sim (-1)^m \mathbf{e}(s)^{-1}\Phi_s(-1)(x-1)^{-a}$$

Using [Er53] 10.8 (13) (observe the normalisation [Er53] 10.8 (3) in this formula), we get for $s = \rho + m$

$$\Phi_{m+\rho}(-1) = (-1)^m \frac{\Gamma(b+1+m)\Gamma(a+1)}{\Gamma(a+1+m)\Gamma(b+1)} = (-1)^m \frac{\Gamma(s+\frac{b-a+1}{2})\Gamma(a+1)}{\Gamma(s+\frac{a-b+1}{2})\Gamma(b+1)},$$

and the observation on the **c**-function used at the end of the proof of 2.4.1 gives now

$$(-1)^m \Phi_s(-1)\mathbf{e}(s)^{-1} = \mathbf{c}(s)s,$$

which, substituted into the first equation, gives the affirmation. Similar for $a = 0$. ∎

Remarks. 1. One can show, using eqn. (2.1), that the statement of the preceeding proposition is in fact true for all s ($-2s \notin \mathbb{N}$) ([Be94b] 2.4.10).

2. In the case of the sphere S^2 we can express f_s in terms of the Legendre function of the second kind Q_s: $F_{-s}(x) = \mathbf{c}(s)Q_{s-1/2}(x)$, and the classical NEUMANN formula (see introduction) appears as a special case of 2.4.1.

2.5 RAMANUJANS **transform (second version).** — Let us put together the MEHLER and HEINE formulae with HARISH-CHANDRA's spherical function expansion (HC). For $f = k_x$, where $x \in X_{\mathbb{C}}$ is such that $z(x) \notin]-\infty, 1]$ and $k_x(y) = k(x,y)$ is the kernel function defined in 2.1, we have $\tilde{f}(s) = \mathbf{d}(s)\phi_s(x)$ for $|\Re s| < \rho$ (Prop. 2.3.1), and the coefficients $(-1)^m \hat{f}(m+\rho)$ have a holomorphic interpolation by $a(s) := \mathbf{e}(s)f_{-s}(x)$ (Prop. 2.4.1). Using the relation $\phi_s = \mathbf{c}(s)f_s(x) + \mathbf{c}(-s)f_{-s}(x)$ ($2s \notin \mathbb{Z}$) we thus obtain for $|\Re s| < \rho$ and $2s \notin \mathbb{Z}$

$$\tilde{f}(s) = \mathbf{d}(s)\phi_s(x) = \mathbf{b}(s)a(s) + \mathbf{b}(-s)a(-s)$$

with

$$\mathbf{b}(s) := \frac{\mathbf{d}(s)\mathbf{c}(-s)}{\mathbf{e}(s)}. \tag{2.2}$$

Using the formula for $\mathbf{d}(s)$ from prop. 2.3.1 and the first formula for $\mathbf{e}(s)$ given in prop. 2.4.1, we obtain

$$\mathbf{b}(s) = \delta(s)\Gamma(\rho - s)\Gamma(s - \rho + 1)\mathbf{c}(s)\mathbf{c}(-s)$$

$$= \delta(s)\frac{\pi}{\sin\pi(\rho - s)}\mathbf{c}(s)\mathbf{c}(-s),$$

which shows that the **b**-function defined here is the same as defined in section 1.4 by eqn. (1.6).

2.5.1 Theorem. — *Assume f is a K-invariant function on the rank-one symmetric space X_G admitting the following integral representation:*

$$f(x) = (\mathcal{R}_G h)(x) = \int_{X_G} h(y)k(x,y)\,dy$$

where h is a continous and integrable K-invariant function on X_G and k the kernel function defined in 2.1. Then f has a holomorphic continuation defined on a neighbourhood of X_G and on $X_U - A(x_0)$, where $A(x_0)$ is the antipodal set $A(x_0)$ of the base point x_0 in X_U. Then the restriction of f to $X_U - A(x_0)$ is integrable, and the coefficients

$$a(m+\rho) := (-1)^m \hat{f}(\rho + m), \quad m \in \mathbb{N},$$

have a holomorphic interpolation given by

$$a(s) = \mathbf{e}(s)\int_{X_G} h(y)f_{-s}(y)\,dy,$$

with $\mathbf{e}(s)$ defined in prop. 2.4.1. The function a is meromorphic on $\Re s > -\rho$. The non-compact spherical Fourier transform of f exists for $|\Re s| < \rho$, and, for $2s \notin \mathbb{Z}$, satisfies

$$(R) \qquad \tilde{f}(s) = \mathbf{b}(s)a(s) + \mathbf{b}(-s)a(-s)$$

with $\mathbf{b}(s)$ given by one of the formulae (1.6) *or* (2.2).

Proof. — (i) The function

$$(\mathcal{R}_G H)(x) = \int_1^\infty H(x)(x+y)^{-(a+b+1)}(y-1)^a(y+1)^b\,dy$$

is analytic on $\mathbb{C}\backslash]-\infty,-1]$, and behaves like $H(1)(x+1)^{-b}$ as x goes to -1, see 2.2.2. This implies that $\mathcal{R}_G h$ is integrable on X_U. So $\hat{f}$ exists, and we can apply Fubini's theorem to obtain

$$\begin{aligned}
\hat{f}(m+\rho) &= \int_{X_U} \phi_{m+\rho}(u)(\mathcal{R}_G h)(u)\,du \\
&= \int_{X_U}\int_{X_G} \phi_{m+\rho}(u)k(u,x)h(x)\,dx\,du \\
&= \int_{X_G} h(x)\int_{X_U} \phi_{m+\rho}(u)k_x(u)\,du\,dx \\
&= (-1)^m\mathbf{e}(m+\rho)\int_{X_G} h(x)f_{-(m+\rho)}(x)\,dx \\
&= (-1)^m a(m+\rho),
\end{aligned}$$

using NEUMANN's formula 2.4.1. In order to see that the function

$$\frac{a(s)}{\mathbf{e}(s)} = \int_{X_G} h(x) f_{-s}(x)\,dx = \int_1^\infty H(z) F_{-s}(z)(z-1)^a (z+1)^b\,dz$$

is well defined and holomorphic on $\Re s > -\rho$, we remark that $F_{-s}(z) \sim (z-1)^{-a}(z \to 1)$ (see 2.4.2) and $F_{-s}(z) \sim z^{-s-\rho}(z \to \infty)$. In order to determine $\tilde{f}$ we use Fubini's theorem and MEHLER's formula 2.3.1; we obtain for $|\Re s| < \rho$,

$$\begin{aligned}
\tilde{f}(s) &= \int_G \int_G \phi_s(g) k(g,x) h(x)\,dx\,dg \\
&= \int_G h(x) \int_G \phi_s(g) k_x(g)\,dg\,dx \\
&= \mathbf{d}(s) \int_G h(x) \phi_s(g) dg \\
&= \mathbf{d}(s)(\mathcal{F}_G h)(s)
\end{aligned}$$

converges for $|\Re s| < \rho$. Finally, we use (HC) $\phi_s = c(s) f_s + c(-s) f_{-s}$ to obtain the relation

$$\begin{aligned}
\tilde{f}(s) &= \mathbf{d}(s) \int_{X_G} h(x) \phi_s(x)\,dx \\
&= \mathbf{d}(s) \int_{X_G} h(x) \left(\mathbf{c}(s) f_s(x) + \mathbf{c}(-s) f_{-s}(x)\right) dx \\
&= \mathbf{d}(s)(\mathbf{c}(s) \frac{a(-s)}{\mathbf{e}(-s)} + \mathbf{c}(-s) \frac{a(s)}{\mathbf{e}(s)}) \\
&= \mathbf{b}(s) a(s) + \mathbf{b}(-s) a(-s)
\end{aligned}$$

with $\mathbf{b}(s) = \frac{\mathbf{d}(s)\mathbf{c}(-s)}{\mathbf{e}(s)}$. ∎

2.5.2 Remark. — Our theorems 1.5 and 2.5.1 give a version of RAMANUJAN's transform (as introduced by the diagram in section 0.3) read "from right to left", but they do not give a formula for the transform "from left to right". Let us give at least a partial answer to this problem. One may ask for the relation between our formula (R_1)

$$\tilde{f}(s) = \mathbf{b}(s) a(s) + \mathbf{b}(-s) a(-s)$$

and the decomposition of $\tilde{f}(s)$ into the "partial spherical Fourier transforms"

$$\tilde{f}(s) = \int_{X_G} \mathbf{c}(-s) f_{-s}(x) f(x) dx + \int_{X_G} \mathbf{c}(s) f_s(x) f(x) dx,$$

where $f_{\pm s}$ are the functions appearing in Harish-chandra's spherical function expansion. (By prop. 2.4.2 and the following remark, f_s is locally integrable at the origin, so the "partial transforms" are well-defined for $|\Re s| < \rho$ if f is bounded.) In the framework of the above theorem these two decompositions are indeed the same; i.e. one can show that, for all f described by the theorem and $\Re s > -\rho$,

$$(R_1') \qquad a(s) = \frac{\mathbf{c}(-s)}{\mathbf{b}(s)} \int_{X_G} f_{-s}(x) f(x)\, dx.$$

This can be read as a formula for the Fourier coefficients of f which are given by $(-1)^m a(m+\rho)$:

$$(-1)^m \int_{X_U} \phi_{m+\rho}(u) f(u) du = \frac{\mathbf{c}(-s)}{\mathbf{b}(s)} \int_{X_G} f_{-s}(x) f(x) dx|_{s=m+\rho}.$$

However, this interpolation is described in terms of the "partial spherical Fourier transform" and not in terms of the usual (symmetric) spherical Fourier transform, and recovering the partial spherical Fourier transform directly from the symmetric one would involve a kernel on $\mathfrak{a}^* \times \mathfrak{a}^*$ of the form $\int_{X_G} f_{-s}(x)\phi_\lambda(x)dx$.

As for the proof of the above relation (R_1'), it is enough to show it for the functions $f = k_y$, $y \in X_G - Kx_0$. Knowing that in this case $a(s) = \mathbf{e}(s) f_{-s}(y)$ (prop. 2.4.1), we have to show that for $\Re s > -\rho$,

$$\int_{X_G} k_y(x) f_{-s}(x) dx = \mathbf{d}(s) f_{-s}(x),$$

which is a sort of "Mehler's formula of the second kind" (compare with 2.3.1). In fact, the proof can be carried out along the same lines as our proof of 2.3.1, using eqn. (2.1) and the fact that $[f_{-s}, k_y](r) \to 0$ $(r \to 0)$.

Appendix

We shall give in this appendix a very rough analog of a theory by Lassalle ([La78]) on the holomorphic continuation of Fourier series on a compact symmetric space, transferred into the framework of the *non-compact dual*. If $G = NAK$ is the Iwasawa-decomposition associated to the non-compact riemannian symmetric space $X_G = G/K$ (see section 0.2) and $N_\mathbb{C}$, $A_\mathbb{C} \subset U_\mathbb{C}$ are the complexifications of the corresponding groups, then it is clear that the orbit $N_\mathbb{C} A_\mathbb{C} \cdot x_0$ is an open neighbourhood of $X_G = NA \cdot x_0$ in $X_\mathbb{C}$ (it is even dense). The following proposition is stated for the rank-one case but can easily be generalized to the general-rank case. We use the identification of $\mathfrak{a}^*$ (and of its dual) with $\mathbb{R}$ by means of the bigger root β, see section 1.1. The function a^s is defined on X_G by $a^s(x) = e^{s \log a(x)} = e^{sH(x)}$, see section 0.2. This function has a holomorphic continuation onto the neighbourhood $N_{\mathbb{C}} A_\epsilon \cdot x_0$ of X_G, where $A_\pi = \exp(\mathfrak{a} \oplus i] - \pi, \pi[)$. In fact, it is well-known that the function a^{-1} is a matrix-function of a finite-dimensional $U_{\mathbb{C}}$-representation (it is the highest weight vector in the first non-trivial spherical representation) and is thus holomorphic on the whole of $X_{\mathbb{C}}$ (this is essentally what is needed to prove lemma 0.2.1; see [He94].) The holomorphic function a^s for $s \in \mathbb{C}$ may thus be defined on $N_{\mathbb{C}} A_\pi \cdot x_0$ by

$$a^s(na \cdot x_0) = (a^{-1}(na \cdot x_0))^{-s} = (e^{-\log a})^{-s} = e^{s \log a}$$

with $\log a \in \mathfrak{a} \oplus i]\pi, \pi[$. (We might also define a^s to be a holomorphic function on the universal cover of $N_{\mathbb{C}} A_{\mathbb{C}}$ and would then not need the precautions on the domain of $\log a$.)

A.1 Proposition. — *Let F be a function on $\mathbb{R}$ such that $|F(s)| < Ce^{-\epsilon|s|}$ for some $C < \infty$ and $\epsilon > 0$. Then*

(i)

$$I(x) := \int_{-i\infty}^{i\infty} F(s) a^{s+\rho}(x) \frac{ds}{\mathbf{c}(s)\mathbf{c}(-s)}$$

defines a holomorphic function on the neighbourhood $N_{\mathbb{C}} A_\epsilon \cdot x_0$ of X_G, where $A_\epsilon = \exp(\mathfrak{a} \oplus i] - \epsilon, \epsilon[)$ (assuming $\epsilon < \pi$).

(ii)

$$J(x) := \int_{-i\infty}^{i\infty} F(s) \phi_s(x) \frac{ds}{\mathbf{c}(s)\mathbf{c}(-s)}$$

defines a holomorphic function on a neighbourhood of X_G which is bounded on X_G.

Proof. — *(i)* For real s we can estimate

$$|a^{is}(na \cdot x_0)| = |e^{is \log a}| = e^{-s\Im(\log a)} \leq e^{|s\Im(\log a)|}.$$

Hence for $x = na \cdot x_0 \in N_{\mathbb{C}} A_\epsilon \cdot x_0$:

$$|I(x)| \leq |a^\rho| \int_{-i\infty}^{i\infty} |a^s F(s)| \frac{ds}{\mathbf{c}(s)\mathbf{c}(-s)} \leq |a^\rho| \int_{-i\infty}^{i\infty} e^{|s|(|\Im(\log a)|-\epsilon)} \frac{ds}{\mathbf{c}(s)\mathbf{c}(-s)}$$

converges for $|\Im(\log a)| - \epsilon < 0$, i.e. on $N_{\mathbb{C}} A_\epsilon$ where it defines a holomorphic function.

(ii) follows from (i) by averaging over the K-orbits after having chosen a suitable K-invariant neighbourhood of X_G inside $N_{\mathbb{C}} A_\epsilon \cdot x_0$. (Such neighbourhoods exist as is easily seen using compactness of K.) As $\int_K a^{\rho(\log(a(kx)))} dk = \phi_0(x)$ is bounded on X_G, the above estimations imply that $J(x)$ is bounded on X_G. $\square$

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