# CONCEPTUAL DIFFERENTIAL CALCULUS
# PART I: FIRST ORDER LOCAL LINEAR ALGEBRA

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Abstract. We give a rigorous formulation of the intuitive idea that a differentiable map should be the same thing as a *locally*, or *infinitesimally, linear* map: just as a linear map respects the operations of addition and multiplication by scalars in a vector space or module, a *locally linear map* is defined to be a map respecting two canonical operations $*$ and $\bullet$ living "over" its domain of definition. These two operations are composition laws of a *canonical groupoid* and of a *scaled action category*, respectively, fitting together into a *canonical double category*. Local linear algebra (of first order) is the study of such double categories and of their morphisms; it is a purely algebraic and conceptual (i.e., categorical and chart-independent) version of first order differential calculus. In subsequent work, the higher order theory (using higher multiple categories) will be investigated.

## Introduction

The present work continues the line of investigations on general differential calculus and general differential geometry started with [BGN04, Be08, Be13, BeS14, Be14]. Combining it with ideas present in work of Nel [Nel88] and in synthetic differential geometry (see [Ko10, MR91]), we obtain a purely algebraic and categorical presentation of the formal rules underlying differential calculus. The results can be read on two different levels:

- even for ordinary (finite-dimensional, real or complex) manifolds $M$, the construction of a canonical *first order difference groupoid* $M^{\{1\}}$ and of a *first order double category* $\overline{M^{\{1\}}}$ seem to be new – indeed, they contain as a special case a new and more conceptual construction of Connes' *tangent groupoid* ([Co94], II.5), and hence our theory may be of some interest in non-commutative geometry and quantization (see, e.g., [La01]),
- and these constructions open the way for a general, "conceptual", approach to calculus and manifolds over any commutative base ring.

The term "conceptual differential calculus" is an allusion to the title of the book [LaSch09], in the sense that "conceptual" means "categorical". Most of the concepts we are going to use (in particular, *double categories* and *double groupoids*) go back to work of Charles Ehresmann, see [E65]; but, while Ehresmann applied them to the output of differential calculus (i.e., to differential geometry), I shall advocate here to apply them already on the level of the input (i.e., to the calculus itself).

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0.1. **Topological differential calculus.** "Usual" differential calculus is not intrinsic, in the sense that it takes place in a chart domain, and not directly on a manifold: the usual difference quotient, for a function $f : U \to W$, defined on a subset $U$ of a vector space $V$, at a point $x \in U$ in direction $v$ and with $t \neq 0$,

$$F(x, v, t) := \frac{f(x + tv) - f(x)}{t} \tag{0.1}$$

depends on the vector space structure, hence on a chart, and it cannot directly be defined on a manifold. Thus, one first has to develop "calculus in vector spaces", from which one then extracts "invariant" information, in order to define manifolds and structures living on them; our work [BGN04, Be08] is no exception to this rule – in Section 1 of the present work, we recall that approach, which we call *topological differential calculus* (cf. [Be11]).

0.2. **Conceptual differential calculus: the groupoid approach.** The path from topological calculus to the conceptual version to be presented here has been quite long, and I refer the reader wishing to have more ample motivation and heuristic explanations to my books and papers given in the reference list (in particular, the attempts to solve the problems listed in [Be08b] have played an important rôle). Also, it would take too much room to mention here all the work that influenced this approach, foremost synthetic differential geometry (see, e.g., [Ko10, MR91]).

In a nutshell, our categorical approach can be summarized as follows: intuitively, we think of a manifold, or of a more general "smooth space", as a space $M$ that is locally, or infinitesimally, linear. Seen algebraically,

- a linear space, $V$, is defined by two laws, $+$ and $\cdot$, living <u>*in*</u> the space $V$, meaning that these laws are *everywhere defined* on $V \times V$, resp. on $\mathbb{K} \times V$ (here, $\mathbb{K}$ is the base field or ring),
- saying that $M$ is locally linear amounts to saying that $M$ is defined by two laws, $*$ and $\bullet$, living <u>*over*</u> the space $M$, in the sense that they are *not everywhere defined* and live in a certain *bundle,* $M^{\overline{\{1\}}}$, *over* $M \times \mathbb{K}$.

More precisely, $*$ is a *groupoid law* and $\bullet$ a *category law*; and the compatibility of $+$ and $\cdot$ in $V$ generalizes to the compatibility of $*$ and $\bullet$, meaning that the whole structure forms a *(small) double category.*[1] In fact, the law $*$ generalizes Connes' tangent groupoid ([Co94], II.5): as in Connes' construction, for each $t \in \mathbb{K}$, the fiber in $M^{\overline{\{1\}}}$ over $M \times \{t\}$ is still a groupoid; for $t = 0$ we get the usual tangent bundle $TM$ (with its usual vector bundle structure), and for invertible $t$, we get a copy of the *pair groupoid over* $M$. Our construction is natural and does not proceed by taking (as in [Co94]) a disjoint union of groupoids: if $M$ is a Hausdorff manifold, it is obvious from our construction that we get an *interpolation between the pair groupoid and the tangent bundle of* $M$ (Theorem 2.11) .

Starting with the difference quotient (0.1), it is indeed quite easy to explain how to arrive at these concepts – see Sections 2 and 3: multiplying by $t$ in (0.1), we get the notion of *difference factorizer* (terminology following Nel, [Nel88]). Analyzing

[1] I do not assume the reader to have any knowledge in category theory since I am myself not a specialist in this domain – full definitions are given in the appendices A, B, C. I apologize in advance to experts in category theory for a possibly somewhat ideosyncratic presentation.

further this concept, we realize that "addititve" difference factorizers are in 1:1-correspondence with morphisms of a certain $*$-groupoid; if the difference factorizer is, moreover, "homogeneous", it corresponds to a morphism of a certain $(*, \bullet)$-double category. Summing up, the formal concept corresponding to usual mappings of class $C^1$ is precisely the one of morphisms of $(*, \bullet)$-double categories: *maps of class $C^1$ are thus the same as infinitesimally linear maps.* This comes very close to the point of view of synthetic differential geometry (achieved there by topos-theoretic methods).

0.3. **Laws of class $C^1_{\mathbb{K}}$.** Once these observations are made, we can generalize $C^1$-*maps* by $C^1$-*laws over an arbitrary base ring* $\mathbb{K}$: given a set-map $\underline{f} : \underline{U} \to \underline{W}$, a $C^1_{\mathbb{K}}$-*law over* $\underline{f}$ is a morphism of double categories $f^{\{1\}} : U^{\{1\}} \to W^{\{1\}}$ (Section 4). Although $f^{\{1\}}$ need not be uniquely determined by $\underline{f}$ (just like a polynomial need not be determined by its underlying polynomial map), we think of $f^{\{1\}}$ as a sort of *derivative* of $f$. Indeed, a differentiable map, in the usual sense, gives rise to a (unique) continuous law. We prove also that every polynomial law (in the general sense of [Ro63]) is a law of class $C^1_{\mathbb{K}}$ (Theorem 4.15). Using general principles on the construction of manifolds (Appendix D), we explain in Section 5 how these concepts carry over to *general* $C^1$-*manifold laws over* $\mathbb{K}$. For instance, the *Jordan geometries* defined over an arbitrary commutative base ring $\mathbb{K}$ ([Be14c]) are manifold laws.

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0.4. **Tentative description of the sequel.** In Part II of this work [Be$\rho\tau$], we shall define *laws of class $C^n$* and prove that they are morphisms of $2n$-fold categories, and, in particular, of *$n$-fold groupoids*, and study their structure. Topics for further work include: revisiting notions of differential geometry (and of synthetic differential geometry), in particular, *connection theory*, *Lie groups* and *symmetric spaces*, from the groupoid viewpoint (in this context, the paper [BB11] is highly relevant); a conceptual version of the *simplicial approach* presented in [Be13], and, finally, the very intriguing topic of possible *non-commutativity of the base ring*: as noted in [Be08b], Problem 8, it is possible to develop most of the first order theory without assuming commutativity of $\mathbb{K}$. Indeed, it turns out, in the present work, that commutativity of $\mathbb{K}$ does not enter before dealing with bilinear maps (Section 4); the construction of the first order double category $M^{\{1\}}$ goes through for possibly non-commutative rings. This makes it clear that commutativity becomes crucial for second and higher order calculus, but not before; and it seems very likely that a careful analysis of this situation may lead to a new conceptual foundation of *super-calculus* (cf. [Be08b], Problem 9).

0.5. **Notation and conventions.** Throughout, the letter $\mathbb{K}$ denotes a *base ring with unit* 1. Unless otherwise stated, this ring is not equipped with a topology and not assumed to be commutative. All $\mathbb{K}$-modules $V, W, \ldots$ are assumed to be *right* $\mathbb{K}$-modules. By definition, a *linear set* is a pair $(U, V)$, where $V$ is a $\mathbb{K}$-module and $U \subset V$ a non-empty subset. The linear set $(\{0\}, \{0\})$ will be denoted by 0 ("terminal object"). Informally, by *local linear algebra* we mean the theory of linear sets, their prolongations and morphisms, as developed in this work.

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## 1. Difference factorizers and differential calculus

1.1. **Difference factorizer.** In order to get rid of the division by the scalar $t$ in the difference quotient (0.1), we define, following a terminology used by Nel [Nel88]:

**Definition 1.1** (Difference factorizer)**.** *Let $U \subset V$ be a linear set (cf. conventions above). As in* [BGN04, Be08, Be11]*, we define its* first prolongation *by*

$$\boxed{U^{[1]} := U^{[1]}_{\mathbb{K}} := U^{[1]}_{V,\mathbb{K}} := \{(x,v,t) \in V \times V \times \mathbb{K} \mid \ x \in U,\, x + vt \in U\}}.$$

*The* non-singular part *of the first prolongation is the set where $t$ is invertible:*

$$(U^{[1]})^{\times} := \{(x,v,t) \in U^{[1]} \mid \ t \in \mathbb{K}^{\times}\}, \tag{1.1}$$

*For a map $f$ from $U$ to a $\mathbb{K}$-module $W$, a* difference factorizer for $f$ *is a map*

$$\boxed{f^{[1]} : U^{[1]} \to W \ \textit{such that}\ \forall (x,v,t) \in U^{[1]} : f(x+vt) - f(x) = f^{[1]}(x,v,t) \cdot t}.$$

When $(x, v, t)$ belongs to the non-singular part, then $f^{[1]}(x, v, t)$ is necessarily given by (0.1), and the proof of the following relations is straightforward: $\forall s, t \in \mathbb{K}^{\times}$,

$$f^{[1]}(x,0,t) = 0, \tag{1.2}$$

$$f^{[1]}(x, v + v', t) = f^{[1]}(x + vt, v', t) + f^{[1]}(x, v, t), \tag{1.3}$$

$$f^{[1]}(x, vs, t) = f^{[1]}(x, v, st)\, s, \tag{1.4}$$

and, if $g$ and $f$ are composable, then, for $t \in \mathbb{K}^{\times}$,

$$(g \circ f)^{[1]}(x,v,t) = g^{[1]}\big(f(x), f^{[1]}(x,v,t), t\big). \tag{1.5}$$

Now, difference factorizers are not unique – e.g., the values for $t = 0$ are not determined by the condition.[2] Differential calculus is a means to assign a well-defined value when $t = 0$. We recall briefly the main ideas, following [BGN04]; as in [Be11] we will call this theory *topological differential calculus*:

### 1.2. **Topological differential calculus.**

**Definition 1.2.** *The* assumptions of topological differential calculus *are:* $\mathbb{K}$ *is a* topological ring having a dense unit group $\mathbb{K}^\times$, *and* $V, W$ *are* topological $\mathbb{K}$-modules. *Maps* $f : U \to W$ *are assumed to be defined on open subsets* $U \subset V$.

**Definition 1.3.** *We say that* $f : U \to W$ is of class $C^1_{\mathbb{K}}$ *if* $f$ *admits a* continuous *difference factorizer* $f^{[1]}$. *Because of density of* $\mathbb{K}^\times$ *in* $\mathbb{K}$*, such a difference factorizer is unique, if it exists, and hence we can define the* first differential of $f$ at $x$ *by*

$$df(x)v := f^{[1]}(x, v, 0).$$

The philosophy of differential calculus can be put with the words of G. W. Leibniz (quoted in the introduction of [Be08]): "The rules of the finite continue to hold in the infinite" – properties valid for difference factorizers and invertible scalars $t$ continue to hold for "singular" scalars, in particular for the most singular value $t = 0$. For instance, by density and continuity, identities (1.2) – (1.5), continue to hold for $t = 0$, proving the "usual" properties of the differential, *linearity* and *chain rule*:

$$\begin{aligned} df(x)(v + v') &= df(x)v + df(x)v', \\ df(x)(vs) &= (df(x)v)s, \\ d(g \circ f)(x)v &= dg(f(x))\big(df(x)v\big) \end{aligned} \tag{1.6}$$

which then allow to define manifolds having a linear tangent bundle, and so on. What we need for such "invariant" constructions is essentially only a "functorial" rule like the cain rule; this permits to define bundles, carrying structure according to what is preserved under coordinate changes (cf. Appendix D).

## 2. The first order difference groupoid and its morphisms

### 2.1. **The first order difference groupoid.**

**Definition 2.1.** *If* $(U, V)$ *is a linear set, we shall henceforth use the notation*[3]

$$U^{\{1\}} := U^{[1]}, \qquad (U^{\{1\}})^\times := (U^{[1]})^\times,$$

*and we define two surjections called "source" and "target"*

$$\pi_0 : U^{\{1\}} \to U \times \mathbb{K}, \quad (x, v; t) \mapsto (x; t)$$
$$\pi_1 : U^{\{1\}} \to U \times \mathbb{K}, \quad (x, v; t) \mapsto (x + vt; t)$$

[2] If $\mathbb{K}$ is a field, then $t = 0$ is the only "exceptional" value; but it will be very important to allow more general rings. Note that the case of an integral domain, like $\mathbb{K} = \mathbb{Z}$, and free $\mathbb{K}$-modules, behaves much like the case of a field, in the sense that $f^{[1]}(x, v, t)$ is unique for all $t \neq 0$.

[3] The deeper reason for this change of notation will only appear at second order, when iterating the constructions: in [BGN04], the index [2] is used, but in Part II we shall rather use $\{1, 2\}$.

*and one injection called "zero section" or "unit section"*

$$z : U \times \mathbb{K} \to U^{\{1\}}, \quad (x,t) \mapsto (x,0,t).$$

Obviously, $z$ is a *bisection* of the projections, i.e., $\pi_0 \circ z = \mathrm{id}_U = \pi_1 \circ z$.

**Lemma 2.2.** *Assume given a map $f : U \to W$. Then there is a 1:1-correspondence between difference factorizers of $f$ and maps $f^{\{1\}} : U^{\{1\}} \to W^{\{1\}}$ that commute with source and target maps and coincide with $f$ on the base, in the sense that*

$$\text{for } \sigma = 0,1 : \ \pi_\sigma \circ f^{\{1\}} = (f \times \mathrm{id}_{\mathbb{K}}) \circ \pi_\sigma : \qquad \begin{array}{ccc} U^{\{1\}} & \xrightarrow{f^{\{1\}}} & W^{\{1\}} \\ \downdownarrows & & \downdownarrows \\ U \times \mathbb{K} & \xrightarrow{f \times \mathrm{id}_{\mathbb{K}}} & W \times \mathbb{K} \end{array}$$

*Namely, the difference factorizer $f^{[1]}$ corresponds to the map*

$$f^{\{1\}} : U^{\{1\}} \to W^{\{1\}}, \quad (x,v,t) \mapsto \Big( f(x), f^{[1]}(x,v,t); t \Big).$$

*Moreover, $f^{[1]}$ satisfies condition (1.2) if, and only if, $f^{\{1\}}$ commutes with $z$ in the sense that $f^{\{1\}} \circ z = z \circ (f \times \mathrm{id}_{\mathbb{K}})$.*

*Proof.* Assume $f^{\{1\}} : U^{\{1\}} \to W^{\{1\}}$ is a map. The condition $\pi_0 \circ f^{\{1\}} = (f \times \mathrm{id}_{\mathbb{K}}) \circ \pi_0$ is equivalent to the existence of a map $F$ such that, for all $(x,v,t) \in U^{\{1\}}$,

$$f^{\{1\}}(x,v,t) = \big( f(x), F(x,v;t); t \big),$$

and then the condition $\pi_1 \circ f^{\{1\}} = (f \times \mathrm{id}_{\mathbb{K}}) \circ \pi_1$ is equivalent to

$$f(x) + F(x,v;t) \cdot t = f(x+vt),$$

for all $(x,v,t) \in U^{\{1\}}$. Thus $f^{\{1\}}$ commutes with projections iff $f^{[1]}(x,v,t) := F(x,v,t)$ is a difference factorizer. Finally, the condition $f^{\{1\}}(x,0,t) = (f(x),0,t)$ is equivalent to $F(x,0,t) = 0$, for all $t$, that is, (1.2). □

**Definition 2.3.** *Given a map $f : U \to W$, a map $f^{\{1\}} : U^{\{1\}} \to W^{\{1\}}$ as in the lemma will be called a* map over $f$, *and $f$ will be called the* base map *of $f^{\{1\}}$.*

**Definition 2.4.** *For $a = (x,v,t), a' = (x',v',t') \in U^{\{1\}}$ such that $\pi_1(a) = \pi_0(a')$ (so $t = t'$ and $x' = x + vt$), we define*

$$\boxed{a' * a := (x+vt, v', t) * (x,v,t) := (x, v'+v, t)}.$$

*Note that $a' * a$ belongs again to $U^{\{1\}}$. Indeed, $x + (v+v')t = x' + v't$, that is,*

$$\pi_0(a' * a) = \pi_0(a), \qquad \pi_1(a' * a) = \pi_1(a'). \tag{2.1}$$

**Theorem 2.5.** *The data $\big(\pi_0, \pi_1 : U^{\{1\}} \downdownarrows U \times \mathbb{K}, z, *\big)$ define a* groupoid. *This groupoid is a bundle of groupoids over $\mathbb{K}$, i.e., for every fixed value of $t \in \mathbb{K}$, we have a groupoid $(\pi : U_t \downdownarrows U, z, *)$,*

$$U_t := \{(x,v) \mid x \in U, x+vt \in U\} \downdownarrows U, \qquad (x',v') * (x,v) = (x, v'+v),$$

$$U_t \times_U U_t = \{((x',v'),(x,v) \in U_t \times U_t \mid x' = x+vt, \, t' = t\}.$$

*Proof.* We check the defining properties of a groupoid (Appendix A): let $a = (x, v, t)$, $a = (x', v', t')$, $a'' = (x'', v'', t'')$. As noted above, (2.1) holds. To check associativity,

$$(x'', v'', t'') * ((x', v', t') * (x, v, t)) = (x'', v'', t'') * (x, v' + v, t) = (x, v'' + (v' + v), t),$$

$$((x'', v'', t'') * (x', v', t')) * (x, v, t) = (x', v'' + v', t') * (x, v, t) = (x, (v'' + v') + v, t),$$

and hence associativity of $*$ follows from associativity of addition in $(V, +)$. Next,

$$(x, 0, t) * (x', v, t) = (x', 0 + v, t) = (x', v, t),$$

$$(x, v, t) * (x, 0, t) = (x, v + 0, t) = (x, v, t),$$

hence $z(x, t)$ is a unit for $*$. We show that $(x + vt, -v, t)$ is an inverse of $(x, v, t)$:

$$(x, v, t) * (x + vt, -v, t) = (x + vt, 0, t),$$

$$(x + vt, -v, t) * (x, v, t) = (x, 0, t).$$

It is obvious from these formulae that, for any fixed $t$, we get again a groupoid. □

**Definition 2.6.** *The groupoid* $(\pi_0, \pi_1, U^{\{1\}} \downdownarrows U \times \mathbb{K}, z, *)$ *defined by the theorem is called the* first order difference groupoid of $U$. *The symbol* $U^{\{1\}}$ *will often be used both to denote the morphism set and the groupoid itself, and we use* $(U^{\{1\}})^\times$ *for the groupoid with underlying morphism set* $(U^{\{1\}})^\times$ *defined by (1.1).*

**Theorem 2.7** (The groupoids $U_0$ and $U_1$, and Connes' tangent groupoid)**.**

(1) *The groupoid* $U_0$ *is a "group bundle"* $TU := U_0 = U \times V$ *over the base* $U$, *with fiber the group* $(V, +)$.

(2) *The groupoid* $U_1$ *is isomorphic to the* pair groupoid $U \times U$ over $U$. *(See Example A.1: composition on* $U \times U$ *is* $(x, y) \circ (y, z) = (x, z)$.*)*

(3) *More generally, for every* $t \in \mathbb{K}^\times$, *the groupoid* $U_t$ *is isomorphic to the pair groupoid over* $U$, *via*

$$\Phi_t : U_t \to U \times U, \quad (x, v) \mapsto (y, x) := (x + vt, x).$$

*The* non-singular groupoid $(U^{\{1\}})^\times$ *is, via the map* $(x, v, t) \mapsto (x + vt, x, t)$, *isomorphic to the direct product* $U \times U \times \mathbb{K}^\times$ *of the pair groupoid of* $U$ *with the trivial groupoid of* $\mathbb{K}^\times$.

(4) *If* $\mathbb{K}$ *is a field, then* $U^{\{1\}}$ *is isomorphic to a disjoint union of groupoids*

$$TU \,\dot\cup\, [(U \times U) \times \mathbb{K}^\times].$$

*Proof.* (1) is obvious from the formulae. To prove (3), note that $\mathrm{pr}_2 \circ \Phi_t(x, v) = x = \pi_0(x, v)$, $\mathrm{pr}_1 \circ \Phi_t(x, v) = x + vt = \pi_1(x, v)$, and

$$\begin{aligned}\Phi_t((x + tv, v') * (x, v)) &= \Phi_t(x, v + v') = (x + (v' + v)t, x) \\ &= (x + v't + vt, x + vt) \circ (x + vt, x) = \Phi_t(x + tv, v') \circ \Phi_t(x, v).\end{aligned}$$

Concerning units, note that $\Phi_t(x, 0) = (x, x)$, proving that $\Phi_t$ is a morphism. It is bijective since $(y, x) \mapsto (x, (y - x)t^{-1})$ is an inverse map. Finally, (2) is the case $t = 1$ of (3), and (4) follows from (3) since, for a field, $\mathbb{K} = \{0\} \,\dot\cup\, \mathbb{K}^\times$. □

For $\mathbb{K} = \mathbb{R}$, the construction from Part (4) corresponds to Connes' construction of the *tangent groupoid*, [Co94], II.5. Note that our construction gives, when $V \cong \mathbb{R}^n$, ready-made the topology defined by Connes in loc.cit., p. 103.

*Remark* 2.1 (Pregroupoids). By forgetting the units, $U^{\{1\}}$ defines (just like any groupoid) a *pregroupoid* (see A.6): the ternary product $[a'', a', a] = a'' * (a')^{-1} * a$ is explicitly given by $[a'', a', a] = (x, v'' - v' + v, t)$.

### 2.2. Morphisms of first order difference groupoids.

**Theorem 2.8.** *Given a map $f : U \to W$, there is a 1:1-correspondence between*

*(1) difference factorizers of $f$ satisfying Condition (1.3) for all $t \in \mathbb{K}$,*
*(2) morphisms of groupoids $f^{\{1\}} : U^{\{1\}} \to W^{\{1\}}$ over $f$.*

*Proof.* In view of Lemma 2.2, it only remains to show that $f^{\{1\}}$ preserves $*$ iff $f^{[1]}$ satisfies (1.3). We compute

$$\begin{aligned} f^{\{1\}}((x', v', t') * (x, v, t)) &= f^{\{1\}}(x, v' + v, t) = (f(x), f^{[1]}(x, v' + v, t), t) \\ f^{\{1\}}(x', v', t') * f^{\{1\}}(x, v, t) &= (f(x), f^{[1]}(x + vt, v', t), t) * (f(x), f^{[1]}(x, v, t), t) \\ &= \big(f(x), f^{[1]}(x + vt, v', t) + f^{[1]}(x, v, t), t\big) . \end{aligned}$$

Thus equality holds iff (1.3) holds for $f^{[1]}$. Finally, note that Condition (1.2) follows from (1.3) by taking $v = 0$ there. □

*Example* 2.1. Every $\mathbb{K}$-linear map $f : V \to W$ gives rise to a morphism

$$f^{\{1\}} : V^{\{1\}} = V^2 \times \mathbb{K} \to W^{\{1\}} = W^2 \times \mathbb{K}, \quad (x, v; t) \mapsto (f(x), f(v); t).$$

Indeed, since, by linearity, $f(x + vt) - f(x) = f(v)t$, the map $f^{[1]}(x, v; t) = f(v)$ is a difference factorizer, and it satisfies (1.2) since $f$ is linear.

*Example* 2.2. If $f(x) = \alpha x + b$ is an affine map, then $f^{\{1\}}(x, v; t) = (\alpha x + b, \alpha v; t)$ is a groupoid morphism.

**Theorem 2.9** (General morphisms)**.** *Assume given two maps $f : U \to W$ and $\varphi : \mathbb{K} \to \mathbb{K}'$. Then the pair of maps*

$$\begin{aligned} U^{\{1\}}_{\mathbb{K}} \to W^{\{1\}}_{\mathbb{K}'}, &\quad (x, v; t) \mapsto \big(f(x), F(x, v; t); \varphi(t)\big), \\ U \times \mathbb{K} \to W \times \mathbb{K}', &\quad \big(x, t) \mapsto (f(x), \varphi(t)\big) \end{aligned}$$

*is a groupoid morphism if, and only if, $F$ is a $\varphi$-*twisted difference factorizer*, i.e.,*

$$\forall (x, v, t) \in U^{\{1\}} : \qquad f(x) + F(x, v; t) \cdot \varphi(t) = f(x + vt)$$

*which satisfies, whenever defined, the condition corresponding to (1.3):*

$$F(x + vt; v', t) + F(x, v; t) = F(x, v' + v; t).$$

*Proof.* By the arguments given in the proof of Lemma 2.2, compatibility with $\pi_0$ is equivalent to the existence of a map $F$ as in the defining formula from the theorem, and compatibility with $\pi_1$ then is equivalent to saying that $F$ is a $\varphi$-twisted difference factorizer of $f$. As in the preceding proof it is seen that compatibility with $*$ then amounts to the last condition stated in the theorem. □

**Definition 2.10.** *We say that a groupoid morphism $U^{\{1\}} \to W^{\{1\}}$ is* of the first kind*, or:* spacial*, if it is of the above form with $\mathbb{K} = \mathbb{K}'$ and $\varphi = \mathrm{id}_{\mathbb{K}}$, and* of the second kind*, or:* internal*, if it is of the above form with $f = \mathrm{id}_U$.*

*Example* 2.3. For $f = \mathrm{id}$, and $F(v,t) := F(x,v;t)$ independent of $x$, the conditions read

$$F(v,t) \cdot \varphi(t) = vt, \qquad F(v',t) + F(x,t) = F(v'+v,t).$$

For instance, taking, for $s \in \mathbb{K}^\times$ fixed, $\varphi(t) := st$ and $F(v,t) := vs^{-1}$, the conditions are satisfied (giving rise to *scaling automorphisms*, see next chapter).

*Example* 2.4. If $\mathbb{K} = \mathbb{C}$ and $U = V = W = \mathbb{C}^n$, then complex conjugation $f(z) = \overline{z}$, $F(z,v;t) = \overline{v}$, $\varphi(t) = \overline{t}$ defines a groupoid automorphism. More generally, every ring automorphism $\varphi$ of $\mathbb{K}$ together with a $\varphi$-conjugate linear map $f$ gives rise, in the same way, to a groupoid morphism $(x,v;t) \mapsto (f(x), f(v); \varphi(t))$.

2.3. **The topological case, and first difference groupoid of a manifold.** Recall from Subsection 1.2 the framework of topological differential calculus. In this case, the preceding results carry over to the manifold level without any difficulties:

**Theorem 2.11.** *Assume $\mathbb{K}$ is a topological ring with dense unit group, $V, W$ Hausdorff topological $\mathbb{K}$-modules and $M, N$ are Hausdorff $C^1_\mathbb{K}$-manifolds modelled on $V$, resp. on $W$.*

*(1) A $C^1_\mathbb{K}$-map $f : U \to W$ induces a morphism of groupoids $f^{\{1\}} : U^{\{1\}} \to W^{\{1\}}$.*
*(2) To the manifold $M$ we can associate a bi-bundle $\pi_0, \pi_1 : M^{\{1\}} \downdownarrows M \times \mathbb{K}$, carrying a canonical continuous groupoid structure.*
*(3) The groupoid $M^{\{1\}}$ is a bundle of groupoids over $\mathbb{K}$, and hence, for all $t \in \mathbb{K}$, the fiber $M_t$ over $t$ is a continuous groupoid with object set $M$.*
*(4) The groupoid $M_0$ is the usual tangent bundle (additive group bundle), and $M_1$ is isomorphic to the pair groupoid over $M$. More generally, the groupoid $(M^{\{1\}})^\times$ lying over $\mathbb{K}^\times$ is naturally isomorphic to the direct product of the pair groupoid of $M$ with the trivial groupoid of $\mathbb{K}^\times$. If $\mathbb{K}$ is a field, then $M^{\{1\}}$ is the disjoint union of that groupoid with the tangent bundle.*
*(5) Let $f : M \to N$ be a map. Then $f$ is of class $C^1_\mathbb{K}$ if, and only if, it extends to a continuous morphism of groupoids $f^{\{1\}} : M^{\{1\}} \to N^{\{1\}}$. Put differently: there is a 1:1-correspondence between $C^1$-maps $M \to N$ and continuous groupoid morphisms $M^{\{1\}} \to N^{\{1\}}$ of the first kind.*
*(6) There is a natural homeomorphism of bundles over $M \times N \times \mathbb{K}$*

$$(M \times N)^{\{1\}} \quad \cong \quad M^{\{1\}} \times_\mathbb{K} N^{\{1\}}.$$

*In other words, for all $t \in \mathbb{K}$, we have $(M \times N)_t \cong M_t \times N_t$.*

*Proof.* (1) If $f$ is of class $C^1$, then it admits a continuous difference factorizer $f^{[1]}$. As said in Section 1, such a difference factorizer satisfies (1.2) and (1.3), and hence, by Theorem 2.8, it induces a morphism of groupoids $f^{\{1\}}$.

(2) Let us first describe the construction of the set $M^{\{1\}}$: via Theorem D.4, $M$ is described by "local data" $(V_{ij}, \phi_{ij})_{(i,j)\in I^2}$. By definition of a $C^1$-manifold, the transition maps $\phi_{ij}$ are $C^1$, hence we get local data $((V_{ij})^{\{1\}}, (\phi_{ij})^{\{1\}})_{(i,j)\in I^2}$. Again by Theorem D.4, such data define a manifold which we denote by $M^{\{1\}}$. By the same theorem, the natural projections $(V_{ij})^{\{1\}} \to V_{ij} \times \mathbb{K}$ define projections $\pi_\sigma : M^{\{1\}} \to M \times \mathbb{K}$. In the same way we get the unit section $M \times \mathbb{K} \to M^{\{1\}}$. The products $*_i$ defined for each $(U_i)^{\{1\}}$ are compatible with transition maps, and hence

coincide over intersections $U_{ij} = U_i \cap U_j$. We have to show that these groupoid structures fit together and define a groupoid law $* : M^{\{1\}} \times_{M \times \mathbb{K}} M^{\{1\}} \to M^{\{1\}}$. To this end, consider $a, b \in M^{\{1\}}$ such that $u := \pi_1(a) = \pi_0(b)$. Let $x := \pi_0(a)$ and $y := \pi_1(b) \in M$. By Lemma D.3, we can find a chart domain $U \subset M$ containing $u, x$ and $y$. Now define $a * b \in U^{\{1\}}$ with respect to this chart. As explained above, this does not depend on the chart, and we are done.

(3) – (6): This now follows from (2) and the corresponding local statements in Theorem 2.5 and in Lemma 2.7. Note that the argument from (2), using Lemma D.3 and the Hausdorff property,[4] shows that $M_1 \cong M \times M$ (pair groupoid). For the proof of (6), the local version is, for chart domains $U \subset V$ and $S \subset W$,

$$\begin{aligned}(U \times S)^{\{1\}} &= \{(x, u; y, v; t) \in (U \times S) \times (V \times W) \times \mathbb{K} \mid (x, y) + t(u, v) \in U \times S\} \\ &= \{(x, u; y, v; t) \mid (x, u, t) \in U^{[1]}, (y, v, t) \in S^{[1]}\} = U^{[1]} \times_{\mathbb{K}} S^{[1]},\end{aligned}$$

and this naturally carries over to the level of manifolds. □

## 3. Scalar action, and the double category

3.1. **The scaling morphisms.** We have seen that the first order difference groupoid $U^{\{1\}}$ takes account of the additive aspect (1.3) of tangent maps. Now let us deal with multiplicative aspects, i.e., with the homogeniety condition

$$f^{[1]}(x, vs, t) = f^{[1]}(x, v, st) \cdot s. \tag{1.4}$$

**Theorem 3.1** (The scaling morphisms)**.** *Fix a couple of scalars $(s, t) \in \mathbb{K}^2$. Then there is a morphism of groupoids $U_{st} \to U_t$, given by*

$$\phi_{s,t} : U_{st} \to U_t,\ (x, v; st) \mapsto (x, vs; t), \qquad \textit{with base map } \mathrm{id}_U.$$

*A groupoid morphism of the type $f^{\{1\}} : U^{\{1\}} \to W^{\{1\}}$ commutes with all morphisms of the type $\phi_{s,t}$ if, and only if, its difference factorizer satisfies relation (1.4). If $\mathbb{K}$ is a topological ring and $M$ a Hausdorff $C^1_{\mathbb{K}}$-manifold, then the morphisms $\phi_{s,t}$ carry over to globally defined continuous groupoid morphisms $M_{st} \to M_t$.*

*Proof.* Clearly, we have $\phi_{s,t} \circ \pi_0 = \pi_0 \circ \phi_{s,t}$, and the condition $\phi_{s,t} \circ \pi_1 = \pi_1 \circ \phi_{s,t}$ holds since $x + v(st) = x + (vs)t$. The condition $\phi_{s,t}(a' * a) = \phi_{s,t}(a') * \phi_{s,t}(a)$ is equivalent to

$$(v' + v)st = v'st + vst,$$

that is, to distributivity of the $\mathbb{K}$-action on $V$. Finally, $\phi_{s,t}(x, 0, st) = (x, 0, t)$, so units are preseved. Thus $\phi_{s,t}$ is a morphism. Given $f : U \to W$, we compute

$$\begin{aligned} f^{\{1\}} \circ \phi_{s,t}(x, v; st) &= (f(x), f^{[1]}(x, vs; t), t), \\ \phi_{s,t} \circ f^{\{1\}}(x, v; st) &= (f(x), f^{[1]}(x, v; st)s, t), \end{aligned}$$

and the last claims follow. (Note that $\mathbb{K}$ need not be commutative for all this.) □

[4] If we drop the Hausdorff assumption, then the same arguments show that $M_1$ is an open neighborhood of the diagonal in $M \times M$; this open neighborhood will in general not be a groupoid, but a *local groupoid*, in the sense of [Ko97].

*Remark* 3.1. When $s$ is invertible, we get the *scaling automorphism* of $M^{\{1\}}$, given by $(x, v; t) \mapsto (x, vs; s^{-1}t)$ (see Example 2.3). For $s = -1$, we get the important automorphism $(x, v; t) \mapsto (x, -v; -t)$.

We interprete $\phi_{s,t}$ as a *scaling morphism*, where $s$ is the scalar acting; morphisms with different scaling level $t$ on target spaces have to be distinguished. It is quite remarkable that such structure fits together with the groupoid structure from the preceding chapter into a *double category*, which we will define next.

3.2. **The first order double category.** We will define a (small) double category

$$
\begin{array}{ccc} C_{11} & \rightrightarrows & C_{01} \\ \downdownarrows & & \downdownarrows \\ C_{10} & \rightrightarrows & C_{00} \end{array} :
\qquad
\begin{array}{ccc} U^{\overline{\{1\}}} & \overset{\pi}{\rightrightarrows} & U \times \mathbb{K} \times \mathbb{K} \\ \partial \downdownarrows & & \downdownarrows \partial \\ U^{\{1\}} & \overset{\pi}{\rightrightarrows} & U \times \mathbb{K} \end{array}
\tag{3.1}
$$

(see Appendix C for definitions), as follows: for a linear set $(U, V)$, define its *first double prolongation* by

$$
\begin{aligned}
U^{\overline{\{1\}}} &:= \{(x, v; s, t) \in V^2 \times \mathbb{K}^2 \mid x \in U,\ x + vst \in U\} \\
&= \{(x, v; s, t) \in V^2 \times \mathbb{K}^2 \mid (x, v; st) \in U^{\{1\}}\}.
\end{aligned}
\tag{3.2}
$$

This comes ready-made with the following projections (the last two have already been defined):

$$
\begin{aligned}
\partial_0 : U^{\overline{\{1\}}} \to U^{\{1\}}, &\quad (x, v; s, t) \mapsto (x, v; st) \\
\partial_1 : U^{\overline{\{1\}}} \to U^{\{1\}}, &\quad (x, v; s, t) \mapsto (x, vs; t) \\
\partial_0 : U \times \mathbb{K}^2 \to U \times \mathbb{K}, &\quad (x; s, t) \mapsto (x; st) \\
\partial_1 : U \times \mathbb{K}^2 \to U \times \mathbb{K}, &\quad (x; s, t) \mapsto (x; t) \\
\pi_0 : U^{\overline{\{1\}}} \to U \times \mathbb{K}^2, &\quad (x, v; s, t) \mapsto (x; s, t) \\
\pi_1 : U^{\overline{\{1\}}} \to U \times \mathbb{K}^2, &\quad (x, v; s, t) \mapsto (x + vst; s, t) \\
\pi_0 : U^{\{1\}} \to U \times \mathbb{K}, &\quad (x, v; t) \mapsto (x; t) \\
\pi_1 : U^{\{1\}} \to U \times \mathbb{K}, &\quad (x, v; t) \mapsto (x + vt; t)
\end{aligned}
$$

**Lemma 3.2.** *For* $i, j \in \{0, 1\}$*:* $\boxed{\partial_i \circ \pi_j = \pi_j \circ \partial_i : U^{\overline{\{1\}}} \to U \times \mathbb{K}}$

*Proof.* By direct computation,

$$
\begin{aligned}
\partial_1 \pi_0(x, v; s, t) &= (x, t) = \pi_0 \partial_1(x, v; s, t), \\
\partial_0 \pi_0(x, v; s, t) &= (x, st) = \pi_0 \partial_0(x, v; s, t) \\
\partial_0 \pi_1(x, v; s, t) &= (x + vst, st) = \pi_1 \partial_0(x, v; s, t) \\
\partial_1 \pi_1(x, v; s, t) &= (x + vst, t) = \pi_1 \partial_1(x, v; s, t)
\end{aligned}
$$

□

Next define "unit (resp. zero) sections"

$$\begin{aligned} z_\pi : U \times \mathbb{K} \to U^{\{1\}}, &\quad (x;t) \mapsto (x,0;t) \\ z_\partial : U \times \mathbb{K} \to U \times \mathbb{K}^2, &\quad (x;t) \mapsto (x;1,t) \\ z_\partial : U^{\{1\}} \to U^{\overline{\{1\}}}, &\quad (x,v;t) \mapsto (x,v;1,t) \\ z_\pi : U \times \mathbb{K}^2 \to U^{\overline{\{1\}}}, &\quad (x;s,t) \mapsto (x,0;s,t) \end{aligned}$$

It is immediately checked that $\boxed{z_\pi \circ z_\partial = z_\partial \circ z_\pi : U \times \mathbb{K} \to U^{\overline{\{1\}}} : \ (x,t) \mapsto (x,0;1,t)}$:

$$\begin{array}{ccc} U^{\overline{\{1\}}} & \leftarrow & U \times \mathbb{K}^2 \\ \uparrow & & \uparrow \\ U^{\{1\}} & \leftarrow & U \times \mathbb{K} \end{array}$$

**Lemma 3.3.** *The maps $z$ are bisections of the projections defined above, that is, for $i = 0,1$,* $\boxed{\partial_i \circ z_\partial = \mathrm{id},\ \pi_i \circ z_\pi = \mathrm{id}}$.

*Proof.* Immediate, since 0 appears in the definition of $z_\pi$ and 1 in the one of $z_\partial$. □

**Lemma 3.4.** *For $i = 0,1$, we have* $\boxed{\partial_i \circ z_\pi = z_\pi \circ \partial_i : U \times \mathbb{K}^2 \to U^{\{1\}}}$.

*Proof.* For $i = 0$, $z_\pi \partial_0(x;s,t) = z_\pi(x;st) = (x,0;st) = \partial_0(x,0;s,t) = \partial_0 z_\pi(x;s,t)$ and similarly for $i = 1$. □

**Lemma 3.5.** *For $i = 0,1$, we have* $\boxed{\pi_i \circ z_\partial = z_\partial \circ \pi_i : U^{\{1\}} \to U \times \mathbb{K}^2}$

*Proof.* $z_\partial \pi_1(x,v;s) = z_\partial(x+vs;s) = (x+vs;1,s) = \pi_1(x,v;1,s) = \pi_1 z_\partial(x,v;s)$ for $i = 1$. Similarly for $i = 0$. □

Now we define composition of morphisms. In the following formulae, we assume that $a = (x,v;s,t), a' = (x',v';s',t') \in U^{\overline{\{1\}}}$. The two compositions $*$ are "additive" and the two compositions $\bullet$ are "multiplicative":

(1) if $\pi_1(a) = \pi_0(a')$ (so $s' = s, t' = t, x' = x + vst$) , we define $a' * a \in U^{\overline{\{1\}}}$:

$$\boxed{a' * a = (x',v';s,t) * (x,v;s,t) = (x+vst,v';s,t) * (x,v;s,t) = (x,v+v';s,t)}.$$

(2) For $(x,v;t),(x',v',t') \in U^{\{1\}}$ such that $\pi_1(x,v,t) = \pi_0(x',v',t')$, (so $t' = t$, $x' = x + vt$), as in the preceding section,

$$\boxed{(x',v';t) * (x,v;t) = (x,v+v';t)}.$$

(3) If $\partial_1(a) = \partial_0(a')$ (so $x = x'$, $v' = vs$ and $t = s't'$), then define $a' \bullet a$

$$\boxed{a' \bullet a = (x,v';s',t') \bullet (x,v;s,t) = (x,vs,s',t') \bullet (x,v,s,s't') = (x,v;ss',t')}.$$

(4) If $\partial_1(x;s,t) = \partial_0(x';s',t')$ (so $x' = x$ and $t = s't'$), let

$$\boxed{(x;s',t') \bullet (x;s,t) = (x;ss',t')}.$$

**Theorem 3.6** (First order double category)**.**

*(1) The data $(U^{\overline{\{1\}}}, U^{\{1\}}, U \times \mathbb{K}^2, U \times \mathbb{K}, \pi, \partial, z, *, \bullet)$ define a double category which we denote by $\mathbf{U}^{\overline{\{1\}}}$ and indicate by a diagram of the form (3.1).*

(2) *Morphisms of double categories* $\mathbf{f} : \mathbf{U}^{\overline{\{1\}}} \to \mathbf{W}^{\overline{\{1\}}}$ *which are trivial on* $\mathbb{K}$ *are in 1:1-correspondence with maps* $f : U \to W$ *together with a difference factorizer* $f^{[1]}$ *satisfying (1.3) and (1.4).*

(3) *The unique map* $U \to 0$ *induces a canonical morphism of the* $\bullet$*-category to the left action category of* $(\mathbb{K}, \cdot)$*:*

$$\begin{array}{ccccc} U^{\overline{\{1\}}} & \overset{\pi}{\rightrightarrows} & U \times \mathbb{K} \times \mathbb{K} & \to & \mathbb{K} \times \mathbb{K} \\ \partial \downdownarrows & & \downdownarrows \partial & & \downdownarrows \\ U^{\{1\}} & \overset{\pi}{\rightrightarrows} & U \times \mathbb{K} & \to & \mathbb{K} \end{array}$$

(4) *The maps* $j(x, v; s, t) := (x + vst, -v; t, s)$*, resp.* $j(x, v; t) := (x + vt, -v; t)$*,* $j(x; s, t) = (x, s, t)$*, and* $j(x; t) = (x; t)$*, define an isomorphism of double categories* $(*, \bullet) \to (*^{op}, \bullet)$*.*

(5) *The inverse image of the left action category* $((\mathbb{K}^\times)^2, \mathbb{K}^\times)$ *under the projections from Item (3) forms a double groupoid, denoted by* $(\mathbf{U}^{\overline{\{1\}}})^\times$*, which is isomorphic to the double groupoid given by the direct product of categories* $U \times U$ *(pair groupoid) and* $\mathbb{K}^\times \times \mathbb{K}^\times$ *(pair groupoid).*

(6) *If* $\mathbb{K}$ *is a topological ring and* $M$ *a Hausdorff* $C^1$*-manifold, then there is a continuous double category* $M^{\overline{\{1\}}}$ *over* $M \times \mathbb{K}$*, and statements analoguous to those of Theorem 2.11 hold. Continuous morphisms* $M^{\overline{\{1\}}} \to N^{\overline{\{1\}}}$ *which are trivial on* $\mathbb{K}$ *are in bijection with maps* $f : M \to N$ *of class* $C^1_{\mathbb{K}}$*.*

*Proof.* (1) We check that properties (1) – (9) from Theorem C.1 hold. We have already checked the compatibility conditions for target and source projections and for the unit sections (Lemmas above), and we have seen in the preceding section that $(U^{\{1\}}, *)$ is a category. Similar computations show that $(U^{\overline{\{1\}}}, *)$ is a category, too. For any fixed $x$, $(U^{\overline{\{1\}}}, \partial, \bullet)$ corresponds to the *scaled action category* from Lemma B.1, Appendix B (with $S$ the monoid $(\mathbb{K}, \cdot)$), and hence we have $\bullet$-categories. Let us show that projections $\pi, \partial$ are morphisms between the respective categories. We write $a' = (x', v'; s', t'), a = (x, v; s, t)$, and when writing compositions $a' \bullet a$ and $a' * a$, it is understood that these compositions are defined.

$$\begin{aligned} \pi_0(a' \bullet a) &= \pi_0(x, v; ss', t') = (x; ss', t') = (x', s', t') \bullet (x, s, t) = \pi_0(a') \bullet \pi_0(a) \\ \pi_1(a' \bullet a) &= \pi_1(x, v; ss', t') = (x + vss't'; ss', t') \\ &= (x + v's't'; s', t') \bullet (x + vst; s, t) = \pi_1(a') \bullet \pi_1(a) \\ \partial_0(a' * a) &= \partial_0(x, v' + v; s, t) = (x, v' + v; st) = \partial_0(a') * \partial_0(a) \\ \partial_1(a' * a) &= \partial_1(x, v' + v; s, t) = (x, (v' + v)s; t) = (x, v's + vs; s) = \partial_1(a') * \partial_1(a) \end{aligned}$$

In the last line we used distributivity in $V$. Next, the bisections $z$ are functors: $z_\partial(a' * a) = z_\partial(a') * z_\partial(a), z_\pi(b' \bullet b) = z_\pi(b') \bullet z_\pi(b')$. Indeed,

$$(x, v' + v; 1, t) = (x, v'; 1, t) * (x, v; 1, t),$$
$$(x, 0; ss', t) = (x, 0; s', t') \bullet (x, 0; s, t).$$

Finally, let us prove the *interchange law* (C.1):

$$\begin{aligned} &\big((x', v'; s', t') * (x, v; s, t)\big) \bullet \big((y', w'; p', q') * (y, w; p, q)\big) \\ = \;& (x, v + v'; s, t) \bullet (y, w + w'; p', q') \\ = \;& (y, w + w'; ps, t) \end{aligned}$$

$$\begin{aligned} &= (y', w'; p's', t') * (y, w; ps, t) \\ &= \big((x', v'; s', t') \bullet (y', w'; p', q')\big) * \big((x, v; s, t) \bullet (y, w; p, q)\big) \end{aligned}$$

This proves that $U^{\overline{\{1\}}}$ is a double category.

(2) Let $\mathbf{f}$ be a morphism, that is, a double functor from $\mathbf{U}$ to $\mathbf{W}$, and assume that it is trivial on $\mathbb{K}$. Denote by $f : U \to W$ the corresponding map on the base and by $f^{\{1\}} : U^{\{1\}} \to W^{\{1\}}$ and $f^{\overline{\{1\}}} : U^{\overline{\{1\}}} \to W^{\overline{\{1\}}}$ the corresponding maps. As in the proof of Theorem 2.8 and Lemma 2.2, we get $f^{\{1\}}(x, v; t) = (f(x), f^{[1]}(x, v, t), t)$ with a difference factorizer $f^{[1]}$ satisfying (1.2). From compatiblity with $\pi$ we gt

$$f^{\overline{\{1\}}}(x, v; s, t) = \big(f(x), F(x, v; s, t); s, t\big)$$

with some map $F : U^{\overline{\{1\}}} \to W$. From $\partial_0 \circ f^{\overline{\{1\}}} \to f^{\{1\}} \circ \partial_0$, it follows that $F(x, v; s, t) = f^{[1]}(x, v, st)$, and from $f^{\{1\}} \circ \partial_1 = \partial_1 \circ f^{\overline{\{1\}}}$ we now get $f^{[1]}(x, sv; t) = f^{[1]}(x, v; st)\, s$, that is (1.4). If these conditions hold, the property $f^{\overline{\{1\}}}(a' \bullet a) = f^{\overline{\{1\}}}(a') \bullet f^{\overline{\{1\}}}(a)$ is proved without further assumptions. Recall that (1.3) corresponds to the property $f^{\{1\}}(a' * a) = f^{\{1\}}(a') * f^{\{1\}}(a)$. Finally, all computations can be reversed, so that a base map $f$ together with a difference factorizer satisfying (1.3), (1.4) defines a double functor $\mathbf{f}$.

(3) This is proved by direct computation (cf. Lemma B.2), or by using that the constant map $U \to 0$ induces a morphism (Lemma 4.9).

(4) The map $j$ is the inversion map of the $*$-groupoids. The statement holds more generally for double categories two of whose edges are groupoids; in the present case it can of course also be checked by direct computations.

(5) The trivialization map is

$$\mathbf{U}^\times \to U^2 \times (\mathbb{K}^\times)^2, \quad (x, v; s, t) \mapsto (y, x; u, t) := (x + vst, x; st, t)$$

with inverse map $(y, x; u, t) \mapsto (x, v; s, t) = (x, (y - x)u^{-1}; ut^{-1}, t)$.

(6) The same arguments as in the proof of Theorem 2.11 apply. □

**Theorem 3.7** (General morphisms)**.** *Assume given a map $f : U \to W$ and two maps $\varphi : \mathbb{K} \to \mathbb{K}$, $\psi : \mathbb{K} \to \mathbb{K}$. Then a map of the form*

$$U^{\overline{\{1\}}} \to W^{\overline{\{1\}}}, \quad (x, v; s, t) \mapsto (f(x), G(x, v; s, t); \varphi(s), \psi(t))$$

*is a morphism of double categories if and only if, whenever defined:*

*(1) there is a map $F$ such that $G(x, v; s, t) = F(x, v; st)$,*
*(2) $f(x + vst) = f(x) + F(x, v; st)\, \varphi(s)\psi(t)$,*
*(3) $\varphi(st) = \varphi(s)\psi(t)$,*
*(4) $F(x + vt, v', t) + F(x, v, t) = F(x, v' + v, t)$,*
*(5) $F(x, vs; t) = F(x, v; st)\varphi(s)$.*

*Proof.* Similar to the proof of Theorem 2.9. □

**Definition 3.8.** *A morphism with $\varphi = \psi = \mathrm{id}_{\mathbb{K}}$ is called* of the first kind (spacial)*, and a morphism with $f = \mathrm{id}$ is called* of the second kind (internal)*.*

One can give examples similar to those following Theorem 2.9: conjugate-linear maps define morphisms (then $\varphi = \psi$ must be a ring automorphism), and there are scaling automorphisms (then $\varphi(t) = \lambda t$, $\psi(t) = t$, $F(x, v, t) = v\lambda^{-1}$ for $\lambda \in \mathbb{K}^\times$).

## 4. Laws of class $C^1$ over arbitrary rings

4.1. **Definition and first properties.** In this section we define the framework of *local linear algebra*: we develop (first order) "calculus" over arbitrary base rings $\mathbb{K}$. Just like *polynomial laws* generalize polynomial maps, *laws of class $C^1$* generalize usual differentiable maps. In Part II, laws of class $C^n$ and $C^\infty$ will be defined.

**Definition 4.1** ($C^1_{\mathbb{K}}$-laws). *Let $(U, V_1)$, $(W, V_2)$ be linear sets. A $C^1_{\mathbb{K}}$-*law between *$U$* and *$W$ is a morphism of the first kind between double categories, $\mathbf{f} : \mathbf{U}^{\overline{\{1\}}} \to \mathbf{W}^{\overline{\{1\}}}$. Thus $\mathbf{f}$ is given by four set-maps*

$$\mathbf{f}^{\overline{\{1\}}} : U^{\overline{\{1\}}} \to W^{\overline{\{1\}}}, \qquad \mathbf{f}^{\{1\}} : U^{\{1\}} \to W^{\{1\}},$$
$$f \times \mathrm{id}_{\mathbb{K}} \times \mathrm{id}_{\mathbb{K}} : U \times \mathbb{K}^2 \to W \times \mathbb{K}^2, \qquad f \times \mathrm{id}_{\mathbb{K}} : U \times \mathbb{K} \to W \times \mathbb{K},$$

*satisfying the conditions from Theorem 3.6. Equivalently, $\mathbf{f}$ is given by a base map $f : U \to W$ and a difference factorizer $\mathbf{f}^{[1]}$ satisfying (1.3) and (1.4). We then say that $\mathbf{f}$ is a $C^1_{\mathbb{K}}$-law* over *$f$. Obviously, linear sets with $C^1_{\mathbb{K}}$-laws as morphisms form a (big) concrete category which we denote by* $\underline{C^1_{\mathbb{K}}\text{-linset}}$. *The set of all $C^1_{\mathbb{K}}$-laws from $U$ to $W$ will be denoted by $C^1_{\mathbb{K}}(U, W)$.*

**Definition 4.2** (Underlying $C^1_{\mathbb{Z}}$-law). *With notation as above, $\mathbf{f}$ has an* underlying $C^1_{\mathbb{Z}}$-law $\mathbf{f}_{\mathbb{Z}}$*, by restricting scalars in (1.3) and (1.4) to $\mathbb{Z}$.*

*Example* 4.1. If $\mathbb{K}$ is a topological ring and $V, W$ topological $\mathbb{K}$-modules, then a $C^1_{\mathbb{K}}$-map (in the sense of topological differential calculus) $f : U \to W$ gives rise to a law $\mathbf{f} : \mathbf{U} \to \mathbf{W}$. Indeed, the continuous difference factorizer of $f$ gives rise to a (continuous) morphism of double categories $f^{\overline{\{1\}}}$, see Theorem 3.6, Item (5). We call this the *law defined by $f$*.

*Remark* 4.1. We use the boldface letters in order to stress that $\mathbf{f}$ is in general not uniquely determined by the base map $f$. For instance, if $U = \{x_0\}$ is a singleton, then $f$ is a constant map, whence $f^{[1]}(x_0, v, t) = 0$ for all $t \in \mathbb{K}^\times$, and the values $f^{[1]}(x_0, v, 0)$ can be chosen independently of $f$.

If there is no risk of confusion, we will occasionally switch back to the notation $f^{\overline{\{1\}}}$, $f^{\{1\}}$ instead of $\mathbf{f}^{\overline{\{1\}}}$, $\mathbf{f}^{\{1\}}$, and, keeping in mind that these need not be determined by $f$, we nevertheless think of $f^{\overline{\{1\}}}$ as a sort of "first derivative of $f$".

**Definition 4.3** (Tangent map). *Given a $C^1_{\mathbb{K}}$-law $\mathbf{f}$ with base map $f$ and difference factorizer $\mathbf{f}^{[1]}$, and if $t \in \mathbb{K}$ is fixed, we write*

$$\mathbf{f}_t : U_t = \{(x, v) \in U \times V, x + vt \in U\} \to W \times W, \quad (x, v) \mapsto (f(x), \mathbf{f}^{[1]}(x, v, t)).$$

*For $t = 0$, this map is called the* tangent map *of $\mathbf{f}$, also denoted by*

$$T\mathbf{f} := \mathbf{f}_0 : TU := U \times V \to TW := W \times W,$$
$$(x, v) \mapsto T\mathbf{f}(x)v := (f(x), d\mathbf{f}(x)v) := (f(x), \mathbf{f}^{[1]}(x, v; 0)).$$

By definition of composition of morphisms, we have the functorial rule

$$(\mathbf{g} \circ \mathbf{f})_t = \mathbf{g}_t \circ \mathbf{f}_t, \tag{4.1}$$

which for $t = 0$ is the "chain rule"

$$T(\mathbf{g} \circ \mathbf{f}) = T\mathbf{g} \circ T\mathbf{f}. \tag{4.2}$$

Written out in terms of the difference factorizers, Equation (4.1) reads

$$(\mathbf{g} \circ \mathbf{f})^{[1]}(x, v, t) = \mathbf{g}^{[1]}(f(x), \mathbf{f}^{[1]}(x, v, t), t). \tag{4.3}$$

**Lemma 4.4.** *Let $\mathbf{f} : \mathbf{U} \to \mathbf{W}$ be a $C^1$-law and $x \in U$. Then the differential $d\mathbf{f}(x) : V \to W$ is a $\mathbb{K}$-linear map.*

*Proof.* As said above, $\mathbf{f}^{[1]}$ satisfies (1.3), (1.4). For $t = 0$, this yields the claim. □

4.2. **Constant laws.** The following will carry over to general manifolds and spaces:

**Definition 4.5.** *A* constant $C^1_{\mathbb{K}}$-law *is a law $\mathbf{f} : \mathbf{U} \to \mathbf{W}$ such that $\mathbf{f}^{\{1\}} = z_\pi \circ f \circ \pi_0$.*

**Lemma 4.6.** *A law $\mathbf{f}$ is constant if, and only if, its difference factorizer vanishes: $\mathbf{f}^{[1]} = 0$. There is a 1:1-correspondence between constant laws and constant maps:*

$$\boxed{f^{\overline{\{1\}}}(x, v, s, t) = (c, 0; s, t)}.$$

*Proof.* The first statement is obvious from the formula defining $\mathbf{f}^{\{1\}}$. Note that $\mathbf{f}^{[1]} = 0$ satisfies (1.2) – (1.4), hence indeed defines a law.

Whenever $\mathbf{f}^{[1]} = 0$, the base map must be constant since then $f(x + v) - f(x) = \mathbf{f}^{[1]}(x, v, 1) = 0$ whenever $x, x + v \in U$. Conversely, when the base map is constant, then the zero map certainly is a possible difference factorizer for $f$. □

*Remark* 4.2. If $\mathbf{f}^{[1]} = 0$, then $d\mathbf{f} = 0$, but the converse need not hold. Note that, even in "usual" ultrametric calculus this need not be true – cf. remarks in [BGN04].

**Lemma 4.7.** *Let $(U, V)$ be a linear set. Then there is a unique $C^1_{\mathbb{K}}$-law from $U$ to the linear set $0 = (\{0\}, \{0\})$ (Conventions 0.5), given by $f^{\overline{\{1\}}}(x, v; s, t) = (0, 0; s, t)$.*

*Proof.* The law is induced by the constant map $U \to 0$ (cf. Part (3) of Th. 3.6). □

4.3. **Linear laws.** The following uses the linear structure of $V$ and $W$:

**Definition 4.8.** *A* linear $C^1_{\mathbb{K}}$-law *is a $C^1_{\mathbb{K}}$-law $\mathbf{f} : \mathbf{V} \to \mathbf{W}$ such that the map $\mathbf{f}^{\overline{\{1\}}} : V^2 \times \mathbb{K}^2 = V^2 \oplus \mathbb{K}^2 \to W^2 \oplus \mathbb{K}^2$ is $\mathbb{K}$-linear.*

**Lemma 4.9.** *There is a 1:1-correspondence between $\mathbb{K}$-linear maps $f : V \to W$ and linear $C^1_{\mathbb{K}}$-laws $\mathbf{f} : \mathbf{V} \to \mathbf{W}$, given by*

$$\boxed{f^{\overline{\{1\}}}(x, v; s, t) = (f(x), f(v); s, t)}.$$

*In particular, a linear law is uniquely determined by its base map.*

*Proof.* A $\mathbb{K}$-linear map $f : V \to W$ gives rise to a morphism $f^{\overline{\{1\}}}(x, v; s, t) := (f(x), f(v); s, t)$ (see Example 2.1), and obviously this map is $\mathbb{K}$-linear.

Conversely, if $\mathbf{f}$ is a linear law, then $(f(x), 0; 0, 0) = \mathbf{f}^{\overline{\{1\}}}(x, 0; 0, 0)$ is linear in $x$, hence $f$ is linear, and and similarly $\mathbf{f}^{[1]} : V^2 \oplus \mathbb{K} \to W$ is also linear. Thus we have, for all $t \in \mathbb{K}$,

$$\begin{aligned}\mathbf{f}^{[1]}(x, v, t) &= \mathbf{f}^{[1]}\big((x, v, 1) + (0, 0, t - 1)\big) = \mathbf{f}^{[1]}(x, v, 1) + \mathbf{f}^{[1]}(0, 0, t - 1) \\ &= \mathbf{f}^{[1]}(x, v, 1) + 0 = f(x + v) - f(x) = f(v),\end{aligned}$$

and hence, for a linear law, $\mathbf{f}$ is uniquely determined by its base map $f$. □

*Example* 4.2. The addition map $a : V \times V = V \oplus V \to V$ is $\mathbb{K}$-linear, hence corresponds to the linear law, called the *addition law of* $V$,

$$a^{\overline{\{1\}}}((x, y), (u, v); s, t) = (x + y, u + v; s, t).$$

For fixed $t, s$, this is addition in $V^2$.

*Example* 4.3. The diagonal map $\delta : V \to V \times V = V \oplus V$, $x \mapsto (x, x)$ is linear. It corresponds to the *diagonal law*

$$\delta^{\overline{\{1\}}}(x, v; s, t) = ((x, x), (v, v); s, t)$$

For fixed $(s, t)$, this is the diagonal imbedding of $V^2$ in $V^4$.

4.4. **Bilinear laws, and algebra laws.** Two preliminary remarks:

(1) Note that we have, for $M \subset V_1$ and $N \subset V_2$, like in Item (6) of Theorem 2.11, a natural isomorphism of bundles over $M \times N \times \mathbb{K}^2$

$$(M \times N)^{\overline{\{1\}}} \quad \cong \quad M^{\overline{\{1\}}} \times_{\mathbb{K}^2} N^{\overline{\{1\}}}$$

by identifying $((x, y), (u, v); s, t)$ with $((x, u), (y, v); s, t)$ (which projects to $(x, y; s, t)$). We will use these identifications frequently.

(2) In this subsection (and in the following ones) we have to assume that $\mathbb{K}$ is commutative (and then we prefer to write modules as *left* modules).

**Theorem 4.10.** *Assume $f : V_1 \times V_2 \to W$ is a $\mathbb{K}$-bilinear map. Then the following formulae define a $C^1_{\mathbb{K}}$-law $\mathbf{f} : \mathbf{V}_1 \times \mathbf{V}_2 \to \mathbf{W}$:*

$$\begin{aligned}\mathbf{f}^{\{1\}}((x, u), (y, v); t) &= \big(f(x, y), f(x, v) + f(u, y) + tf(u, v); t\big) \\ \mathbf{f}^{\overline{\{1\}}}((x, u), (y, v); s, t) &= \big(f(x, y), f(x, v) + f(u, y) + stf(u, v); s, t\big)\end{aligned}$$

*Proof.* Since $f$ is bilinear, $f((x, y)+t(u, v))-f(x, y) = t(f(x, v)+f(u, y)+tf(u, v))$, hence $f^{[1]}((x, y), (u, v), t) = f(x, v) + f(u, y) + tf(u, v)$ is a difference factorizer for $f$. It satisfies (1.3) and (1.4), hence $\mathbf{f}$ indeed defines a $C^1$-law. □

**Definition 4.11.** *A* bilinear law *is a law coming from a bilinear map, as in the theorem. If, moreover, $V_1 = V_2 = W$, then the law is called an* algebra law.

Thus, by definition, there is a bijection between bilinear base maps and their bilinear laws. For any fixed $t$, the map $f_t$ is again bilinear; however, $\mathbf{f}^{\{1\}}$, seen as a polynomial, is already of degree 3. If $\mathbf{f}$ is an algebra law, we often write $x \cdot y := f(x, y)$

(which does not mean that we assume the product to be associative), and then the formula for $f_t$ reads

$$(x,u)\cdot(y,v) := f_t\big((x,u),(y,v)\big) = (xy, xv+uy+tuv). \tag{4.4}$$

**Theorem 4.12.** *Assume $f : V\times V \to V$, $(x,y)\mapsto f(x,y) = x\cdot y$ is a bilinear map. For fixed $t\in\mathbb{K}$, the set of composable elements in the category $(V_t,*)$,*

$$V_t \times_V V_t = \{((x',v'),(x,v)) \in V_t\times V_t \mid x' = x+tv\}$$

*is a subalgebra of the direct product algebra $V_t\times V_t$, and the law $*$ of this category,*

$$\alpha : V_t\times_V V_t \to \mathbb{K}_t, \quad (x',v';x,v)\mapsto (x',v')*(x,v) = (x, v'+v)$$

*is a morphism of $\mathbb{K}$-algebras.*

*Proof.* The first claim follows from computing in $V_t\times V_t$

$$((x',v');(x,v))\cdot((y',w');(y,w)) = ((x'y', x'w'+v'y'+tv'w');(xy, xw+vy+tvw))$$

and noting that $x'y' = (x+tv)(y+tw) = xy + t(vy+xw+tvw)$. Now we prove that $\alpha$ is a morphism of algebras:

$$\begin{aligned}
\alpha((x',v';x,v)\cdot(y',w';y,w)) &= (xy, x'w'+v'y'+tv'w'+xw+vy+tvw)\\
&= (xy, xw'+tvw'+v'y+tv'w+tv'w'+xw+vy+tvw)\\
\alpha(x',v';x,v)\cdot\alpha(y',w';y,w) &= (x,v+v')\cdot(y,w+w')\\
&= (xy, x(w+w')+(v+v')y+t(v+v')(w+w')).
\end{aligned}$$

Both terms coincide, hence $\alpha : V_t\times_V V_t\to V_t$ is an algebra morphism. □

The last claim of the theorem is a kind of *interchange law* (cf. equation (C.1)):

$$\big((x',v')\cdot(y',w')\big) * \big((x,v)\cdot(y,w)\big) = \big((x',v')*(x,v)\big)\cdot\big((y',w')*(y,w)\big) \tag{4.5}$$

Consider the following commutative diagram:

$$\begin{array}{ccc}
V^{\{1\}} & \to & 0^{\{1\}} = \mathbb{K}\\
\downarrow\downarrow & & \downarrow \mathrm{id}_{\mathbb{K}}\\
V\times\mathbb{K} & \to & 0\times\mathbb{K} = \mathbb{K}
\end{array} \tag{4.6}$$

The two vertical arrows indicate small categories (where the law $*$ on $0^{\{1\}}$ is trivial, but the one on $V^{\{1\}}$ is not), and the two horizontal arrows indicate bundles of products $\cdot$ indexed by $\mathbb{K}$. The whole thing satisfies properties similar to the ones of a small double category, except that $\cdot$ need not be associative or unital. If, however, the product $\cdot$ on $V$ is associative and unital, then $(V,\cdot)$ is a monoid, hence a small category with one object, and the theorem implies that diagram (4.6) defines a small double category with products $*$ and $\cdot$ . For the special case $V=\mathbb{K}$, with its bilinear ring product, we can identify the product on $V_t$ in terms of truncated polynomial rings, as follows:

**Lemma 4.13** (The ring laws)**.** *Let $a : \mathbb{K}\times\mathbb{K}\to\mathbb{K}$ and $m : \mathbb{K}\times\mathbb{K}\to\mathbb{K}$ be addition and multiplication in the (commutative) ring $\mathbb{K}$. For any $t\in\mathbb{K}$, these define maps*

$$a_t : \mathbb{K}_t\times\mathbb{K}_t\to\mathbb{K}_t, \qquad m_t : \mathbb{K}_t\times\mathbb{K}_t\to\mathbb{K}_t.$$

*Identifying $\mathbb{K}_t$ with $\mathbb{K}^2$, the maps $a_t$ and $m_t$ define a ring structure on $\mathbb{K}^2$, which is isomorphic to the ring $\mathbb{K}[X]/(X^2-tX)$ with $\mathbb{K}$-basis $[1]$ and $[X]$.*

*Proof.* The ring structure on $\mathbb{K}[X]/(X^2 - tX)$ is given by $[X^2] = t[X]$ whence

$$(x[1] + u[X])(y[1] + v[X]) = xy[1] + (xv + uy + tuv)[X].$$

Comparing with (4.4), we see that the bilinear product on $\mathbb{K}_t$ is given by the same formula, and hence $\mathbb{K}_t$ is a ring, isomorphic to $\mathbb{K}[X]/(X^2 - tX)$. □

The neutral element of $\mathbb{K}_t$ is $e = (1, 0)$. We identify $\mathbb{K}$ with the subalgebra $\mathbb{K}e$ and thus consider $\mathbb{K}_t$ as $\mathbb{K}$-algebra. According to Theorem 4.12, for fixed $t \in \mathbb{K}$, the set $\mathbb{K}_t \times_{\mathbb{K}} \mathbb{K}_t = \{((x', v'), (x, v) \in \mathbb{K}_t \times \mathbb{K}_t \mid x' = x + tv\}$ is a subalgebra of the direct product algebra $\mathbb{K}_t \times \mathbb{K}_t$, and the law $*$ of this category,

$$* : \mathbb{K}_t \times_{\mathbb{K}} \mathbb{K}_t \to \mathbb{K}_t, \quad (x', v'; x, v) \mapsto (x', v') * (x, v) = (x, v' + v)$$

is a morphism of $\mathbb{K}$-algebras.

*Example* 4.4 (Module laws). Left multiplication by scalars, $m_V : \mathbb{K} \times V \to V$, $(\lambda, v) \mapsto \lambda v$, is $\mathbb{K}$-bilinear, hence gives rise to a law

$$m_V^{\overline{\{1\}}} : \mathbb{K}^{\overline{\{1\}}} \times_{\mathbb{K}^2} V^{\overline{\{1\}}} \to V^{\overline{\{1\}}}.$$

For any $t \in \mathbb{K}$, we get a map $(m_V)_t : \mathbb{K}_t \times V_t \to V_t$. Writing explicitly the formulae, one sees that this map describes the action of the ring $\mathbb{K}_t = \mathbb{K}[X]/(X^2 - tX)$ on the scalar extended module $V_t = V \otimes_{\mathbb{K}} (\mathbb{K}[X]/(X^2 - tX))$, given by $(r, s) \cdot (x, v) = (rx, rv + sx + tsv)$. We call this the $\mathbb{K}$-*module law of* $V$.

4.5. **Polynomial laws.** Informally, a polynomial law is given by a map together with all possible scalar extensions. We recall from [Ro63] the relevant definitions (see also [Lo75], Appendix); the base ring $\mathbb{K}$ is assumed to be commutative.

**Definition 4.14.** *Denote by* $\underline{\mathrm{Alg}_{\mathbb{K}}}$ *the (big) concrete category of unital commutative* $\mathbb{K}$*-algebras and* $\underline{\mathrm{Set}}$ *the concrete category of sets and mappings. Any* $\mathbb{K}$*-module* $V$ *gives rise to a functor* $\underline{V} : \underline{\mathrm{Alg}_{\mathbb{K}}} \to \underline{\mathrm{Set}}$ *by associating to* $\mathbb{A}$ *the scalar extended module* $V_{\mathbb{A}} = V \otimes_{\mathbb{K}} \mathbb{A}$ *and to* $\phi : \mathbb{A} \to \mathbb{B}$ *the induced map* $\phi_V := \mathrm{id} \otimes_{\mathbb{K}} \phi : V_{\mathbb{A}} \to V_{\mathbb{B}}$.

*A* polynomial law between $V$ and $W$ *is defined to be a natural transformation* $P : \underline{V} \to \underline{W}$*, i.e., for every* $\mathbb{K}$*-algebra* $\mathbb{A}$ *we have a map* $P_{\mathbb{A}} : V_{\mathbb{A}} \to W_{\mathbb{A}}$*, compatible with algebra morphisms* $\phi : \mathbb{A} \to \mathbb{B}$ *in the sense that* $P_{\mathbb{B}} \circ \phi_V = \phi_W \circ P_{\mathbb{A}}$*. We say that* $P_{\mathbb{A}} : V_{\mathbb{A}} \to W_{\mathbb{A}}$ *is a* scalar extension *of the* base map $P_{\mathbb{K}} : V \to W$.

**Theorem 4.15** (Polynomial laws are $C^1$-laws)**.** *Every polynomial law* $P : \underline{V} \to \underline{W}$ *gives rise to a* $C^1_{\mathbb{K}}$*-law* $\mathbf{P} : \mathbf{V} \to \mathbf{W}$*. More precisely, if* $P$ *is a polynomial law, letting* $P_r := P_{\mathbb{K}[X]/(X^2 - rX)} : V_r \to W_r$*, the map* $P^{\overline{\{1\}}}$ *is obtained by*

$$P^{\overline{\{1\}}}(x, v; s, t) := (P_{st}(x, v); s, t).$$

*In particular, the tangent map* $TP : TV \to TW$ *is given by scalar extension by* dual numbers $\mathbb{K}[X]/(X^2)$*, and the differential* $dP(x) : V \to W$*,* $v \mapsto P_0(x, v)$ *is* $\mathbb{K}$*-linear.*

*Proof.* Let us prove that $P^{\overline{\{1\}}}$, defined as in the theorem, is a $C^1_{\mathbb{K}}$-law. This is done by showing that all relevant maps are induced by algebra morphisms. Note first that the two projections

$$\pi_0 : \mathbb{K}_t \to \mathbb{K}, \ (x, v) \mapsto x, \qquad \pi_1 : \mathbb{K}_t \to \mathbb{K}, \ (x, v) \mapsto x + tv$$

are algebra morphisms, and they induce the two projections $\pi_i := V_{\pi_i} : V_t \to V$. Thus, from the definition of polynomial laws, we get $\pi_i \circ P_t = P_{\mathbb{K}} \circ P_i$, hence

$$P^{[1]}(x,v,t) := \mathrm{pr}_2(P_t(x,v))$$

is a difference factorizer for the base map $P_{\mathbb{K}} : V \to W$. Let us show that $P^{[1]}$ satisfies (1.4). For any $(s,t) \in \mathbb{K}^2$, the map $\phi_{s,t} : \mathbb{K}_{st} \to \mathbb{K}_t$, $(x,v) \mapsto (x,sv)$ is an algebra morphism, as is immediately checked. It induces a map $\Phi_{s,t} : V_{st} \to V_t$, and by definition of polynomial laws, we then have $P_t \circ \Phi_{s,t} = \Phi_{s,t} \circ P_{ts}$. The computation given in the proof of Theorem 3.1 shows that then (1.4) holds.

Let us prove that $P^{[1]}$ satisfies (1.3). By Theorem 4.12, $* : \mathbb{K}_t \times_{\mathbb{K}} \mathbb{K}_t \to \mathbb{K}_t$ is an algebra morphism, and this morphism induces the category law $* : V_t \times_V V_t \to V_t$. By definition of a polynomial law, $P$ commutes with the induced maps, which means that $P_t(a' * a) = P_t(a') * P_t(a)$, or, equivalently, that $P_t$ satisfies (1.3).

Finally, for $r = 0$, $P_0$ is obtained by scalar extension with $\mathbb{K}_0 = \mathbb{K}[X]/(X^2)$. [5] $\square$

*Remark* 4.3. Constant, linear, and bilinear $C^1$-laws obviously come from polynomial laws, in the way described by the theorem.

*Remark* 4.4 (Formal laws). In the second part of his long paper [Ro63], Roby defines and investigates *formal laws* ("lois formelles"). They generalize formal power series. We will show in subsequent parts of this work that *formal laws are laws of class* $C^\infty$. The underlying linear set is $(\{0\}, V)$ (since 0 is they only point where all formal series converge).

## 5. Manifold laws of class $C^1$

5.1. **$C^1$-manifold laws over $\mathbb{K}$.** Using the general principles from Appendix D, subsets of $V$ can be glued together by using a specified set of laws ("atlas law"):

**Definition 5.1.** *Let $\mathbb{K}$ be a topological ring and $V$ a topological $\mathbb{K}$-module; we do not assume here that $\mathbb{K}^\times$ is dense in $\mathbb{K}$, so in particular the discrete topology on $\mathbb{K}$ and $V$ is admitted. A $C^1_{\mathbb{K}}$-*manifold law *modelled on $V$, denoted by $\mathbf{M}$, is given by $C^1_{\mathbb{K}}$-laws $(\mathbf{V}, \mathcal{T}_{\mathbf{V}}, (\mathbf{V}_{ij}, \phi_{ij})_{(i,j)\in I^2})$ in the sense of Theorem D.4, that is, base sets and base maps of these data form "gluing data" as described in that theorem, and likewise for the other components of the laws. For each $U = V_{ij}$, the topology on $U^{\{1\}}$ shall be the initial topology with respect to $\pi_1, \pi_0 : U^{\{1\}} \to U$. The manifold law is called* handy *if the base manifold $(V, (V_{ij}, \phi_{ij}))$ is handy in the sense of Definition D.2.*

*A $C^1_{\mathbb{K}}$-*law $\mathbf{f}$ *between manifold laws $\mathbf{M}$ and $\mathbf{N}$ is given by a family of $C^1_{\mathbb{K}}$-laws $\mathbf{f}_{ij}$ such that all components of the law are is an Theorem D.4.*

**Theorem 5.2.** *Let $\mathbf{M}$ be a manifold law over $\mathbb{K}$, modelled on $V$. Then there are primitive manifolds $M^{\{1\}}$, $\overline{M^{\{1\}}}$ together with projections and injections fitting into*

[5] The proof given here generalizes the one from [Be14] proving linearity of $dP(x)$. The proof of linearity given in [Ro63] (and in [Lo75]) is different, using heavily the decomposition of a polynomial into homogeneous parts, which is not adapted to the case $r \neq 0$.

*the diagram*

$$\begin{array}{ccccc} M^{\overline{\{1\}}} & \overset{\pi}{\rightrightarrows} & M \times \mathbb{K} \times \mathbb{K} & \to & \mathbb{K}^2 \\ \partial \downdownarrows & & \downdownarrows \partial & & \downdownarrows \\ M^{\{1\}} & \overset{\pi}{\rightrightarrows} & M \times \mathbb{K} & \to & \mathbb{K}\,. \end{array}$$

*There are also partially defined products $*$ and $\bullet$ defining, if the manifold law is handy, the structure of a double category on $M^{\overline{\{1\}}}$. If $\mathbf{N}$ is another manifold law over $\mathbb{K}$, modelled on $W$, then $C^1_{\mathbb{K}}$-laws $\mathbf{f}$ between $\mathbf{M}$ and $\mathbf{N}$ are precisely the morphisms of the structure defined by projections, injections and partially defined laws $*$ and $\bullet$.*

*Proof.* Existence of the sets $M^{\{1\}}$ and $M^{\overline{\{1\}}}$, as well as of the projections and injections, follows directly from the above definition combined with Theorem D.4. As in the proof of Theorems 3.6 and 2.11, the same holds for the definition of locally defined products $*$ and $\bullet$. As in these proofs, the only point that needs attention is when checking if $M^{\{1\}} \times_{(M\times\mathbb{K})} M^{\{1\}}$ is stable under $*$: for this, we infer the assumption that the manifold law is handy, as in the proof of Theorem 2.11. □

*Remark* 5.1. In the general case (handy or not), we obtain a *local small double category* $M^{\overline{\{1\}}}$. To avoid technicalities, we do not give formal definitions here (see, e.g., [Ko97]): the groupoid law is no longer defined on all of $M^{\{1\}} \times_{(M\times\mathbb{K})} M^{\{1\}}$, but only on an open neighborhood of the zero section; and similarly for $M^{\overline{\{1\}}}$.

5.2. **Towards more general categories of spaces.** We postpone the general theory of manifold laws to subsequent parts of this work. In guise of a conclusion, let us, however, already mention that this very general category of manifolds still has the drawbacks that the category of usual manifolds already has: (1) the lack of *inverse images*, (2) it is *not cartesian closed*:

*Remark* 5.2 (Inverse images). In general, the inverse image of a small [double] category (and of a [double] groupoid) under a morphism is again a small [double] category (resp. a [double] groupoid). Therefore, if $\mathbf{f} : \mathbf{M} \to \mathbf{N}$ is a morphism, and $c \in N$ a fixed element, then the inverse image of the "isolated point $c$" (sub-double category $\mathbf{c} := \{(c,0;s,t) \mid s,t \in \mathbb{K}\} \cong \mathbb{K}^2$) in $\mathbf{N}$ under $\mathbf{f}$ is again a sub-double category of $\mathbf{M}$. Explicitly, on the chart level, if $\mathbf{f} : U^{\overline{\{1\}}} \to W^{\overline{\{1\}}}$ is a law and $c \in W$ a fixed element, the inverse image

$$\begin{aligned} \mathbf{f}^{-1}(\mathbf{c}) &= \{(x,v;s,t) \in U^{\overline{\{1\}}} \mid f^{\overline{\{1\}}}(x,v;s,t) = (c,0;s,t)\} \\ &= \{(x,v;s,t) \in U^{\overline{\{1\}}} \mid f(x) = c,\ f^{[1]}(x,v,st) = 0\} \end{aligned} \tag{5.1}$$

is a sub-double category. However, $\mathbf{f}^{-1}(\mathbf{c})$ will in general not be a manifold: it may be a "singular space".

*Remark* 5.3 (Cartesian closedness). If $M$ and $N$ are usual manifolds, then the set $C^1(M,N)$ of $C^1$-morphisms from $M$ to $N$ is in general not a manifold. The same problem arises for any other kind of manifolds. On the other hand, the space of mappings from $U$ to a $\mathbb{K}$-module $W$ is always a linear space, with pointwise defined structure, having the additive maps as subspace (this is true for *commutative* groups

$(W,+)$ and fails for general groups). The following result can be interpreted by saying that the "locally linear maps" share this property (and the proof shows that commutativity of $(W,+)$ enters here in the same way):

**Theorem 5.3** (The double category structure on the set of $C^1$-laws)**.**

*(1) The set* $\mathrm{Hom}_{\mathbb{K}}(U^{\{1\}}, W^{\{1\}})$ *of groupoid morphisms of the first kind between* $U^{\{1\}}$ *and* $W^{\{1\}}$ *carries a natural groupoid structure, namely, the* pointwise groupoid structure *inherited from* $W^{\{1\}}$.

*(2) If* $\mathbb{K}$ *is commutative, then the set* $C^1_{\mathbb{K}}(U,W)$ *of* $C^1$*-laws from* $U$ *to* $W$ *carries a natural structure of small double category, given by the pointwise structure. In particular, for* $U=W$ *and* $t=0$*, we get the natural linear structure on the* space of vector fields.

*Proof.* (1) Let $\mathbf{f},\mathbf{g} \in \mathrm{Hom}_{\mathbb{K}}(U^{\{1\}}, W^{\{1\}})$. The two projections are $\mathbf{f} \mapsto \pi_i(\mathbf{f}) := \pi_i \circ \mathbf{f}$. Assume $\pi_1(\mathbf{g}) = \pi_0(\mathbf{f})$, that is,

$$\forall c \in U^{\{1\}}: \qquad \pi_1(\mathbf{g}(c)) = \pi_0(\mathbf{f}(c)), \tag{5.2}$$

and define a map by "pointwise product" $\mathbf{f} * \mathbf{g} : U^{\{1\}} \to W^{\{1\}}$, $a \mapsto \mathbf{f}(a) * \mathbf{g}(a)$. Because of (5.2) this is well-defined. We show that $\mathbf{f}*\mathbf{g}$ is again a groupoid morphism: let $a', a \in U^{\{1\}}$ such that $\pi_1(a) = \pi_0(a')$, so $a' * a$ is defined, hence

$$(\mathbf{f}*\mathbf{g})(a'*a) = \mathbf{f}(a'*a) * \mathbf{g}(a'*a) = \mathbf{f}(a') * \mathbf{f}(a) * \mathbf{g}(a') * \mathbf{g}(a).$$

From (5.2) with $c = a' * a$ we get

$$\pi_1 \mathbf{g}(a') = \pi_1 \mathbf{g}(a'*a) = \pi_0 \mathbf{f}(a'*a) = \pi_0 \mathbf{f}(a).$$

On the other hand, since $\mathbf{f}$ is a morphism, from $\pi_1(a) = \pi_0(a')$, it follows with (5.2),

$$\pi_1 \mathbf{f}(a) = \pi_0 \mathbf{f}(a') = \pi_1 \mathbf{g}(a'),$$

so that $\pi_0 \mathbf{f}(a) = \pi_1 \mathbf{g}(a') = \pi_1 \mathbf{f}(a) = \pi_0 \mathbf{g}(a')$. Thus both $\mathbf{f}(a)$ and $\mathbf{g}(a')$ are *endo*morphisms of the same object. Now, since $(W,+)$ is *commutative*, endomorphisms of the same object commute:

$$(x,v',t)*(x,v,t) = (x,v'+v,t) = (x,v+v',t) = (x,v,t)*(x,v',t), \tag{5.3}$$

and hence we get

$$(\mathbf{f}*\mathbf{g})(a'*a) = \mathbf{f}(a') * \mathbf{g}(a') * \mathbf{f}(a) * \mathbf{g}(a) = (\mathbf{f}*\mathbf{g})(a') * (\mathbf{f}*\mathbf{g})(a),$$

proving that $\mathbf{f}*\mathbf{g}$ is again a morphism. Thus $\mathrm{Hom}_{\mathbb{K}}(U^{\{1\}}, W^{\{1\}})$ is stable under the pointwise structures, and by general principles, we get again a structure of the same type, that is, a groupoid. If $\mathbb{K}$ is commutative, then endomorphisms in the $\bullet$-categories also commute with each other, so that the same arguments as above imply that the pointwise product $\mathbf{f} \bullet \mathbf{g}$ belongs to a well-defined category structure on $\mathrm{Hom}_{\mathbb{K}}(\mathbf{U},\mathbf{W})$, which together with the $*$-structure defined before forms a double category, proving (2). $\square$

Summing up, there ought to be some (big) cartesian closed category of local double categories containing $C^1$-manifold laws and their inverse images. This category would, then, be a good candidate for a general notion of "space laws". We come back to this issue as soon as the general $k$-th order theory is developed.

## Appendix A. Categories, groupoids

A.1. **Concrete categories.** Formally, a *concrete category* (abridged: ccat) is defined as a category together with a faithful functor to the category of sets. For our purposes, concrete categories are just a piece of language, and we rather think of a concrete category as given by a certain "type $\mathcal{T}$ of structure defined on sets": objects are "sets with structure of type $\mathcal{T}$", and morphisms are "maps preserving structure". We will denote such concrete categories by underlined roman letters, for instance

- the ccat $\underline{\mathrm{Vect}}_{\mathbb{K}}$ of all $\mathbb{K}$-vector spaces (with linear maps as morphisms),
- the ccat $\underline{\mathrm{Top}}$ of all topological spaces (with continuous maps),
- the ccat $\underline{\mathrm{Man}}_{\mathbb{K}}$ of smooth manifolds over $\mathbb{K}$ (with smooth maps),
- the ccat $\underline{\mathrm{Grp}}$ of all groups, its subcat $\underline{\mathrm{Cgroup}}$ of all commutative groups,
- the ccat $\underline{\mathrm{Ring}}$ of all rings, its subcat $\underline{\mathrm{Field}}$ of all fields,
- the ccat $\underline{\mathrm{Alg}}_{\mathbb{K}}$ of all (associative) $\mathbb{K}$-algebras,
- the ccat $\underline{\mathrm{Cat}}$ of all small cats (see below) and its subcat $\underline{\mathrm{Goid}}$ of all groupoids,
- the ccat $\underline{\mathrm{Set}}$ of all sets (with arbitrary set-maps).

A.2. **Small categories.** A *small category* (abridged: small cat) is given by a pair of sets $(B, M)$, $B$ called the *set of objects* and $M$ called the *set of morphisms*, together with two maps "source" and "target" $\pi_0, \pi_1 : M \to B$ (we shall write $\pi_\sigma$, where $\sigma$ takes two values; instead of 0 and 1 one may also use the values $s$ et $t$, or $+$ and $-$, or others), and map $z : B \to M$ "zero section" or "unit section", and finally a binary composition $g * f$, defined for $(g, f) \in M \times_B M$ where

$$M \times_B M := M \times_{B,\pi} M := \{(g, f) \in M \times M \mid \pi_0(g) = \pi_1(f)\}, \tag{A.1}$$

such that these data satisfy the following properties:

$$\pi_1(g * f) = \pi_1(g), \qquad \pi_0(g * f) = \pi_0(f), \tag{A.2}$$

and the law $*$ is *associative*: whenever $(h, g)$ and $(g, f)$ are in $M \times_B M$, then

$$(h * g) * f = h * (g * f). \tag{A.3}$$

Moreover, $z : M \to B$ is a *bisection* (that is, $\pi_\sigma \circ z = \mathrm{id}_B$ for $\sigma = 0, 1$), such that

$$z(\pi_1(f)) * f = f, \qquad g * z(\pi_0(g)) = g. \tag{A.4}$$

The small cat will be denoted by $(B, M, \pi_\sigma, s, *)$, or, shorter, $(M \downdownarrows B, *)$ or $(B, M)$. Especially in Part II it will be useful to write $(C_0, C_1)$ instead of $(B, M)$, and to call the disjoint union $C := C_0 \dot\cup C_1 = B \dot\cup M$ the *underlying set* of the small cat.

By a *bundle of small cats* $M \rightrightarrows B \to I$ we just mean an indexed family $(M_i, B_i)$ of small cats indexed by a set $I$. The total space $(M, B)$ is then again a small cat.

A.3. **Functors.** A *morphism between small cats* $(M \downdownarrows B, *)$, $(M' \downdownarrows B', *')$, or *(covariant) functor*, is a pair of maps $(F : M \to M', f : B \to B')$ such that

$$f \circ \pi_0 = \pi_1' \circ F, \qquad F \circ z = z' \circ f, \qquad F(h) *' F(g) = F(h * g). \tag{A.5}$$

It is obvious that the composition of functors is again a functor, and that the identity $\mathrm{id}_C$ of the underlying set $C = M \dot\cup B$ is a functor. Identifying $(F, f)$ with $f \dot\cup F : C \to C'$, small cats form a concrete category $\underline{\mathrm{Cat}}$.

A.4. **Groupoids.** Assume $(B, M)$ is a small cat. An element $f \in M$ is *invertible* if there is another one, $g$, such that $\pi_0(g) = \pi_1(f)$ and $\pi_1(g) = \pi_0(f)$ and $g * f = z(\pi_0(f))$ and $f * g = z(\pi_0(g))$. By standard arguments, such a $g$ is unique. It is then called the *inverse of* $f$ and denoted by $g = f^{-1}$. A *groupoid* is a small category in which *every* $f \in M$ is invertible. Groupoids and functors form a concrete category $\underline{\text{Goid}}$. For every groupoid, the *inversion map* $i : M \to M$, $f \mapsto f^{-1}$, together with $\mathrm{id}_B$, is an isomorphism onto the opposite groupoid. See, e.g., [Ma05], for more information on groupoids.

A *group bundle* is a groupoid such that $\pi_1 = \pi_0$. Then the fibers of $\pi_\sigma$ are groups.

*Example* A.1 (The pair groupoid). If $A$ is a set, then $(A \times A, \downdownarrows, A, z, \circ)$ with $\pi_0(x, y) = y$, $\pi_1(x, y) = x$, $z(x) = (x, x)$ and $(x, y) \circ (y, z) = (x, z)$ is a groupoid, called the *pair groupoid* of $A$. The inverse of $(x, y)$ is $(y, x)$. If $R \subset (A \times A)$, then $(R, A, z, \circ)$ is a subgroupoid of the pair groupoid if, and only if, $R$ is an equivalence relation on $A$.

*Example* A.2 (The anchor morphism). For a category $(\pi : M \downdownarrows B, z, *)$, the *anchor*

$$\kappa : M \to B \times B,\ a \mapsto (\pi_1(a), \pi_0(a)), \qquad B \to B,\ a \mapsto a \tag{A.6}$$

is a morphism of $(M, B)$ to the pair groupoid. Indeed, if $\pi_1(a) = \pi_0(a')$,

$$\kappa(a' * a) = (\pi_1(a' * a), \pi_0(a' * a)) = (\pi_1(a'), \pi_0(a)) = \kappa(a') \circ \kappa(a).$$

A.5. **Opposite category, and notation.** The *opposite category (resp. groupoid)* of a category (resp. groupoid) $(\pi_0, \pi_1, M, B, *, z)$ is $(\pi_0^{op}, \pi_1^{op}, M, B, *^{op}, z)$, where

$$\pi_0^{op} := \pi_1, \qquad \pi_1^{op} := \pi_0, \qquad g *^{op} f := f * g. \tag{A.7}$$

Note that each groupoid is isomorphic to its opposite groupoid, via the inversion map (but a category needs not be isomorphic to its opposite category). A *contravariant functor* between categories is a morphism to the opposite category. We are aware that several authors use other conventions concerning notation of the product.[6] However, once a convention is fixed, it should be kept.

A.6. **Pregroupoids.** In every groupoid we may define a ternary product

$$[a'', a', a] := a'' * (a')^{-1} * a, \tag{A.8}$$

whenever $\pi_1(a) = \pi_1(a')$ and $\pi_0(a'') = \pi_0(a')$. In Part II we will need the notion of *pregroupoid* (cf. [Ko10], see also [Be14b]):

**Definition A.1.** *A* pregroupoid *is given by a set $M$ together with two surjective maps $a : M \to A$, $b : M \to B$ and a partially defined* ternary product map

$$M \times_a M \times_b M \to M, \quad (x, y, z) \mapsto [xyz]$$

*where*

$$M \times_a M \times_b M = \{(x, y, z) \in M^3 \mid a(x) = a(y),\ b(y) = b(z)\},$$

[6] See [BHS11], p. 145 Rk 6.1.1 and p. 556: "The first notation is taken from the composition of maps and the second is more algebraic. ... we have used both...". Our convention, given by (A.1), follows the conventions that seem to be most common, see, e.g., `http://ncatlab.org:8080/nlab/show/category` and `http://ncatlab.org/nlab/show/opposite+category`.

*such that these data satisfy*

$$\forall (x,y,z) \in M \times_a M \times_b M : \quad a([xyz]) = a(x),\, b([xyz]) = b(z),$$

*and the* para-associative *and the* idempotent law *hold:*

(PA) $[x[uvw]z] = [[xwv]uz] = [xw[vuz]]$,
(IP) $[xxz] = z = [zxx]$.

Note that such structure only depends on the equivalence relations of fibers defined by $a$ and $b$, hence the sets $A$ and $B$ may be eliminated from the definition by considering $a, b$ just as equivalence relations on $M$ (as done in [Be14b]). A *morphism of pregroupoids* is given by a map $f : M \to M'$ sending fibers of $a$ to fibers of $a'$ and fibers of $b$ to fibers of $b'$ and preserving the ternary product: $f[xyz] = [fx, fy, fz]$. Obviously, pregroupoids and their morphisms form a concrete cat $\underline{\mathrm{Pgoid}}$.

**Theorem A.2.** *The ccat of groupoids is equivalent to the ccat of pregroupoids together with a fixed bisection.*

*Proof.* This observation is due to Johnstone, cf. [Be14b]. □

*Example* A.3. If $A, A'$ are two sets, there is a pregroupoid $(A \times A', \downdownarrows, A, A', [\,,\,,\,])$ with $[(x,y),(u,y),(u,v)] = (x,v)$. It admits a bisection if, and only if, $A$ and $A'$ are equipontentious. See [Be14b] for more on this.

## Appendix B. Scaled monoid action category

B.1. **The action groupoid.** If $S$ is a group and $V \times S \to V$ a right group action, then the following construction of the *action groupoid* is well-known in category theory (see, eg., `http://ncatlab.org/nlab/show/action+groupoid`). The sets of morphisms and objects are defined by $(M, B) = (V \times S, V)$, with two projections $\rho_i : V \times S \to V$,

$$\rho_0(v,g) = v, \qquad \rho_1(v,g) = vg,$$

and product when $vg = v'$

$$(v', g') \bullet (v, g) = (vg, g') \bullet (v, g) = (v, gg').$$

Units are $(v, 1)$, where 1 is the unit of $S$, and the inverse of $(v, g)$ is $(vg, g^{-1})$. In case of a left action, we take $M = S \times V$, $B = V$ and $(g', v') \bullet (g, v) = (g'g, v)$.

B.2. **Monoid action category.** Let $S$ be a monoid acting from the right on a set $V$, via $V \times S \to V$. (In the main text, $S = (\mathbb{K}, \cdot)$ is the multiplicative monoid of a ring $(\mathbb{K}, +, \cdot)$, and $V$ a $\mathbb{K}$-module.) Then, of course, the preceding construction still works, but instead of a groupoid it merely defines a small cat.

*Remark* B.1. In the preceding situation we may define, for any non-empty subset $U \subset V$, a subcategory

$$M_U := \{(v,g) \in V \times S \mid v \in U, vg \in U\}, \qquad B_U := U.$$

Even if $S$ is a group, this need not be a groupoid (the category $M_U$ then rather belongs to the semigroup $\{g \in G \mid U.g \subset U\}$, and not to a group).

B.3. **The scaled action category.** The effect of the construction of the action groupoid is to break up the "single object $V$" into a collection of different objects and to distinguish all the isomorphisms $(v, g)$ when $v$ runs over the set of objects. If $S$ is not a group, we will need a kind of refinement of the monoid action category: we will distinguish various "scaling levels" of $v$ – the couple $(v, k)$ with $k \in S$ should be seen as "the object $v$, scaled at level $k$". Then each $g \in S$ gives rise to morphisms (denoted by $(v; g, k)$) from $v$, scaled at $gk$, to $vg$, scaled at $k$. Finally, a most symmetric formulation of this concept is gotten when the "scale" $k$ lives in a space $K$ on which $S$ acts from the left (later we take $S = K$).

**Lemma B.1.** *Assume $S$ is a monoid acting from the right on $V$ and from the left on a set $K$. The following data $(M, B, \partial, z, \bullet) = (V \times S \times K, V \times K, \partial, z, \bullet)$ define a small cat, called the* scaled action category*:*

$$\partial_0 : V \times S \times K \to V \times K, \quad (v; s, t) \mapsto (v; st)$$
$$\partial_1 : V \times S \times K \to V \times K, \quad (v; s, t) \mapsto (vs; t).$$

*The composition $a' \bullet a$ for $a = (v; s, t), a' = (v'; s', t')$ is defined if $\partial_1(a) = \partial_0(a')$, so $(v', s't') = (vs, t)$, so $v' = vs$, $s't' = t$,*

$$(v'; s', t') \bullet (v; s, t) = (vs; s', t') \bullet (v; s, s't') := (v; ss', t'), \tag{B.1}$$

*and, if $1$ denotes the unit of the monoid $S$, the unit section is defined by*

$$z(v; s) := (v; 1, s). \tag{B.2}$$

*If, moreover, $S$ is a group, then the category $(M, \downdownarrows, B)$ is a groupoid.*

*Proof.* Everything is checked by straightforward computations. For convenience, we give some details: first, note that

$\partial_0(a' \bullet a) = (v, ss't') = (v, st) = \partial_0(a)$

$\partial_1(a' \bullet a) = (vss', t') = (v's', t') = \partial_1(a')$,

and associativity follows from the one of $S$:

$$\big((v''; s'', t'') \bullet (v'; s', t')\big) \bullet (v; s, t) = (v'; s's'', t'') \bullet (v; s, t) = (v; ss's'', t'')$$
$$(v'', s'', t'') \bullet \big((v'; s', t') \bullet (v; s, t)\big) = (v''; s'', t'') \bullet (v; ss', t) = (v; ss's'', t'').$$

The element $(v; 1, s)$ is a unit for the categorial product: $(v'; s', t') \bullet (v; 1, t) = (v; s', t')$ (note $v' = v1 = v$) and $(v', 1, t') \bullet (v; s, t) = (v; s', t) = (v; s, t)$ since $s = s't' = s'$. Note that the morphism $(v; s, t)$ is invertible if, and only if, $s$ is invertible in the multiplicative semigroup of $\mathbb{K}$: $(v'; s, t') \bullet (v, s^{-1}, t) = (v, 1, t')$. □

*Remark* B.2. As above (Remark B.1), for every $U \subset V$, there is a subcategory $\{(v; s, t) \mid v \in U, vst \in U\}$.

**Lemma B.2.** *With notation from the preceding lemma, the following maps of objects and morphisms define a functor from the scaled action category to the left action category of $S$ on $K$:*

$$V \times S \times K \to S \times K, \quad (v, s, t) \mapsto (s, t),$$
$$V \times K \to K, \quad (v, s) \mapsto s.$$

*If the action $V \times S \to V$ admits a fixed point $o$, then we also get a functor in the other sense, via $S \times K \to V \times S \times K$, $(s, t) \mapsto (o; s, t)$.*

*Proof.* The left action category of $S$ acting on itself is given by the morphism set $S \times S$ and object set $S$ and source and target maps

$$\partial_0 : S \times S \to S,\ (s,t) \mapsto st, \qquad \partial_1 : S \times S \to S,\ (s,t) \mapsto s, \tag{B.3}$$

and composition, whenever $t = s't'$,

$$(s', t') \bullet (s,t) = (ss', t) \tag{B.4}$$

From these formulae it is seen that the maps given above define a functor. □

## Appendix C. Small double categories, double groupoids

The following presentation follows [BrSp76] (where letters $H, V$ "horizontal, vertical" are used for our $(C_{11}, C_{10}, C_{01}, C_{00})$). We give full details in order to prepare for the algebraic presentation of small $n$-fold cats in Appendix B of Part II, and we are more tedious than in usual presentations, regarding "size questions".

C.1. **Small cats and groupoids of a given type.** Let $\mathcal{T}$ be a concrete category. A *small cat of type* $\mathcal{T}$ is a set $C$ carrying both a structure of type $\mathcal{T}$ and the structure of a small cat $(C_0, C_1, \pi_\sigma : C_1 \downdownarrows C_0, *)$, so $C = C_0 \dot\cup C_1$, such that all structure maps of the small cat $C$ are compatible with the structure $\mathcal{T}$. This includes the assumption that $C_0, C_1$ and $C_1 \times_{C_0} C_1$ also carry structures of type $\mathcal{T}$ (in practice, one will often check this by first noticing that $C_1 \times C_1$ carries a structure of type $\mathcal{T}$, and then that equalizers as given by (A.1) are again of type $\mathcal{T}$), and hence it makes sense to require that $\pi_\sigma, z$ and $*$ are morphisms for $\mathcal{T}$. A *morphism between two small cats of type* $\mathcal{T}$, say $C$ and $C'$, is a map $f : C \to C'$ which is a functor (for the small cat-structures) and a structure-preserving map for $\mathcal{T}$. This defines a new concrete category $\underline{\mathcal{T}\text{-Cat}}$. In the same way the concrete category $\underline{\mathcal{T}\text{-Goid}}$ of *groupoids ot type* $\mathcal{T}$ is defined.

C.2. **Small double categories.** A *(strict) small double category* (abridged: small doublecat) is a small cat of type $\underline{\text{Cat}}$. In other words, in the preceding paragraph we take $\mathcal{T} = \underline{\text{Cat}}$. This defines a concrete category $\underline{\text{Doublecat}}$. A small doublecat is thus an algebraic structure of a certain type. We wish to give a more explicit description, in the spirit of usual algebra: a small doublecat $C$ is, first of all, a small cat $C = C_1 \dot\cup C_0$ with projections $\pi_\sigma$ and product $*$, and $C_1 = C_{11} \dot\cup C_{10}$ and $C_0 = C_{01} \dot\cup C_{00}$ are in turn two small cats with 4 projections all indicated by the symbol $\partial$ and two products both indicated by $\bullet$. Since $\pi$ restricts to two projections, we also get 4 projections denoted by the symbol $\pi$, and similar for the unit sections, everything fitting into two commutative diagrams:

$$\begin{array}{ccc} C_{11} & \overset{\pi}{\rightrightarrows} & C_{01} \\ \partial \downdownarrows & & \partial \downdownarrows \\ C_{10} & \overset{\pi}{\rightrightarrows} & C_{00} \end{array}, \qquad \begin{array}{ccc} C_{11} & \overset{z_\pi}{\leftarrow} & C_{01} \\ z_\partial \uparrow & & z_\partial \uparrow \\ C_{10} & \overset{z_\pi}{\leftarrow} & C_{00} \end{array}. \tag{C.1}$$

Using this notation, saying that every edge of the square defines a small cat and that the pairs $(\pi_\sigma, \pi_\sigma)$, resp. $(\partial_\sigma, \partial_\sigma)$ are functors, amounts to the following requirements:

(1) $\forall i, j \in \{0,1\}$: $\partial_i \circ \pi_j = \pi_j \circ \partial_i : C_{11} \to C_{00}$,
(2) $z_\pi \circ z_\partial = z_\partial \circ z_\pi : C_{00} \to C_{11}$,

(3) $\forall \sigma \in \{0,1\}$: $\partial_\sigma \circ z_\partial = \mathrm{id}_{C_{\sigma 0}}$, $\pi_\circ \circ z_\pi = \mathrm{id}_{C_{0\sigma}}$,
(4) $\forall \sigma \in \{0,1\}$: $\partial_\sigma \circ z_\pi = z_\pi \circ \partial_\sigma : C_{01} \to C_{10}$, $\pi_\sigma \circ z_\partial = z_\partial \circ \pi_\sigma : C_{10} \to C_{01}$,
(5) the partially defined products $*$ and $\bullet$ are associative,
(6) elements $z_\partial(u)$ are units for $\bullet$ and elements $z_\pi(v)$ are units for $*$,
(7) $\forall \sigma \in \{0,1\}$: the pairs of maps $(\partial_\sigma, \partial_\sigma)$, $(\pi_\sigma, \pi_\sigma)$ preserve partially defined products: $\partial_\sigma(a' * a) = \partial_\sigma(a') * \partial_\sigma(a)$, $\pi_\sigma(b' \bullet b) = \pi_\sigma(b') \bullet \pi_\sigma(b)$,
(8) the maps $z_\partial : (V,*) \to (C,*)$, $z_\pi : (H, \bullet) \to (C, \bullet)$ are sections of $\partial$, resp. of $\pi$, and $z_\partial(a' * a) = z_\partial(a') * z_\partial(a)$, $z_\pi(b' \bullet b) = z_\pi(b') \bullet z_\pi(b)$.

**Theorem C.1.** *Data $(C_{11}, C_{10}, C_{01}, C_{00}, \pi, \partial, z, *, \bullet)$ given by spaces, maps and partially defined products define a small doublecat if, and only if, they satisfy* (1) – (8) *together with the following* interchange law*:*

(9) *whenever both sides are defined (which is the case iff $\pi_1(b) = \pi_0(a) = \pi_0(c) = \pi_1(d)$ and $\partial_0(b) = \partial_1(c) = \partial_0(a) = \partial_1(d)$), then*

$$(a * b) \bullet (c * d) = (a \bullet c) * (b \bullet d).$$

*A morphism of small double cats is given by four maps which define a functor for each of the four small cats forming the edges of (C.1).*

*Proof.* It only remains to show that, in presence of (1) – (8), the interchange law (9) is equivalent to saying that $* : C_1 \times_{C_0} C_1 \to C_1$ defines a morphism for the $(\partial, \bullet)$-cat structure. Here, $C_1 \times C_1$ is the direct product of two $\bullet$-small cats, and the equalizer (A.1) is a small subcat since the projections $\pi_\sigma : C_1 \downdownarrows C_0$ are morphisms, by (7). Thus the product on $C_1 \times_{C_0} C_1$ is given by $(a,b) \bullet (c,d) := (a \bullet c, b \bullet d)$. Then, the map $A := * : C_1 \times_{C_0} C_1 \to C_1$, $(a,b) \mapsto a * b$ is a morphism for $\bullet$ iff

$$A((a,b) \bullet (c,d)) = A(a,b) \bullet A(c,d),$$

that is, the interchange law $(a \bullet c) * (b \bullet d) = (a * b) \bullet (c * d)$ holds. □

**Definition C.2.** *The four small cats forming the edges of the diagram (C.1) are called the* edge cats *of the small doublecat. It is obvious from the description given in the theorem that, exchanging $C_{01}$ and $C_{10}$ and $*$ and $\bullet$, we get again a small doublecat, called the* transposed doublecat*. A small doublecat is called* edge symmetric *if it admits an isomorphism onto its transposed doublecat.*

C.3. **Double groupoids.** A *double groupoid* is a small doublecat such that each of its four edge categories is a groupoid. Equivalently, it is a groupoid of type <u>Goid</u>. This defines a concrete category <u>Doublegoid</u>.

C.4. **Double bundles.** A *double group bundle* is a double groupoid such that $\pi_0 = \pi_1$ and $\partial_0 = \partial_1$. E.g., *double vector bundles* (see [BM92]) are double group bundles.

C.5. **Opposite double cat.** In a double cat, we may replace each of the pairs of vertical, resp. horizontal, cats, by its opposite pair, and we get again a double cat. Thus we get altogether 4 double cats: $(*, \bullet)$, $(*^{op}, \bullet)$, $(*, \bullet^{op})$, and $(*^{op}, \bullet^{op})$. In general, they are not isomorphic among each other; however, if $*$ or $\bullet$ belongs to a groupoid, then inversion is an isomorpism onto $*^{op}$, resp. onto $\bullet^{op}$, and compatible with the other law.

## Appendix D. Primitive manifolds

The following is a generalization of the usual definition of atlas of a manifold. It formalizes the idea that a space $M$ "is modelled on a space $V$" (which may be a linear space, or not – linearity of the model space is not needed for the following).

**Definition D.1.** *Let $V$ be a topological space that will be called* model space *(the topology need not be separated or non-discrete). A* primitive manifold modelled on *$V$ is given by $(M, \mathcal{T}, V, (U_i, \phi_i, V_i)_{i\in I})$, where $(M, \mathcal{T})$ is a topological space, $(U_i)_{i\in I}$ an open cover of $M$, so $M = \cup_{i\in I} U_i$ and the $U_i$ are open and non-empty, and and $\phi_i : U_i \to V_i$ are homeomorphisms onto open sets $V_i \subset V$. We then also say that $\mathcal{A} = (U_i, \phi_i, V_i)_{i\in I}$ is an* atlas on $M$ with model space $V$*. The atlas is called* maximal *if it contains all compatible charts (defined as usual in differential geometry, see e.g.,* [Hu94]*). The primitive manifold is called a* Hausdorff manifold *if its topology is Hausdorff and if the atlas is maximal.*
*A* morphism of primitive manifolds $(M, \mathcal{T}, \mathcal{A})$, $(M', \mathcal{T}', \mathcal{A}')$ *is a continuous map $f : M \to M'$. Thus primitive manifolds form a concrete category* $\underline{\text{Pman}}$.

*Remark* D.1. One may suppress the topology $\mathcal{T}$ from this definition by considering on $M$ the *atlas topology*, which is the coarsest topology such that chart domains are open (topology generated by the $U_i$). This is the point of view taken in a first version of this work; however, including the topology $\mathcal{T}$ as additional datum gives more freedom and is closer to usual definitions.

**Definition D.2.** *An atlas $\mathcal{A}$, and the primitive manifold $M$, are called* handy *if, for any finite collection $x_1, \ldots, x_n \in M$, there exists a chart $(U_i, \phi_i)$ such that $x_1, \ldots, x_n \in U_i$.*

**Lemma D.3.** *A Hausdorff manifold modelled on a topological group $V$ is handy.*

*Proof.* Let $n = 2$. If $x_1 = x_2$, there is nothing to show. If $x_1 \neq x_2$, choose disjoint charts $(U', \phi')$ around $x_1$ and $(U'', \phi'')$ around $x_2$. By shrinking chart domains if necessary and using translations on $V$, we may assume that $V' := \phi'(U')$ and $V'' := \phi''(U'')$ are disjoint. But then the disjoint union $(U' \cup U'', \phi' \cup \phi'', V' \cup V'')$ is a chart (by maximality of the atlas) with the required properties (note that connectedness is not required for chart domains), and we are done. For $n > 2$, one proceeds in the same way. □

Given an atlas, we let for $(i,j) \in I^2$,

$$U_{ij} := U_i \cap U_j \subset M, \qquad V_{ij} := \phi_j(U_{ij}) \subset V_j, \tag{D.1}$$

and the *transition maps* belonging to the atlas are defined by

$$\phi_{ij} := \phi_i \circ \phi_j^{-1}|_{V_{ji}} : V_{ji} \to V_{ij}. \tag{D.2}$$

They are homeomorphisms satisfying the *cocycle relations*

$$\begin{aligned} \phi_{ii} = \mathrm{id}_{V_i} \quad &\text{and} \quad \phi_{ij} \circ \phi_{jk} = \phi_{ik} : V_{kji} \to V_{ijk}, \\ &\text{where} \quad V_{abc} = \phi_a(U_a \cap U_b \cap U_c) = \phi_{ab}(V_{cb} \cap V_{ab}). \end{aligned} \tag{D.3}$$

**Theorem D.4** (Reconstruction from local data)**.** *The data of a primitive manifold $(M, \mathcal{T}, \mathcal{A})$ are equivalent to the data $(V, \mathcal{T}_V, (V_{ij}, \phi_{ij})_{(i,j)\in I^2})$, where*

- *$I$ is a (discrete) index set,*
- *$(V, \mathcal{T}_V)$ is a topological space (the model space),*
- *$V_{ij} \subset V$, for $(i,j) \in I^2$, are open subsets,*
- *$\phi_{ij} : V_{ji} \to V_{ij}$ are homeomorphisms satisfying the cocycle relations* (D.3).

*Morphisms of primitive manifolds then are the same as families of continuous maps*

$$f_{ij} : V_{jj} \to V'_{ii}, \qquad (i,j) \subset I' \times I, \text{ such that } f_{k\ell} = \phi'_{ki} \circ f_{ij} \circ \phi_{j\ell} : V_{\ell j} \to V_{ki}.$$

*Proof.* Given a primitive manifold $M$, the data $(V, \mathcal{T}_V, (V_{ij}, \phi_{ij})_{(i,j)\in I^2})$ are defined as above, and if $f : M \to M'$ is a morphism, we let $f_{ij} := \phi'_i \circ f \circ (\phi_j)^{-1}$.

Conversely, given $(V, \mathcal{T}_V, (V_{ij}, \phi_{ij})_{(i,j)\in I^2})$, define $M$ to be the quotient $M := S/\sim$, where $S := \{(i,x) | x \in V_{ii}\} \subset I \times V$ with respect to the equivalence relation $(i,x) \sim (j,y)$ if and only if $(\phi_{ij})(y) = x$. We then put $V_i := V_{ii}$, $U_i := \{[(i,x)], x \in V_i\} \subset M$ and $\phi_i : U_i \to V_i, [(i,x)] \mapsto x$. The topology on $M$ is defined to be the topology generated by all $(\phi_i)^{-1}(X)$ with $X$ open in $V_i$ and $i \in I$. Moreover, given a family $f_{ij}$ as in the theorem, the map

$$f : M \to M', \quad [(j,x)] \mapsto [(i, f_{ij}(x))]$$

is well-defined and continuous. All properties are now checked in a straightforward way; we omit the details (cf. [Hu94], Section 5.4.3). □

*Example* D.1. If all $U_{ij}$ are empty for $i \neq j$, then $M$ is just the disjoint union of the sets $V_i := V_{ii}$, and there are no transition conditions. In particular, when $|I| = 1$, we see that every open subset $U \subset V$ is a manifold, with a single chart.

*Remark* D.2. The preceding theorem is used to "globalize" local functorial constructions: if such a construction transforms local data, as described in the theorem, into other such local data, then, again by the theorem, such a construction carries over to the manifold level. For instance, in topological differential calculus, this construction permits to define the *tangent bundle* $TM$ of a manifold $M$, and, much more generally, the *Weil bundle* $FM$ of $M$ for any *Weil functor* $F$ (see [BeS14, Be14]). In the present work, it is applied to the local construction $U \mapsto U^{\{1\}}$.

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