

CONCEPTUAL DIFFERENTIAL CALCULUS PART II: CUBIC HIGHER ORDER CALCULUS

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ABSTRACT. Following the programme set out in Part I of this work, we develop a *conceptual higher order differential calculus*. The “local linear algebra” defined in Part I is generalized by “higher order local linear algebra”. The underlying combinatorial object of such higher algebra is the *natural n-dimensional hypercube*, and so we qualify this calculus as “cubic”. More precisely, we define two versions of conceptual cubic calculus: “full” and “symmetric cubic”. The theory thus initiated sheds new light on several foundational issues.

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1. INTRODUCTION

1.1. “**Die Gesetze des Endlichen**”. In preceding work ([Be15, Be08]), I quoted the following phrase by G.W. Leibniz: “Die Gesetze des Endlichen gelten im Unendlichen weiter” (“the rules of the finite continue to hold in the infinitesimal”), but I should admit that I really started to understand its meaning while working on the results to be presented here. When teaching differential calculus, we usually hasten to take the limit – “the infinitesimal” – and don’t ask ourselves what “the rules of the finite” are. Implicitly, we take for granted that these rules are kind of trivial and known by everybody.

In the end, it may well appear that these “rules of the finite” look trivial; but they are certainly not known by everybody. They are trivial in the sense that they “just”

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deal with iterated directed products – but to say that, we need the subtle concept of n -fold groupoid. I guess that such structures underly huge parts of mathematics. Some of the following is still tentative, and *notation* is an important issue: I am not yet sure to have found the best one. For these and other reasons, the present version of this work will not be submitted for publication in a mathematical journal.

I will start by introducing three different kinds of higher order calculus, then explain what their intrinsic, or conceptual, version is, and finally say some words about the relation between “the finite” and “the infinite”. Let me stress once again that this is far from being a complete theory, but rather a first picture of the tip of an iceberg whose shape is still unknown, and the present work would have reached its goal if the reader felt that something interesting and new is going on here.

1.2. The threefold way of higher order calculus. The *first order conceptual calculus* developed in [Be15] (quoted as Part I) is based on a groupoid interpretation of the *first order difference quotient map*, or *slope*, of a map $f : U \rightarrow W$,

$$(1.1) \quad f^{[1]}(x, v, t) = \frac{f(x + tv) - f(x)}{t}.$$

Invertible scalars t represent “das Endliche” and non-invertible scalars t represent “das Unendliche” (with $t = 0$ as most singular value). Recall that, in a topological context (cf. Part I), f is called *of class C^1* if the slope $f^{[1]}$ extends to a *continuous map* (still denoted by $f^{[1]}$), for the value $t = 0$ included. At higher order, there are three different ways to iterate this procedure (Section 2):

- *symmetric cubic calculus* (closest to usual calculus),
- *simplicial calculus* (closer to the spirit of algebraic geometry),
- *full cubic calculus* (which includes both preceding approaches).

Briefly, *full cubic calculus*, as developed in [BGN04], consists of iterating the procedure in the “brutal way”, that is, by looking at higher order slopes $f^{[n]}$ defined by $f^{[2]} := (f^{[1]})^{[1]}$, and so on. These maps are very hard to understand, so let’s propose a “tamed” version of the iteration procedure: fix a scalar t and consider the map

$$(1.2) \quad f_t^{[1]} : (x, v) \mapsto f^{[1]}(x, v, t),$$

Iterating this, we define $f_{t_1, t_2}^{[2]} := (f_{t_1}^{[1]})_{t_2}^{[1]}$, and so on: at each step, we fix a scalar t_k , take the slope with this fixed value of scalar, and then look what happens when the multi-scalar (t_1, \dots, t_n) tends to zero (Section 2.2). In fact, this amounts to consider a sub-structure of the “full” $f^{[n]}$, which turns out to contain already “most” of the information of the full map, and having the advantage that it is much easier understandable (Theorem 2.3). In particular, there is a natural action of the symmetric group S_n preserving the structure (“Schwarz’ theorem”), whence the term *symmetric cubic calculus*. However, for proving the general Taylor formula from [BGN04], the symmetric cubic setting is not sufficient, and for this reason we have developed in [Be13] a *simplicial differential calculus*. It should be possible and important (in particular, in order to relate the present work to approaches used in algebraic geometry) to develop a *conceptual and categorical formulation of simplicial calculus* – however, to keep the present work in reasonable bounds, this is left for subsequent work.

1.3. **Conceptual version: the three symbols G^n , Gsy^n , Gsi^n .** I have chosen the sans serif letter G , reminding the term *groupoid*, combined with n standing for the set $n = \{1, \dots, n\}$, to denote certain *n-fold groupoids*, which are the conceptual objects corresponding to the three calculi. More precisely,

- G^nU is the *full cubic n-fold groupoid* of the domain U ,
- Gsy^nU is the *symmetric n-fold groupoid* of U ,
- Gsi^nU will denote a simplicial (pre)groupoid version of the preceding.

Before going ahead, I should reassure the reader who is not a specialist in category theory and who has never before met *n-fold groupoids* and categories: I have been in the same situation, and so the present text is also designed to introduce a newcomer to these gadgets. In this respect, I hope that the presentation given here (Appendices A and B) may be useful: *small n-fold categories* and *groupoids* are presented as “algebraic structures” in the usual sense, defined by sets, structure maps and identities, so a small *n-fold category* is given by

- a family of sets C_α , one for each *vertex* α of the natural hypercube $\mathcal{P}(n)$,
- a family of small categories, one for each *edge* (β, α) of the hypercube $\mathcal{P}(n)$,
- such that for each *face* of the hypercube these data define a small double category (as defined in Part I, appendix C).¹

Seen that way, small *n-fold categories* are computable: for Gsy^nU formulae are explicit and fairly simple (Theorem 4.3), whereas for G^nU they are far more complicated – we make some effort to present them in algorithmic form, that could be programmed on a machine (Theorems 5.2, 5.3, 5.4).

Having said this, I hope the general reader is willing to accept that the iteration procedure, leading from first order to higher order conceptual calculus, is entirely “canonical”, that is, following from general principles; once the concepts are clear, most proofs are “by straightforward induction”: for $n = 1$, all three groupoids coincide with the groupoid $U^{\{1\}}$ defined in Part I, so, if Q is one of the symbols G, Gsy , then $Q^1U = U^{\{1\}}$. For $n > 1$, to compute structure data of Q^nU , we just have to apply a copy of Q^1 to the data of $Q^{n-1}U$. However, since the object set of G^1U is a cartesian product $U \times \mathbb{K}$, we need to know how Q behaves with respect to cartesian products and general pullbacks. At this stage, the difference between the three calculi comes in: the key notion to understand this is the one of *scaleoid*.

1.4. **Scaleoids, pullbacks and cartesian products.** By “terminal object”, we mean the zero-subset $\{0\}$ of the zero \mathbb{K} -module $\{0\}$; we just denote it by 0 . The *n-fold groupoid* Q^n0 is called the *n-th order scaleoid*. More precisely:

- in symmetric calculus, $Gsy^n0 \cong \mathbb{K}^n$, and since variables t_k are kept constant at each iteration step, the groupoid structure here is the trivial one,
- in full calculus, the top vertex of G^n0 is \mathbb{K}^{2^n-1} , and the *n-fold groupoid* structure is non-trivial and rather complicated.

¹ After having written this text, I found an essentially equivalent presentation, designed for category theorists, in Section 2 of [FP10]. See also [GM12], where a similar description is given for *n-fold vector bundles*, using a quite fancy terminology of “hops” and “runs”, cf. Th. 4.7.

To relate scaleoids with cartesian products, a basic property of our symbols Q^n is that *they are compatible with pullbacks* (see Appendix E): they satisfy the rule

$$(1.3) \quad Q^n(A \times_C B) = Q^n A \times_{Q^n C} Q^n B.$$

The usual cartesian product $A \times B$ is the pullback over a point $A \times_0 B$, whence

$$(1.4) \quad Q^n(A \times B) = Q^n A \times_{Q^n 0} Q^n B.$$

Now, in symmetric calculus, the groupoid $Gsy^n 0$ is \mathbb{K}^n with trivial structure, so we can fix scalars, get an n -fold groupoid $Gsy_{\mathbf{t}}^n U$ for fixed \mathbf{t} and use the preceding relation in the form

$$(1.5) \quad Gsy_{\mathbf{t}}^n(A \times B) = Gsy_{\mathbf{t}}^n A \times Gsy_{\mathbf{t}}^n B,$$

expressing that $Gsy_{\mathbf{t}}^n$ is a *product preserving functor*, which is a well-established concept in differential geometry (see [KMS93]). In particular, this relation implies that the philosophy of *scalar extension*, known from algebraic geometry and used in our preceding work [Be08, BeS14, Be14], wholly applies in the present context. In full cubic calculus, since $G^n 0$ is highly non-trivial, computations cannot be untied so easily, and their description, though entirely explicit, is a bit tedious (Theorem 5.2). Certainly, the abstract algebraic structure of the full scaleoid $G^n 0$ deserves a more profound study in its own right.

1.5. Iterated pair groupoids, and the rules of the finite. We consider the n -fold groupoid $G^n U$ as an n -fold *magnification of the “space”* U . It is a kind of n -th order version of *Connes’ tangent groupoid* (cf. Part I, Theorem 2.7), realizing an interpolation between the n -th order tangent bundle and the n -fold pair groupoid: recall that applying the natural iteration procedure to the pair groupoid $M \times M$ over a set M , one gets the n -fold *pair groupoid* $PG^n M$ (Appendix C). For $\mathbf{t} = (1, \dots, 1)$, and more generally for $\mathbf{t} = (t_1, \dots, t_n)$ such that each t_i is invertible, we have a natural isomorphism of n -fold groupoids

$$(1.6) \quad Gsy_{\mathbf{t}}^n U \cong PG^n U.$$

In full calculus, the situation is similar but more subtle (Theorem 3.8). Thus one may say that “the n -fold pair groupoid governs the rules of the finite”. On the “infinitesimal side”, for $t = (0, \dots, 0)$, we get the *iterated tangent bundle*

$$(1.7) \quad Gsy_{(0, \dots, 0)}^n U = T^n U$$

which is not only an n -fold groupoid, but an “ n -fold group bundle” since all target and source maps coincide for $\mathbf{t} = 0$ (Theorem 4.7).

1.6. Symmetry. The general cubic n -fold groupoid $G^n U$ is *not edge-symmetric* (in contrast to those considered by Brown and Higgins [BH81]). It is a non-trivial fact that $Gsy^n U$ is *edge-symmetric* (Theorem 4.3), and this symmetry can be interpreted by saying that the “rule of the finite”, stating that the n -fold pair groupoid is edge-symmetric, continues to hold for all parameters \mathbf{t} . By the way, this is the most natural proof of *Schwarz’ theorem* on the symmetry of second differentials, as given in topological differential calculus ([BGN04, Be08]). In full calculus, the “rules of the finite” are more complicated: there ought to be something replacing Schwarz’ theorem – what it is, is not yet clear to me.

1.7. Homogeneity. The next “rule of the finite” is *homogeneity under scalars*. For $n = 1$, it has been formulated conceptually as a *structure of double category* $U^{\{1\}}$ (Part I). By the general iteration procedure, in full cubic calculus this leads to define a *small $2n$ -fold category* that we denote by $\mathbf{G}^n U$ (the letter \mathbf{G} is used although this is not a groupoid). Everything said so far about \mathbf{G}^n applies here, *mutatis mutandis*, and the structure is, of course, even more complicated. However, since homogeneity makes no sense for a fixed scalar t , this construction cannot be implemented into symmetric calculus, and we have to add homogeneity “by hand” as a property that can be respected (or not) by the morphisms of $\mathbf{G}\mathbf{sy}^n U$.

1.8. C^n - and C^∞ -laws. *Laws of class C^n over \mathbb{K}* are now defined in a canonical way: they are morphisms of k -fold groupoids for $k = 0, \dots, n$, each compatible with the preceding, and likewise for $n = \infty$. These concepts can be defined with respect to each of the three symbols \mathbf{G} , $\mathbf{G}\mathbf{sy}$, $\mathbf{G}\mathbf{si}$. For $\mathbb{K} = \mathbb{R}$, or other topological base rings, and domains U open in topological \mathbb{K} -modules V , smooth maps give rise to such laws, in a unique way (Theorem 3.16). We also prove that *polynomial laws* (in the sense of Roby, cf. Part I) give rise to laws of class C^∞ (Theorem 3.14). The definition of *C^∞ -manifold laws* follows the pattern given in Part I (see Section 3.8).

Every law has a “finite part” which respects the “rules of the finite”, since it is uniquely determined by its basic set-map f (Theorem 3.15). However, the law need not respect all “rules of the infinitesimal”: if it respects scalar actions, we call it *homogeneous*; if it respects symmetry, we call it *symmetric*. According to “Leibniz principle”, the laws induced by smooth maps in topological calculus do respect all rules. At a first glance, it might appear to be a drawback of our approach that such properties *are not automatic* for abstract laws; however, in view of a deeper understanding, it rather opens the very interesting question of “classifying” such rules and of understanding “how many independent ones” of them exist, and it points the way towards calculi having different rules than the classical ones, such as *supercalculus*.

1.9. Towards geometry: the neighbor relation. In the present work, I concentrate on developing the *formalism*: the n -fold groupoid $\mathbf{G}^n M$ and $2n$ -fold small category $\mathbf{G}^n M$ are canonical and chart-independent objects coming with any “smooth space” M , and they contain all information about its “smooth structure”. Once the formal theory is developed, we may ask for its “geometric interpretation”. Such geometric interpretation will be more “synthetic” than the usual, rather analytic, differential geometry, since we directly work with invariant and chart-independent structures. Indeed, we shall be able to use, in our framework, the language of *synthetic differential geometry* (SDG, see [Ko06, Ko10, MR91]) – indeed, I believe that the present approach yields a new and very interesting model of SDG. Such topics will be treated in subsequent work. However, to understand the present text, it may help the reader to have in mind some kind of “geometric picture”, and therefore I shall already here say some words on topics that will be taken up in later work.

Let us call “primitive points” the elements of the point set M underlying our smooth space, and think of the n -fold groupoids $\mathbf{G}^n M$ as representing higher “stages” or “levels of definition” of M , whence the proposition of terminology to call $\mathbf{G}^n M$

the n -th *magnification* of the “space M ” (for a most general theory, one will have to speak of *local n -fold groupoids*, cf. Part I, but this does not change much in the following). More precisely, a point a has “stage” or “level k ” if it belongs to a vertex set $\mathbf{G}^{\alpha;n}M$ for a vertex $\alpha \in \mathcal{P}(n)$ of order $|\alpha| = k$; so the primitive points are of stage 0. Every point $a \in \mathbf{G}^n M$ has a *scale*, which is its image $\mathbf{t} = \mathbf{t}_a$ in the scaleoid $\mathbf{G}^n 0$ under the canonical morphism $\mathbf{G}^n M \rightarrow \mathbf{G}^n 0$. A point a is called *finite* if its scale \mathbf{t}_a belongs to the finite part $\mathbf{G} \mathbf{f}^n 0$ of the scaleoid, and *infinitesimal* else.

The *first order neighbor relation*, which Kock in [Ko10] considers “the main actor” of his presentation of SDG, can be defined as follows: say that two points a, b of same level k and belonging to a common vertex set $\mathbf{G}^{\alpha;n}M$ are *first order neighbors*, $a \sim_1 b$, if there exists an edge (β, α) such that a and b have same image under the source projection of the edge category defined by this edge: $\pi_0^{\beta, \alpha; n}(a) = \pi_0^{\beta, \alpha; n}(b)$. This relation is symmetric and reflexive, but not transitive since there are k such edge projections! In symmetric calculus, neighbors have the same scale; in full cubic calculus, the situation is more intricate since the “magnification procedure” is not only applied to the space, but also to the scale itself. Similarly, the *second neighbor relation*, $a \sim_2 b$, is defined by using *double source projections* coming from faces with top vertex α , and so on. As in SDG, it follows immediately that $a \sim_k b, b \sim_\ell c$ implies $a \sim_{k+\ell} c$. When a, b, c are infinitesimal (with scale essentially zero), then heuristic arguments show that these neighbor relations coincide indeed with those used by Kock in [Ko06, Ko10]. The advantage, compared to SDG, is that the same terminology, and all arguments, literally also apply for finite scale – all “synthetic reasoning” applies in a perfectly rigorous way if we start on the “finite side”, by formulating things carefully in a language using natural groupoid structures, and then letting scales “tend to zero”. To carry this out, on the finite side, we may think of “points of level k ” in different ways: with regard to points of higher level, they are just “points” (objects); but with respect to points of lower level, they rather are “arrows” or “segments”, namely, the “point a ” is interpreted as the segment or arrow from its source $x = \pi_0(a)$ to its target $y = \pi_1(a)$. If targets and sources match, such segments can be composed: this operation is natural, and it is preserved by all smooth maps. If targets and sources don’t match, there is no natural composition; however, there are unnatural compositions: this is what *connection theory* is about (subsequent work). The groupoid aspect of connection theory, going back to Charles Ehresmann ([E65, KPRW07]), is strongly stressed in SDG ([Ko10]), but I have the impression that only with the present approach it really becomes clear why it is so natural: the very foundations of calculus rely on it. I believe that many other topics of local differential geometry will be amenable, in a similar way, to a “synthetic” approach giving a link between “the finite” and “the infinitesimal”.

Notation. We use small sans serif letters to denote the following finite subsets of \mathbb{N} :

$$1 = \{1\}, \quad 2 = \{1, 2\}, \quad 3 = \{1, 2, 3\}, \dots, \quad n = \{1, \dots, n\}.$$

We let also $n' = \{1', \dots, n'\}$ be a formal copy of n , disjoint from n , and

$$\bar{n} := n \cup n' = \{1, 1', 2, 2', \dots, n, n'\}.$$

The power set of a set A is denoted by $\mathcal{P}(A)$. In particular, $\mathcal{P}(\mathfrak{n})$ is the *natural n-hypercube*, and $\mathcal{P}(\bar{\mathfrak{n}})$ is the natural $2n$ -hypercube (Appendix A). The natural n -hypercube is the index set for n -fold categories (Appendix B). To simplify notation, for elements of a \mathbb{K} -module V , instead of $v_\emptyset, v_{\{1\}}, v_{\{1,2\}}, \dots$ we throughout write v_0, v_1, v_{12}, \dots , and likewise for elements of \mathbb{K} .

Acknowledgment. I gratefully acknowledge that I have a permanent position at a public university. I also acknowledge having received the advice that long range and novel work is not suitable for mathematical journals, and that one should better wait and collect material until it is presentable in monograph form – I shall follow this advice (and I just hope that the delay will not be longer than the usual length of a human life). Anyways, comments and suggestions are welcome, and I thank the reader in advance for letting me know.

2. THE THREEFOLD WAY OF HIGHER ORDER CALCULUS

2.1. Full cubic calculus. Recall from Part I the definitions of $U^{[1]}$, $(U^{[1]})^\times$, and $f^{[1]}$. The “full” cubic calculus ([BGN04]) is based on direct iteration of (1.1).

Definition 2.1 (Full cubic C^n). *Under the assumptions of topological differential calculus (i.e., \mathbb{K} a topological base ring with dense unit group, V, W topological \mathbb{K} -modules, U open in V ; cf. Part I), a map $f : U \rightarrow W$ is called of class C^n if it is C^{n-1} and if $f^{[n-1]}$ is C^1 ; so the map $f^{[n]} := (f^{[n-1]})^{[1]}$ exists and is continuous on its domain of definition*

$$U^{[n]} = (U^{[n-1]})^{[1]}.$$

The *full n-th order difference quotient map* $f^{[n]}$ has a “cubic structure”: it is a combination of values of f at 2^n *evaluation points*. We use “cubic notation”: for $n = 1$, instead of (x, v, t) , we write (v_0, v_1, t_1) , and $f^{[2]} = (f^{[1]})^{[1]}$ is given by

$$\begin{aligned} f^{[2]}((v_0, v_1, t_1), (v_2, v_{12}, t_{12}), t_2) &= \frac{1}{t_2} \left(f^{[1]}((v_0, v_1, t_1) + t_2(v_2, v_{12}, t_{12})) - f^{[1]}(v_0, v_1, t_1) \right) \\ (2.1) \quad &= \frac{f(v_0 + t_2v_2 + (t_1 + t_2t_{12})(v_1 + t_2v_{12})) - f(v_0 + t_2v_2)}{t_2(t_1 + t_2t_{12})} - \frac{f(v_0 + t_1v_1) - f(v_0)}{t_2t_1} \end{aligned}$$

For $n > 2$, it is rather hard to give a “closed formula” for $f^{[n]}$.

2.2. Symmetric cubic calculus. Things are considerably simplified if, at each step, one “derives” only in direction of the “space variables” v , and suppresses derivation in direction of the “time variables” t :

Definition 2.2 (Symmetric $C^{n, \text{sym}}$). *Under assumptions as above, for fixed $t \in \mathbb{K}$, let $U_t := \{(v_0, v_1) \in V^2 \mid v_0 \in U, v_0 + tv_1 \in U\}$ and write, for a C^1 -map $f : U \rightarrow W$,*

$$f_t^{[1]} : U_t \rightarrow W, \quad (v_0, v_1) \mapsto f^{[1]}(v_0, v_1, t).$$

By induction, for fixed $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{K}^n$, we put, whenever defined,

$$f_{\mathbf{t}}^{[n]} := f_{t_1, \dots, t_n}^{[n]} := (f_{t_1, \dots, t_{n-1}}^{[n-1]})_{t_n}^{[1]} : U_{\mathbf{t}}^{[n]} := (\dots (U_{t_1})_{t_2} \dots)_{t_n} \rightarrow W,$$

and we say that f is $C^{n,\text{sym}}$ if it is $C^{n-1,\text{sym}}$, and if the map $f_{(t_1,\dots,t_{n-1})}^{[n-1]}$ is C^1 (so that, in particular, $f_{\mathbf{t}}^{[n]}$ is defined and continuous).

At second order, this amounts to taking $t_{12} = 0$ in (2.1):

$$(2.2) \quad \begin{aligned} f_{t_1,t_2}^{[2]}(v_0, v_1, v_2, v_{12}) &= f^{[2]}((v_0, v_1, t_1), (v_2, v_{12}, 0), t_2) \\ &= \frac{f(v_0 + t_1 v_1 + t_2 v_2 + t_1 t_2 v_{12}) - f(v_0 + t_1 v_1) - f(v_0 + t_2 v_2) + f(v_0)}{t_1 t_2} \end{aligned}$$

This formula generalizes: using notation about hypercubes from Appendix A,

Theorem 2.3. *For invertible scalars t_1, \dots, t_n and $\mathbf{v} = (v_\alpha)_{\alpha \in \mathcal{P}(n)} \in U_{\mathbf{t}}^{[n]}$ with $v_\alpha \in V$, we have,*

$$f_{\mathbf{t}}^{[n]}(\mathbf{v}) = \frac{1}{\prod_{i=1}^n t_i} \sum_{\alpha \in \mathcal{P}(n)} (-1)^{\ell_\alpha + 1} \cdot f\left(\sum_{\beta \in \mathcal{P}(\alpha)} t_{\beta_1} \cdots t_{\beta_\ell} v_\beta\right).$$

Proof. By induction: for $n = 1$, $t = t_1$, $\mathcal{P}(1) = \{\{1\}, \emptyset\}$, we get $f_t^{[1]}(v_\emptyset, v_1) = \frac{1}{t}(f(v_\emptyset + t v_1) - f(v_\emptyset))$, as we should. Assume the formula holds at rank n . In order to compute the formula for $f_{t_1, \dots, t_{n+1}}^{[n+1]}$, compute $h_{t_{n+1}}^{[1]}$ for $h = f_{t_1, \dots, t_n}^{[n]}$. Each of the 2^n “old” evaluation points $x_\alpha := \sum_{\beta \in \mathcal{P}(\alpha)} t_{\beta_1} \cdots t_{\beta_\ell} v_\beta$ gives rise to another “new” evaluation point $x_\alpha + t_{n+1} v_{\alpha \cup \{n+1\}}$, which together with the 2^n “old” evaluation points give the 2^{n+1} evaluation points from the claim. Mind also the sign-changes and division by t_{n+1} in order to get the formula from the claim at rank $n + 1$. \square

The formula may be qualified as *cubic*, and (in contrast to (2.1)) has an obvious symmetry property with respect to the action of the permutation group S_n acting on indices, whence the term “symmetric”. Moreover, the usual higher order differentials $d^n f(v_0)$ appear when all $t_i = 0$ and all $v_\alpha = 0$ whenever α has more than one digit, and then the S_n -symmetry gives Schwarz’ lemma (symmetry of higher differentials), cf. [BGN04]. Thus the symmetric cubic calculus allows to recover easily certain notions of usual calculus.

2.3. Simplicial calculus. In [Be13], I take as departure point an “explicit formula” which can be seen as a simplicial analog of the explicit formula from Theorem 2.3. We shall give a conceptual interpretation in subsequent work.

Theorem 2.4. *Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , V, W locally convex topological vector spaces, and $f : U \rightarrow W$ a map defined on an open set of V . Then the following are equivalent:*

- (1) f is of class C^n in the “usual” sense (Michal-Bastiani, cf. [BGN04]),
- (2) f is $C^{n,\text{sym}}$ in the sense defined above,
- (3) f is C^n in the full cubic sense defined above,
- (4) f is $C^{<n>}$ as defined in [Be13].

Proof. The equivalence of (1) and (3) is shown in [BGN04] (and for the implication (1) \Rightarrow (3) we need the specific assumptions on \mathbb{K}, V , and W). It is clear that “full C^n ” is stronger than “symmetric C^n ”, so (3) \Rightarrow (2). By induction, one shows that (2) \Rightarrow (1); indeed, (2) implies existence and continuity of $f_0^{[1]}$, which amounts to (1) (existence and joint continuity of all directional derivatives). As shown in [Be13],

Th. 2.6., we have (3) \Rightarrow (4) for any topological ring. Finally, (4) \Rightarrow (1) since by [Be13], Cor. 2.8, the usual differentials are partial maps of $f^{<k>}$. \square

If V is finite-dimensional, all of these properties are equivalent to any of the classical definitions of the class C^n . For a general topological ring we have (3) \Rightarrow (2) \Rightarrow (1), but the converses fail in general. For the converse of (3) \Rightarrow (4), we do not know.

3. THE GENERAL PATTERN

We define the symbols G^n , $G\text{fi}^n$, $G^{\bar{n}}$ and $G\text{fi}^{\bar{n}}$, and derive their basic properties.

3.1. Zero order. For linear sets $U \subset V$, $U' \subset V'$ and a map $f : U \rightarrow U'$, we let

$$G^{\emptyset}U := G^{\bar{\emptyset}}U := U, \quad G^{\emptyset}f := G^{\bar{\emptyset}}f := f.$$

3.2. First order groupoids and double cats. We introduce first order notation. Recall (Appendix A) that the *1-cube* has two vertices: \emptyset and 1 , corresponding to the sets of objects, resp. of morphisms, and one edge: $(\emptyset, 1)$, corresponding to the projections from morphisms to objects. The *extended 1-cube* has four vertices, and so there will be four vertex sets labelled by the four vertices $\emptyset, \{1\}, \{1'\}, \{1, 1'\}$ of $\bar{1} = \{1, 1'\}$; there are four pairs of projections, four injections and four compositions corresponding to the four edges. The edges $(\emptyset, \{1\})$ and $(\{1'\}, \{1', 1\})$ are *of the first kind*, and the edges $(\emptyset, \{1'\})$ and $(\{1\}, \{1, 1'\})$ are *of the second kind*.

Definition 3.1. For a linear set $U \subset V$, we define sets, zero section, projections

$$\begin{aligned} G^{1;1}U &= U^{\{1\}} = \{(v_0, v_1, t_1) \in V^2 \times \mathbb{K} \mid v_0 \in U, v_0 + v_1 t_1 \in U\}, \\ G^{\emptyset;1}U &= U \times \mathbb{K} \end{aligned}$$

$$z^{\emptyset,1;1}, : G^{\emptyset;1}U \rightarrow G^{1;1}U, \quad (v_0, t_1) \mapsto (v_0, 0, t_1)$$

$$\pi_0^{\emptyset,1;1} : G^{1;1}U \rightarrow G^{\emptyset;1}U \quad (v_0, v_1, t_1) \mapsto (v_0, t_1),$$

$$\pi_1^{\emptyset,1;1} : G^{1;1}U \rightarrow G^{\emptyset;1}U \quad (v_0, v_1, t_1) \mapsto (v_0 + t_1 v_1, t_1),$$

and composition given on $U^1 \times_{U \times \mathbb{K}} U^1$ by

$$(v'_0, v'_1, t_1) * (v_0, v_1, t) = (v_0, v_1 + v'_1, t).$$

These data define a groupoid, denoted by $U^{\{1\}}$ in Part I,

$$G^1U := (G^{\emptyset;1}U, G^{1;1}U, \pi_{\sigma}^{\emptyset,1;1}, z^{\emptyset,1;1}, *).$$

For fixed $t_1 = t$, we use the notation G_t^1U for the groupoid obtained by freezing the last variable. Its morphism set is

$$G_t^{1;1}U = U_t = \{(v_0, v_1) \in V^2 \mid (v_0, v_1, t) \in U^1\}.$$

The description by triples (v_0, v_1, t_1) will be called the affine coordinates of G^1U .

We shall give explicit formulae with respect to affine coordinates also at higher order; however, note that affine coordinates are not an intrinsic feature (in geometric terms, they depend on the flat affine connection given by a chart domain).

Definition 3.2. For a linear set $U \subset V$, we define vertex sets

$$\begin{aligned} \mathbf{G}^{\bar{1};\bar{1}}U &= U^{\bar{\{1\}}} = \{(v_0, v_1, s_1, t_1) \in V^2 \times \mathbb{K}^2 \mid v_0 \in U, v_0 + v_1 s_1 t_1 \in U\}, \\ \mathbf{G}^{\emptyset;\bar{1}}U &= U \times \mathbb{K}, \\ \mathbf{G}^{1;\bar{1}}U &= \mathbf{G}^{1;1}U = U^{\{1\}}, \\ \mathbf{G}^{\{1'\};\bar{1}}U &= U \times \mathbb{K}^2, \end{aligned}$$

and edge projections $\pi_{\sigma}^{\beta,\alpha;\bar{1}} : \mathbf{G}^{\alpha;\bar{1}}U \rightarrow \mathbf{G}^{\beta;\bar{1}}U$ for each of the four edges (β, α) , as described in Part I, Section 3.2, as well as zero sections and four compositions, as defined in loc. cit. These data define a small double category $\mathbf{G}^{\bar{1}}U$ (and denoted by $U^{\bar{\{1\}}}$ from Part I). Sometimes, as in Part I, projections and composition corresponding to the two edges of second kind will be denoted by ∂ and \bullet .

Definition 3.3. Recall the linear set $0 = (0, 0)$ (zero \mathbb{K} -module). The first order scaleoid is $\mathbf{G}^1 0 = (\mathbb{K}, \mathbb{K})$ (which has a trivial groupoid structure), resp. $\mathbf{G}^{\bar{1}} 0$ (which reduces to the left action category $(\mathbb{K}, \mathbb{K}^2)$ of \mathbb{K}). Note that there are canonical morphisms of groupoids (and $\mathbf{G}^1 0$ is indeed a terminal object in the category of linear sets and C^1 -laws, see Def. 3.6 below)

$$0_U : \mathbf{G}^1 U \rightarrow \mathbf{G}^1 0, \quad \bar{0}_U : \mathbf{G}^{\bar{1}} U \rightarrow \mathbf{G}^{\bar{1}} 0.$$

Definition 3.4. The finite part of $\mathbf{G}^1 U$, denoted by $\mathbf{Gfi}^1 U$, is the subgroupoid lying over the invertible scalars \mathbb{K}^\times , and denoted by $(U^{\{1\}})^\times$ in Part I. The finite part of $\mathbf{G}^{\bar{1}} U$, denoted by $\mathbf{Gfi}^{\bar{1}} U$, is the sub-doublecat lying over $\mathbb{K}^\times \times \mathbb{K}^\times \Downarrow \mathbb{K}^\times$, and denoted by $(U^{\bar{\{1\}}})^\times$ in Theorem 3.6 of Part I. Recall from that theorem that the finite part is isomorphic to the direct product of the pair groupoid of U with the one of \mathbb{K}^\times .

Theorem 3.5. Under the assumptions of topological differential calculus, for a map $f : U \rightarrow W$ the following are equivalent:

- (1) f is of class $C_{\mathbb{K}}^1$,
- (2) f induces a continuous morphism of groupoids $\mathbf{G}^1 f : \mathbf{G}^1 U \rightarrow \mathbf{G}^1 W$ such that $0_W \circ \mathbf{G}^1 f = 0_U$,
- (3) f induces a continuous morphism of small double cats $\mathbf{G}^{\bar{1}} f : \mathbf{G}^{\bar{1}} U \rightarrow \mathbf{G}^{\bar{1}} W$ such that $\bar{0}_W \circ \mathbf{G}^{\bar{1}} f = \bar{0}_U$.

Proof. The two vertex maps of $\mathbf{G}^1 f$ will be denoted by

$$\mathbf{G}^{1;1} f : \mathbf{G}^{1;1} U \rightarrow \mathbf{G}^{1;1} W, \quad \mathbf{G}^{\emptyset;1} f : \mathbf{G}^{\emptyset;1} U \rightarrow \mathbf{G}^{\emptyset;1} W,$$

and likewise for the four vertex maps of $\mathbf{G}^{\bar{1}} f$. As seen in Part I, in affine coordinates, $\mathbf{G}^{1;1} f$ is necessarily given by

$$f^{\{1\}}(v_0, v_1, t_1) = (f(v_0), f^{\{1\}}(v_0, v_1, t_1), t_1),$$

where $f^{\{1\}}$ is a *difference factorizer* of f . If f is C^1 , a continuous difference factorizer exists, and it is automatically additive and homogeneous, hence f induces morphisms of groupoids and of double cats. Conversely, if a continuous $\mathbf{G}^1 f$ exists, then a continuous $f^{\{1\}}$ exists, and f is C^1 . \square

Definition 3.6. Let \mathbb{K} be an arbitrary ring and $(U, V), (U', V')$ be linear sets. A law of class C^1 is a morphism $\mathbf{f}^1 = (\mathbf{f}^{0;1}, \mathbf{f}^{1;1})$ from \mathbf{G}^1U to \mathbf{G}^1U' such that $0_{U'} \circ \mathbf{f}^1 = 0_U$, and a homogeneous law is a morphism $\mathbf{f}^{\bar{1}}$ from $\mathbf{G}^{\bar{1}}U$ to $\mathbf{G}^{\bar{1}}U'$ such that $\bar{0}_{U'} \circ \mathbf{f}^{\bar{1}} = \bar{0}_U$. The concrete category of linear sets with C^1 -laws is given by linear sets $((U, V); (\mathbf{G}^{0;1}U, \mathbf{G}^{0;1}V); (\mathbf{G}^{1;1}U, \mathbf{G}^{1;1}V))$ and morphisms $(f, \mathbf{f}^{0;1}, \mathbf{f}^{1;1})$. In this ccat, pullbacks are defined by Definition E.2.

Theorem 3.7. Every polynomial law induces a homogeneous law of class C^1 .

Proof. Part I, Theorem 4.15. □

3.3. Copies of k -th generation. We define the symbols $\mathbf{G}^{\{k\}}$ and \mathbb{K}_k to be formal copies of $\mathbf{G}^{\{1\}}$ and \mathbb{K} , together with a label $k \in \mathbb{N}$ remembering its “generation”:

$$(3.1) \quad \begin{aligned} \mathbf{G}^{\{k\}}U &:= (\mathbf{G}^{0;\{k\}}U, \mathbf{G}^{\{k\};\{k\}}U, \pi_{\sigma}^{0,\{k\};\{k\}}, z, *) \\ &:= (U \times \mathbb{K}_{\mathbb{K}}, U^{\{k\}}, \pi_{\sigma}, z, *) \end{aligned}$$

is a formal copy, with label k , of the groupoid $\mathbf{G}^1U = (U \times \mathbb{K}, U^{\{1\}})$ (def. 3.1). We say that this copy is *of k -th generation*. In the same way, for $f : U \rightarrow U'$, the *copy of k -th generation of $f^{\{1\}}$* is

$$(3.2) \quad \mathbf{G}^{\{k\}}f := (f \times \text{id}_{\mathbb{K}}, f^{\{k\}}) : \mathbf{G}^{\{k\}}U \rightarrow \mathbf{G}^{\{k\}}U',$$

Similarly, $\overline{\mathbf{G}^{\{k\}}}$ and $\overline{f^{\{k\}}}$ are defined as a formal copy of $\overline{\mathbf{G}^1}$ and $\overline{f^{\{1\}}}$.

3.4. The n -fold groupoid \mathbf{G}^nU . To define the symbol $\mathbf{G}^n = \mathbf{G}^{\{1, \dots, n\}}$, the idea is simply to apply first $\mathbf{G}^{\{1\}}$, then $\mathbf{G}^{\{2\}}$, up to $\mathbf{G}^{\{n\}}$, to the initial datum (U, V) . At the first step this gives a pair of linear sets, forming a groupoid, at the second step four linear sets, forming a double groupoid, and at the n -th step a hypercube of 2^n linear sets, which form an n -fold groupoid (cf. Appendix B for definitions).

Theorem 3.8 (The full n -fold groupoid). *Let (U, V) be a linear set. Then, for all $n \in \mathbb{N}$, there exist uniquely defined n -fold groupoids*

$$\mathbf{G}^nU = (\mathbf{G}^{\alpha;n}U)_{\alpha \in \mathcal{P}(n)},$$

such that \mathbf{G}^1U is as in Definition 3.1, and \mathbf{G}^nU is the n -fold groupoid obtained by applying $\mathbf{G}^{\{n\}}$ to the data of the $(n-1)$ -fold groupoid $\mathbf{G}^{n-1}U$, that is,

$$\mathbf{G}^nU = \mathbf{G}^{\{1, \dots, n\}}U := \mathbf{G}^{\{n\}}(\mathbf{G}^{\{1, \dots, n-1\}}U) = (\mathbf{G}^{\{n\}} \circ \mathbf{G}^{\{n-1\}} \circ \dots \circ \mathbf{G}^{\{1\}})U.$$

In particular, projections and zero section in the following diagram are morphisms of $n-1$ -fold groupoids:

$$(3.3) \quad \begin{array}{ccc} \mathbf{G}^nU = \mathbf{G}^{\{n\}}(\mathbf{G}^{n-1}U) & & \\ \pi_{\sigma}^n \Downarrow & \uparrow z^n & \\ \mathbf{G}^{n-1}U \times \mathbb{K}_n & & \end{array}$$

Analogous statements hold for the finite part n -fold groupoid \mathbf{Gfi}^nU , which is defined inductively by $\mathbf{Gfi}^nU = \mathbf{Gfi}^{\{n\}}(\mathbf{Gfi}^{\{1, \dots, n-1\}}U) = (\mathbf{Gfi}^{\{n\}} \circ \mathbf{Gfi}^{\{n-1\}} \circ \dots \circ \mathbf{Gfi}^{\{1\}})U$. As n -fold groupoid, it is a product of iterated pair groupoids (Appendix C),

$$\mathbf{Gfi}^nU \cong \text{PG}^nU \times \text{PG}^{\{2,3, \dots, n\}}\mathbb{K}_1^{\times} \times \text{PG}^{\{3,4, \dots, n\}}\mathbb{K}_2^{\times} \times \dots \times \text{PG}^{\{n\}}\mathbb{K}_{n-1}^{\times} \times \mathbb{K}_n^{\times}.$$

Proof. Projections, injections and product $*_1$ of the groupoid $(G^1U, *_1)$ are given by polynomial (quadratic) formulae. Hence we can apply $G^{\{2\}}$ to these data, and we get again data of the same kind, i.e., a groupoid with product $G^{\{2\}}(*_1)$. Note that the product $*_1^{\{2\}}$ is defined on the set $(U^{\{1\}} \times_{U \times \mathbb{K}} U^{\{1\}})^{\{2\}}$, which, according to Theorem E.3, contains the domain of definition

$$D := G^{\{2\}}(U^{\{1\}} \times_{U \times \mathbb{K}} U^{\{1\}}) = G^{\{2\}}U^{\{1\}} \times_{G^{\{2\}}(U \times \mathbb{K})} G^{\{2\}}U^{\{1\}}$$

of $G^{\{2\}}(*_1)$, and hence is well-defined. Once the structure maps are seen to be well-defined, it is immediate from the functorial properties of $G^{\{2\}}$ that $G^{\{2\}}(G^1, *_1)$ is again a groupoid. To show that $G^{\{2\}}G^1U$ is a double groupoid, note that it carries also a groupoid structure with product $*_2$ over the base $G^1U \times \mathbb{K}$, and this groupoid structure is compatible with the preceding one since its structure maps are all of the form $G^{\{2\}}f$ for some f , and such maps are compatible with the $*_2$ -groupoid structure. Theorem E.3 is used to ensure that D is indeed stable under $*_2$.

The claim for general $n \in \mathbb{N}$ is now proved by induction, using exactly the same arguments: applying $G^{\{n\}}$ to the $(n-1)$ -fold groupoid structure of $G^{n-1}U$ yields again an $(n-1)$ -fold groupoid; concerning domains of definitions of partially defined products, this is well defined since the functor $G^{\{n\}}$ acts on pullbacks as defined in Definition E.2. The $(n-1)$ -fold groupoid is of type Goid since all structures are compatible with the groupoid structure $*_n$ coming with the functor $G^{\{n\}}$. Taken together, this shows that G^nU is an n -fold groupoid.

For Gfi , we proceed in the same way by induction, starting with Def. 3.4. As shown in Part I, $\text{Gfi}^1U \cong \text{PG}^1U \times \mathbb{K}_1^\times$, so

$$\text{Gfi}^2U \cong \text{PG}^{\{2\}}(\text{PG}^{\{1\}}U \times \mathbb{K}_1^\times) \times \mathbb{K}_2^\times = \text{PG}^{\{1,2\}}U \times \text{PG}^{\{2\}}\mathbb{K}_1^\times \times \mathbb{K}_2^\times,$$

and so on, by induction. □

The 2^n vertex sets $G^{\alpha;n}U$ of the n -fold groupoid will be described later in more detail (Section 5, Theorem 5.4). For the moment, let us just record the description of the *top vertex set* ($\alpha = n$) and of the *bottom vertex set* ($\alpha = \emptyset$):

Theorem 3.9 (Top and bottom vertex sets). *The top vertex set of G^nU is given by*

$$G^{n;n}U = U^{\{1, \dots, n\}} := (U^{\{1, \dots, n-1\}})^{\{n\}}.$$

Up to notation, this is the set $U^{[n]}$ from Definition 2.1. The bottom vertex set is

$$G^{\emptyset;n}U = U \times \mathbb{K}^n.$$

The finite part n -fold groupoid is the sub- n -fold groupoid of G^nU projecting onto its bottom vertex set

$$\text{Gfi}^{\emptyset;n}U = U \times (\mathbb{K}^\times)^n.$$

Proof. The top vertex set of G^nU is obtained by applying $G^{\{n\}}$ to the top vertex set of $G^{n-1}U$. Since the top vertex set of G^1U is $U^{\{1\}}$, the claim follows by induction. The bottom vertex set of G^nU is a direct product of \mathbb{K} with the bottom vertex set of $G^{n-1}U$. Since the bottom vertex set of G^1U is $U \times \mathbb{K}$ and the one of Gfi^1U is $U \times \mathbb{K}^\times$, the claims about the bottom vertex sets follow again by induction. □

For instance, the top vertex set of \mathbf{G}^2U is given in affine coordinates by

$$(3.4) \quad U^2 = (U^{\{1\}})^{\{2\}} = \left\{ \begin{array}{l} (v_0, v_1, v_2, v_{12}, t_1, t_2, t_{12}) \in V^4 \times \mathbb{K}^3 \mid \\ \begin{array}{l} v_0 \in U \\ v_0 + t_1 v_1 \in U, \\ v_0 + t_2 v_2 \in U \\ v_0 + t_2 v_2 + (t_1 + t_2 t_{12})(v_1 + t_2 v_{12}) \in U \end{array} \end{array} \right\}$$

(compare this with (2.1)!). Note that this formula is “not symmetric in the indices 1, 2”, indicating that the n -fold groupoid \mathbf{G}^nU is *not edge-symmetric*. Therefore, it is important to have notation taking account of issues related to the order of \mathbb{N} :

Definition 3.10. *Let $N = \{a_1, \dots, a_n\} \subset \mathbb{N}$ be a non-empty finite subset. We consider N as ordered, with the natural order inherited from \mathbb{N} : we assume that $a_1 < \dots < a_n$. The n -fold groupoid $\mathbf{G}^N U$ is defined as in the preceding theorem by*

$$\mathbf{G}^N U := (\mathbf{G}^{\{a_n\}} \circ \dots \circ \mathbf{G}^{\{a_1\}})U.$$

For each vertex α , the vertex spaces are denoted by $\mathbf{G}^{\alpha;N}U$; the top vertex set is

$$\mathbf{G}^{N;N}U = U^N = (\dots (U^{\{a_1\}})^{\{a_2\}} \dots)^{\{a_n\}}$$

and the bottom vertex set is $\mathbf{G}^{\emptyset;N}U = U \times \mathbb{K}^n$. For each edge (β, α) , the edge projections and injections are denoted by $\pi_U^{\beta, \alpha; N}$, resp. $z_U^{\beta, \alpha; N}$, and the groupoid law belonging to the edge category is denoted by $*^{\beta, \alpha; N}$. More generally, if (γ, α) is a k -cube in $\mathcal{P}(N)$ (so $\gamma \subset \alpha$ and $|\alpha \setminus \gamma| = k$), there is a unique section

$$z^{\gamma, \alpha; N} : \mathbf{G}^{\gamma;N}U \rightarrow \mathbf{G}^{\alpha;N}U,$$

and there are 2^k projections $\pi_\sigma^{\gamma, \alpha; N} : \mathbf{G}^{\alpha;N}U \rightarrow \mathbf{G}^{\gamma;N}U$ (at each step there is a choice between target and source projection). The section $z^{\emptyset, N; N}$ is called the bottom-up section, and the 2^n projections $\pi^{\emptyset, N; N}$ are called the top-down projections.

This definition makes it obvious that, if $N = A \cup B$ is a “Dedekind cut” (by this we mean: $\forall a \in A, \forall b \in B: a < b$), then

$$(3.5) \quad \mathbf{G}^{A \cup B} = \mathbf{G}^B \circ \mathbf{G}^A, \quad U^{A \cup B} = (U^A)^B.$$

3.5. The $2n$ -fold category $\mathbf{G}^{\bar{n}}U$. To define the symbol $\mathbf{G}^{\bar{n}} = \mathbf{G}^{\{1, 1', \dots, n, n'\}}$, we proceed as above, applying at each step the double-cat rule $\mathbf{G}^{\overline{\{k\}}}$:

Theorem 3.11. *Let (U, V) be a linear set and $n \in \mathbb{N}$. Then there exists a unique $2n$ -fold category, given by data as in Theorem B.2 of the form*

$$\mathbf{G}^{\bar{n}}U = (\mathbf{G}^{\alpha; \bar{n}}U)_{\alpha \in \mathcal{P}(\bar{n})},$$

such that $\mathbf{G}^{\bar{1}}U = \mathbf{G}^{\overline{\{1\}}}U$ is the double category defined above, and

$$\mathbf{G}^{\bar{n}}U = \mathbf{G}^{\overline{\{1, \dots, n\}}}U := \mathbf{G}^{\overline{\{n\}}}(\mathbf{G}^{\overline{\{1, \dots, n-1\}}}U) = (\mathbf{G}^{\overline{\{n\}}} \circ \mathbf{G}^{\overline{\{n-1\}}} \circ \dots \circ \mathbf{G}^{\overline{\{1\}}})U.$$

Proof. The theorem is proved by induction, in the same way as Theorem 3.8, by applying at each step a functor of the type $\mathbf{G}^{\overline{\{k\}}}$ to the structure from the preceding level (cf. Section B.3). \square

The symbols $\mathbf{Gfi}^{\bar{N}}$ and $\mathbf{G}^{\bar{N}}$ can now be defined similarly as in 3.10, and the structure of the finite part can be described similarly as in Theorem 3.8.

3.6. Morphisms, and laws.

Definition 3.12. Assume $(U, V), (U', V')$ are linear sets over \mathbb{K} . A C^n -law between U and U' is given by morphisms of k -fold groupoids $\mathbf{f}^k : \mathbf{G}^k U \rightarrow \mathbf{G}^k U'$ for $k = 0, \dots, n$, compatible with each other, in the sense that they complete diagram (3.3)

$$(3.6) \quad \begin{array}{ccc} \mathbf{G}^k U = \mathbf{G}^{\{k\}}(\mathbf{G}^{k-1} U) & \xrightarrow{\mathbf{f}^k} & \mathbf{G}^k U' \\ \pi_\sigma^n \Downarrow \quad \uparrow z^n & & \Downarrow \quad \uparrow \\ \mathbf{G}^{k-1} U \times \mathbb{K}_k & \xrightarrow{\mathbf{f}^{k-1} \times \text{id}_{\mathbb{K}}} & \mathbf{G}^{k-1} U' \times \mathbb{K}_k. \end{array}$$

The set map $f := \mathbf{f}^0 : U \rightarrow U'$ is called the base map of \mathbf{f}^n . A C^∞ -law between U and U' is given by morphisms of k -fold groupoids $\mathbf{f}^k : \mathbf{G}^k U \rightarrow \mathbf{G}^k U'$ for each $k \in \mathbb{N}$ which are compatible with each other, in the sense explained above. In the same way, homogeneous laws are defined by replacing \mathfrak{n} by $\bar{\mathfrak{n}}$ and morphisms of n -fold groupoids by morphisms of $2n$ -fold categories.

Definition 3.13. We denote by $\underline{C_{\mathbb{K}}^n\text{-linset}}$ (resp. $\underline{C_{\mathbb{K}}^{\bar{n}}\text{-linset}}$) the concrete category of linear sets with (homogeneous) C^n -laws as morphisms: objects of $\underline{C_{\mathbb{K}}^n\text{-linset}}$ are families of linear sets $(U, \mathbf{G}^n U; V, \mathbf{G}^n V)$ and morphisms are C^n -laws (f, \mathbf{f}^n) . Pull-backs in this category are defined by the base linear set $A \times_C B$ (see Definition E.2) and the extended sets

$$\mathbf{G}^n(A \times_C B) := \mathbf{G}^n A \times_{\mathbf{G}^n C} \mathbf{G}^n B$$

(which form again an n -fold groupoid, by induction from Theorem E.3). In particular, for $C = 0$ we get the behaviour of \mathbf{G}^n with respect to cartesian products

$$\mathbf{G}^n(A \times B) = \mathbf{G}^n A \times_{\mathbf{G}^n 0} \mathbf{G}^n B.$$

Similar definitions are given for the ccat of linear sets with homogeneous C^n -laws, with the symbol \mathbf{G}^n replaced by $\mathbf{G}^{\bar{n}}$.

Theorem 3.14. Assume V and W are \mathbb{K} -modules and $P : \underline{V} \rightarrow \underline{W}$ is a polynomial law (cf. Part I, Def. 4.12). Then P induces homogeneous C^∞ -laws.

Proof. By induction from Theorem 3.7. \square

For general laws, the base map does not always determine uniquely the whole law; however, on the level of the finite part, they do:

Theorem 3.15. Every set-map $f : U \rightarrow U'$ induces unique morphisms of finite part n -fold groupoids, resp. $2n$ -fold cats, compatible with each other, such that $f = \mathbf{f}^0$.

Proof. For $n = 1$, recall from Part I, that, over \mathbb{K}^\times , the difference factorizer can be expressed as a difference quotient of the base map, and is uniquely defined in this way. For $n > 1$, the claim now follows by induction. \square

In the topological setting, we can replace $\mathbf{G}f$ by \mathbf{G} , “by density”:

Theorem 3.16. Assume \mathbb{K} is a topological ring with dense unit group and $(U, V), (U', V')$ are non-empty open linear sets in topological \mathbb{K} -modules. Then, for all $n \in \mathbb{N} \cup \{\infty\}$, we have a bijection between

- C^n -maps $f : U \rightarrow U'$,
- continuous C^n -laws $\mathbf{f}^n : \mathbf{G}^n U \rightarrow \mathbf{G}^n U'$,

- continuous homogeneous C^n -laws $\mathbf{f}^{\bar{n}} : \mathbf{G}^{\bar{n}}U \rightarrow \mathbf{G}^{\bar{n}}U'$.

The C^n -laws \mathbf{f}^n , resp. $\mathbf{f}^{\bar{n}}$ are then entirely defined by the base map f .

Proof. For $n = 1$, this is Theorem 3.5, and for $n > 1$, it follows by induction. \square

Definition 3.17. For $f : U \rightarrow U'$ as in Theorem 3.15, the unique laws determined by f will be denoted

$$\mathbf{Gfi}^n f : \mathbf{Gfi}^n U \rightarrow \mathbf{Gfi}^n U', \quad \mathbf{Gfi}^{\bar{n}} f : \mathbf{Gfi}^{\bar{n}} U \rightarrow \mathbf{Gfi}^{\bar{n}} U',$$

and if f is a C^n -map as in Theorem 3.16, the unique induced laws are denoted by

$$\mathbf{G}^n f := \mathbf{f}^n : \mathbf{G}^n U \rightarrow \mathbf{G}^n U', \quad \mathbf{G}^{\bar{n}} f := \mathbf{f}^{\bar{n}} : \mathbf{G}^{\bar{n}} U \rightarrow \mathbf{G}^{\bar{n}} U'.$$

In both situations, we say that these laws are obtained “by deriving the map f ”. We also use such notation and language when f is polynomial law, and especially when f is constant, linear or bilinear.

The “derived laws” of constant, linear and bilinear polynomials are computed by induction, using the results for $n = 1$ in Part I, Section 4.

3.7. Scaleoids, and the terminal map.

Definition 3.18. By scaledoid, we mean the family of n -fold groupoids, resp. of $2n$ -fold categories $\mathbf{G}^n 0, \mathbf{G}^{\bar{n}} 0$, for $n \in \mathbb{N}$, where $0 = (\{0\}, \{0\})$ is the zero-space, seen as linear set over \mathbb{K} .

For $n = 1$, we have $\mathbf{G}^1 0 = (\mathbb{K}, \mathbb{K})$ with trivial groupoid structure, so, by induction, $\mathbf{G}^n 0$ really is an $(n - 1)$ -fold groupoid. Likewise, $\mathbf{G}^{\bar{1}} 0 = (\mathbb{K}, \mathbb{K}^2)$ is the left action category of \mathbb{K} on itself (Part I, Theorem 3.6), and so $\mathbf{G}^{\bar{n}} 0$ really is a small $(2n - 1)$ -fold cat. Its structure is complicated; however, for the finite part, we easily get

Theorem 3.19. The finite part of $\mathbf{G}^n 0$ is isomorphic to a product of iterated pair groupoids

$$\mathbf{Gfi}^n 0 \cong \mathbf{PG}^{\{2,3,\dots,n\}} \mathbb{K}_1^\times \times \mathbf{PG}^{\{3,4,\dots,n\}} \mathbb{K}_2^\times \times \dots \times \mathbf{PG}^{\{n\}} \mathbb{K}_{n-1}^\times \times \mathbb{K}_n^\times.$$

Proof. This follows from Theorem 3.8 for $U = 0$. \square

Theorem 3.20 (The terminal map). Assume (U, V) is a linear set.

- (1) The constant map $0_U : U \rightarrow 0$ induces [homogeneous] C^∞ laws

$$0_U^n : \mathbf{G}^n U \rightarrow \mathbf{G}^n 0, \quad 0_U^{\bar{n}} : \mathbf{G}^{\bar{n}} U \rightarrow \mathbf{G}^{\bar{n}} 0.$$

- (2) For each $x \in U$, the map $0 \rightarrow U, 0 \mapsto x$ induces [homogeneous] C^∞ -laws $\mathbf{G}^n 0 \rightarrow \mathbf{G}^n U$ and $\mathbf{G}^{\bar{n}} 0 \rightarrow \mathbf{G}^{\bar{n}} U$ which are sections of the morphisms from (1).
 (3) Every C^n -law \mathbf{f}^n is compatible with terminal maps in the sense that

$$0_{U'}^n \circ \mathbf{f}^n = 0_U^n.$$

Proof. (1) and (2) follow from Theorem 3.14 and the fact that constant maps are polynomial, and (3) follows from the compatibility condition (3.6): for $n = 1$, it implies that $f^{\{1\}}(x, v, t) = (f(x), f^{\{1\}}(x, v, t), t)$, whence $0_{U'}^1 \circ \mathbf{f}^1 = 0_U^1$, and by induction the claim follows. \square

3.8. C^∞ -manifold laws, and general C^∞ -spaces. Following Subsection 5.1 of Part I, we may now define C^n -manifold laws over \mathbb{K} , and prove that, if the manifold law is handy, it gives rise to an n -fold groupoid $\mathbf{G}^n M$, which we call the n -fold magnification of M , resp. a small $2n$ -fold cat $\mathbf{G}^{\bar{n}} M$, everything projecting to the scaleoid $\mathbf{G}^n 0$, resp. to $\mathbf{G}^{\bar{n}} 0$. Moreover, the finite part $\mathbf{Gf}^n M$ admits a decomposition into iterated pair groupoids, as stated for $M = U$ in Theorem 3.8. If the manifold law is not assumed to be handy, all statements remain valid for *local* structures, as in Part I.

Since the general category of C^∞ -manifolds thus defined is not cartesian closed and lacks existence of inverse images (Part I, Section 5.2), one will wish to define more general categories of C^∞ -spaces: these shall be concrete categories of small $2n$ -fold cats, projecting to the final object “scaleoid”, and containing finite parts of iterated pair groupoids lying over the finite part of the scaleoid. I will come back to such issues in subsequent work.

4. SYMMETRIC CUBIC CALCULUS

4.1. The n -fold groupoids $\mathbf{Gsy}_t^n U$ and $\mathbf{Gsy}^n U$. The basic principle of symmetric cubic calculus is to apply the cat rule \mathbf{G}_t (Definition 3.1) over and over again, by varying the value of t , so that after n steps we get an n -fold cat rule $\mathbf{Gsy}_t^n := \mathbf{G}_{t_n} \circ \dots \circ \mathbf{G}_{t_1}$. On the technical level, the procedure is simpler than in full cubic calculus since each of the functors \mathbf{G}_t preserves cartesian products (they are examples of “product preserving functors” in the sense of [KMS93]), so the discussion of pullbacks from Appendix E can be skipped. As a drawback, implementation of “homogeneity” into the set-up is less natural than in the preceding section, so we will mainly speak about n -fold groupoids and not about $2n$ -fold cats.

Theorem 4.1 (The symmetric cubic n -fold groupoids). *Let (U, V) be a linear set and $n \in \mathbb{N}$, and fix $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{K}^n$. Then there exists a unique n -fold groupoid*

$$\mathbf{Gsy}_t^n U = (\mathbf{Gsy}_t^{\alpha; n} U)_{\alpha \in \mathcal{P}(n)},$$

such that $\mathbf{Gsy}_t^1 U$ is the groupoid defined in Definition 3.1, and $\mathbf{Gsy}_t^n U$ is the n -fold groupoid obtained by applying $\mathbf{G}_{t_n}^{\{n\}}$ to the data of the $(n-1)$ -fold groupoid $\mathbf{Gsy}_{t_1, \dots, t_{n-1}}^{n-1} U$, that is,

$$\mathbf{Gsy}_t^n U = \mathbf{Gsy}_t^{\{1, \dots, n\}} U := \mathbf{G}_{t_n}^{\{n\}} (\mathbf{Gsy}_{t_1, \dots, t_{n-1}}^{\{1, \dots, n-1\}} U) = (\mathbf{G}_{t_n}^{\{n\}} \circ \mathbf{G}_{t_{n-1}}^{\{n-1\}} \circ \dots \circ \mathbf{G}_{t_1}^{\{1\}}) U.$$

Proof. We use the same arguments as in the proof of Theorem 3.8. \square

Next, to describe vertex sets and edge projections we need the following definition. A by-product of the computations will be the non-trivial fact that there is a natural action of the group S_n of permutations on n letters.

Definition 4.2 (The ring action). *For any $\alpha = \{\alpha_1, \dots, \alpha_\ell\} \in \mathcal{P}(n)$, let V_α be a copy of V , on which we let act the ring \mathbb{K}^n (direct product of rings) in the following way:*

$$\forall \mathbf{t} \in \mathbb{K}^n, \forall v_\alpha \in V_\alpha : \quad \mathbf{t} \cdot v_\alpha := t_{\alpha_1} \cdots t_{\alpha_\ell} \cdot v_\alpha.$$

Theorem 4.3 (Structure of symmetric n -fold groupoids). *Let (U, V) be a linear set and $n \in \mathbb{N}$, and fix $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{K}^n$.*

(1) *For $\alpha \in \mathcal{P}(n)$, the vertex set $\mathbf{Gsy}_{\mathbf{t}}^{\alpha;n}U$ of $\mathbf{Gsy}_{\mathbf{t}}^n U$ is given by*

$$U_{\mathbf{t}}^{\alpha;n} := \left\{ (v_{\beta})_{\beta \in \mathcal{P}(\alpha)} \mid \forall \beta \in \mathcal{P}(\alpha) : v_{\beta} \in V_{\beta}, \sum_{\gamma \in \mathcal{P}(\beta)} \mathbf{t} \cdot v_{\gamma} \in U \right\}.$$

The bottom vertex set is $U_{\mathbf{t}}^{\emptyset;n} = U$.

(2) *For an edge (β, α) with $\alpha = \beta \cup \{i\}$, the two edge projections are*

$$\begin{aligned} \pi_0((v_{\gamma})_{\gamma \in \mathcal{P}(\alpha)}) &= (v_{\gamma})_{\gamma \in \mathcal{P}(\beta)} \\ \pi_1((v_{\gamma})_{\gamma \in \mathcal{P}(\alpha)}) &= (v_{\gamma} + t_i v_{\gamma \cup \{i\}})_{\gamma \in \mathcal{P}(\beta)}, \end{aligned}$$

the zero section $z = z_{\beta, \alpha; n}$ comes from the natural inclusion $\mathcal{P}(\beta) \subset \mathcal{P}(\alpha)$:

$$z((v_{\gamma})_{\gamma \in \mathcal{P}(\beta)}) = (v_{\gamma})_{\gamma \in \mathcal{P}(\beta)},$$

and the corresponding groupoid composition law $ = *_{\beta, \alpha; n}$ is given by*

$$(v'_{\gamma})_{\gamma \in \mathcal{P}(\alpha)} * (v_{\gamma})_{\gamma \in \mathcal{P}(\alpha)} = (v_{\gamma}, v_{\gamma \cup \{i\}} + v'_{\gamma \cup \{i\}})_{\gamma \in \mathcal{P}(\beta)}.$$

(3) *The $2^{|\alpha|}$ top-down projections from $U_{\mathbf{t}}^{\alpha;n}$ to the bottom vertex set obtained by composing target and source mappings are given by, for $\beta \in \mathcal{P}(\alpha)$,*

$$\xi_{\alpha, \beta} : U_{\mathbf{t}}^{\alpha;n} \rightarrow U, \quad (v_{\gamma})_{\gamma \in \mathcal{P}(\alpha)} \mapsto x_{\beta} := \sum_{\gamma \in \mathcal{P}(\beta)} \mathbf{t} \cdot v_{\gamma}.$$

(4) *The “flip” or “exchange map” is an isomorphism of double groupoids:*

$$\mathbf{Gsy}_{t_1, t_2}^2 U \rightarrow \mathbf{Gsy}_{t_2, t_1}^2 U, \quad (v_0, v_1, v_2, v_{12}) \mapsto (v_0, v_2, v_1, v_{12}).$$

In particular, when $t_1 = t_2$, the double groupoid is edge-symmetric.

(5) *For any $n \in \mathbb{N}$ and $\sigma \in S_n$, there is a natural isomorphism*

$$\mathbf{Gsy}_{(t_1, \dots, t_n)}^n U \rightarrow \mathbf{Gsy}_{t_{\sigma(1)}, \dots, t_{\sigma(n)}}^n U, \quad (v_{\alpha})_{\alpha \subset n} \mapsto (v_{\sigma(\alpha)})_{\alpha \subset n},$$

and for $t_1 = \dots = t_n = t$, we get an edge-symmetric n -fold groupoid $\mathbf{Gsy}_{\mathbf{t}}^n U$.

(6) *For two linear sets $U \subset V$, $U' \subset W$, and all $\mathbf{t} \in \mathbb{K}^n$, we have a natural isomorphism of n -fold groupoids*

$$\mathbf{Gsy}_{\mathbf{t}}^n(U \times U') \cong \mathbf{Gsy}_{\mathbf{t}} U \times \mathbf{Gsy}_{\mathbf{t}}^n U'.$$

Proof. All assertions are proved by induction. Concerning (1), for $n = 1$, this is simply another notation describing the groupoid U_{t_1} . For $n = 2$, let us give explicit formulae describing the double groupoid

$$(4.1) \quad \begin{array}{ccc} \mathbf{Gsy}_{t_1, t_2}^{2;2} U & \rightrightarrows & \mathbf{Gsy}_{t_2}^{\{2\};2} U \\ \Downarrow & & \Downarrow \\ \mathbf{Gsy}_{t_1}^{1;2} U & \rightrightarrows & U. \end{array}$$

Applying Definition 3.1 in order to compute $U_{t_1, t_2}^{2,2} = (U_{t_1}^{[1]})_{t_2}^{[1]}$, we get (cf. eqn. (3.4))

$$(4.2) \quad U_{t_1, t_2}^{2,2} = \left\{ (v_0, v_1, v_2, v_{12}) \in V^4 \mid \begin{array}{l} v_0 \in U \\ v_0 + t_1 v_1 \in U, v_0 + t_2 v_2 \in U \\ v_0 + t_1 v_1 + t_2 v_2 + t_1 t_2 v_{12} \in U \end{array} \right\}$$

projecting to $U_{t_1, t_2}^{1,2} = \{(v_0, v_1) \mid v_0 \in U, v_0 + t_1 v_1 \in U\}$. On the other hand, $U_{t_1, t_2}^{\{2\}, 2} = \{(v_0, v_2) \mid v_0 \in U, v_0 + t_2 v_2 \in U\}$ is obtained by extending U with respect to t_2 . This describes the four vertex sets. We describe the four pairs of edge projections $\pi_\sigma^{\beta, \alpha; 2}$. For the edge $(\beta, \alpha) = (1, 2) = (\{1\}, \{1, 2\})$, this is given directly by the inductive definition: for $\sigma = 1$ (target),

$$\pi_{t_1, t_2}^{\{1\}, 2}(v_0, v_1, v_2, v_{12}) = (v_0, v_1) + t_2(v_2, v_{12}) = (v_0 + t_2 v_2, v_2 + t_2 v_{12})$$

and for $\sigma = 0$ (source), $\pi_{t_1, t_2}^{\{1\}, 2}(v_0, v_1, v_2, v_{12}) = (v_0, v_1)$. For the edge $(\beta, \alpha) = (\{2\}, \{1, 2\})$, we have to apply the functor \mathbf{G}_{t_2} to the edge projection corresponding to the only non trivial edge of the 1-cube: for $\sigma = 1$ (target) by “deriving” $v_0 + t_1 v_1$, we get

$$\pi_{t_1, t_2}^{\{2\}, 2; 2}(v_0, v_1, v_2, v_{12}) = (v_0 + t_1 v_1, v_2 + t_1 v_{12})$$

(in full cubic calculus, there will be here an additional term $t_{12} v_1 + t_2 t_{12} v_{12}$ to be added in the last component, causing the more complicated structure of full cubic calculus). For $\sigma = 0$ (source), we get $\pi_{t_1, t_2}^{\{2\}, 2; 2}(v_0, v_1, v_2, v_{12}) = (v_0, v_2)$. The other edge projections are, for $\sigma = 1$,

$$\pi_{t_1}^{\emptyset, 1; 2}(v_0, v_1) = v_0 + t_1 v_1, \quad \pi_{t_2}^{\emptyset, \{2\}; 2}(v_0, v_2) = v_0 + t_2 v_2,$$

and for $\sigma = 0$ they just project to v_0 . The groupoid laws are, in all cases, given by usual vector addition in certain components of \mathbf{v} . Indeed, for the edges $(\emptyset, 1)$ and $(1, 2)$ this follows again from the inductive definition (and the definition of $*$ Part I), and for the remaining edges again we have to apply the functor \mathbf{G}_{t_2} to the groupoid law from the 1-cube. But since vector addition is a *linear* map $a : V \times V \rightarrow V$ (the commutativity of $(V, +)$ is important here), we have $a^{\{2\}}(x, u, t_2) = (a(x), a(u), t_2)$ (see Part I), so $\mathbf{G}_{t_2} a(x, u) = (a(x), a(u))$. Note that the four conditions appearing in (4.2) correspond to the four projections of the top vertex set onto the bottom vertex set U , and that (4) is an immediate consequence of the explicit formulae.

For $n \geq 2$, the inductive proof follows exactly the same arguments. \square

In order to state (5) in the form that S_n acts on a certain n -fold groupoid, we incorporate the \mathbf{t} -parameter into the groupoid, as we did for $n = 1$ in Part I:

Definition 4.4. For every vertex $\alpha \in \mathcal{P}(n)$ of the natural n -hypercube, we define

$$(4.3) \quad \mathbf{Gsy}^{\alpha; n} U := \{(\mathbf{v}, \mathbf{t}) \in \prod_{\alpha \in \mathcal{P}(n)} V_\alpha \times \mathbb{K}^n \mid \mathbf{v} \in \mathbf{Gsy}_\mathbf{t}^{\alpha; n} U\}.$$

Note that the bottom vertex, corresponding to $\alpha = \emptyset$, is

$$(4.4) \quad \mathbf{Gsy}^{\emptyset; n} U = U \times \mathbb{K}^n.$$

Projections, zero sections and composition $*$ are defined as in the theorem, and with these structures $\mathbf{Gsy}^n U$ is an edge-symmetric n -fold groupoid. It comes together with a family of morphisms

$$(4.5) \quad \pi_n : \mathbf{Gsy}^n U \rightrightarrows \mathbf{Gsy}^{n-1} U \times \mathbb{K}$$

and a morphism onto the trivial n -fold groupoid \mathbb{K}^n :

$$(4.6) \quad \mathbf{Gsy}^n U \rightarrow \mathbf{Gsy}^n 0 = \mathbb{K}^n, \quad (\mathbf{v}, \mathbf{t}) \mapsto \mathbf{t}$$

The inverse image of $(\mathbb{K}^\times)^n$ under this morphism is called the finite part of $\mathbf{Gsy}^n U$:

$$(4.7) \quad \mathbf{Gsyfi}^{\alpha,n} U := \{(\mathbf{v}, \mathbf{t}) \in \mathbf{Gsy}^{\alpha,n} U \mid \forall i : t_i \in \mathbb{K}^\times\}.$$

Theorem 4.5. *The finite part is a direct product of an n -fold pair groupoid with the trivial n -fold groupoid given by $(\mathbb{K}^\times)^n$:*

$$\mathbf{Gsyfi}^n U \cong \mathbf{PG}^n(U) \times (\mathbb{K}^\times)^n.$$

On each vertex set, the isomorphism is given by the “trivialization maps”

$$\mathbf{Gsy}_t^{\alpha;n} U \rightarrow \mathbf{PG}^{\alpha;n} U = \prod_{\gamma \in \mathcal{P}(\alpha)} U, \quad (v_\gamma)_{\gamma \in \mathcal{P}(\alpha)} \mapsto (x_\gamma)_{\gamma \in \mathcal{P}(\alpha)},$$

where x_γ is defined as in Item (3) of Theorem 4.3.

Proof. For $n = 1$, this is Item (3) of Theorem 2.7, Part I, and for $n \in \mathbb{N}$, the claim follows by induction. In general, if an n -fold groupoid is isomorphic to an n -fold pair groupoid, then the trivialization maps are given by assembling the 2^n top-down projections. \square

4.2. Scalar action: homogeneity. As explained in Part I, the groupoid structure corresponds to vector addition. Concerning multiplication by scalars, for various \mathbf{t} 's, the n -fold groupoids \mathbf{Gsy}_t are related among each other by the following

Theorem 4.6. *Let $\mathbf{t}, \mathbf{s} = (s_1, \dots, s_n) \in \mathbb{K}^n$. Then the family of maps, for $\alpha \in \mathcal{P}(n)$,*

$$\Phi_s^\alpha : \mathbf{Gsy}_{\mathbf{s} \cdot \mathbf{t}}^{\alpha,n} U \rightarrow \mathbf{Gsy}_t^{\alpha,n} U, \quad (v_\gamma)_{\gamma \in \mathcal{P}(\alpha)} \mapsto (\mathbf{s} \cdot v_\gamma)_{\gamma \in \mathcal{P}(\alpha)}$$

defines a morphism Φ_s of n -fold groupoids, where $\mathbf{s} \cdot \mathbf{t} = (s_1 t_1, \dots, s_n t_n)$, and $\mathbf{s} \cdot v_\gamma = s_{\gamma_1} \cdots s_{\gamma_\ell} v_\gamma$ is as in Definition 4.2.

Proof. For $n = 1$, this is Theorem 3.1 of Part I, and for general n , it follows by induction. \square

Remark 4.1. Like in Chapter 3 of Part I, using the action of \mathbb{K}^n from the preceding theorem, one can build an $(n + 1)$ -fold cat out of the n -fold groupoid $\mathbf{Gsy}^n U$. This $(n + 1)$ -fold cat is a subcat of the $2n$ -fold cat $\mathbf{G}^n U$ from the preceding chapter; however, it seems to be less interesting than the whole $2n$ -fold cat. For $U = 0$, the $(n + 1)$ -fold cat reduces to the left action category of \mathbb{K}^n on itself.

Theorem 4.7. *The n -fold tangent bundle of U ,*

$$\mathbf{T}^n U := \mathbf{Gsy}_{(0, \dots, 0)}^n U,$$

is a symmetric n -fold vector bundle ([GM12]), i.e., an edge-symmetric n -fold groupoid such that, for each edge, source and target projections coincide, and each vertex group carries moreover a \mathbb{K} -module structure.

Proof. Theorem 4.3 implies that, for $\mathbf{t} = 0$, we have an edge-symmetric n -fold groupoid whose source and target projections coincide, and the preceding Theorem 4.6 implies that each vertex set carries a \mathbb{K} -module structure which is invariant. \square

4.3. $C^{n,\text{sym}}$ -laws.

Definition 4.8. For $n \in \mathbb{N}$, or $n = \infty$, a $C^{n,\text{sym}}$ -law between linear sets U, U' is a family of morphism of k -fold groupoids, for $k = 0, \dots, n$,

$$\mathbf{f}^k : \text{Gsy}^k U \rightarrow \text{Gsy}^k U'$$

which are compatible with each other: with notation from Equation (4.5), we have

$$\forall k = 1, \dots, n : \quad \pi_k \circ \mathbf{f}^k = (\mathbf{f}^{k-1} \times \text{id}_{\mathbb{K}}) \circ \pi_k.$$

Such a law is called *symmetric* if each \mathbf{f}^k commutes with the S_k -action, and it is called *homogeneous* if it commutes with scalar actions:

$$\forall \mathbf{s} \in \mathbb{K}^n : \quad \mathbf{f}^n \circ \Phi_{\mathbf{s}}^\alpha = \Phi_{\mathbf{s}}^\alpha \circ \mathbf{f}^n.$$

Recall from Definition 2.3 the higher order difference factorizer $f_{\mathbf{t}}^{[n]}$, and from Theorem 2.3 its “explicit formula”. Putting things together, we get explicit and “closed” formulae for all components of $C^{n,\text{sym}}$ -laws. However, such formulae are not “intrinsic”: in geometric terms, they depend on the flat affine connection of the chart domain U .

Theorem 4.9. Let \mathbf{f}^n be a $C^{n,\text{sym}}$ -law with base map $f : U \rightarrow U'$ and assume $\mathbf{t} \in (\mathbb{K}^\times)^n$. Then, for a general vertex $\alpha \in \mathcal{P}(\mathbf{n})$, the vertex map can be expressed in terms of the higher order difference factorizers of f by

$$\text{Gsy}_{\mathbf{t}}^{\alpha;n} f(\mathbf{v}_\alpha) = (f_{\mathbf{t}_\beta}^{[\ell_\beta]}(\mathbf{v}_\beta))_{\beta \subset \alpha},$$

where $\mathbf{v}_\lambda = (v_\mu)_{\mu \subset \lambda}$. If, moreover, \mathbb{K} is a topological base ring and $f : U \rightarrow W$ is $C^{n,\text{sym}}$ (Definition 2.2), then the same formulae define a continuous, symmetric and homogenous $C^{n,\text{sym}}$ -law \mathbf{f}^n , which is entirely determined by its base map f .

Proof. For $n = 1$, a C^1 -law is explicitly described by its top-vertex map

$$\text{G}_{t_1}^{1,1} f((v_0, v_1)) = (f(v_0), f_{t_1}^{[1]}(v_0, v_1, t_1)),$$

Iterating this for $n = 2$, we get

$$\text{Gsy}_{(t_1, t_2)}^{2,2} f((v_0, v_1, v_2, v_{12})) = (f(v_0), f_{t_1}^{[1]}(v_0, v_1), f_{t_2}^{[1]}(v_0, v_2), f_{t_1, t_2}^{[2]}(v_0, v_1, v_2, v_{12})),$$

where $f^{[2]}$ is explicitly given by Equation (2.2). For general \mathbf{n} and $\alpha \in \mathcal{P}(\mathbf{n})$, the computation is the same, by induction. Note that, on the finite part, this law is uniquely determined by the base map f , and moreover, it is symmetric, since the explicit formulae obviously have a symmetric structure.

Now, in a topological setting, if f is $C^{n,\text{sym}}$, then, “by density”, all properties carry over to arbitrary \mathbf{t} . Indeed, by definition of the $C^{n,\text{sym}}$ -property, the law can be defined by induction, and the explicit formulae show that it is continuous. Symmetry now carries over by density, and so does homogeneity. \square

4.4. Scalar extensions, and symmetry of polynomial laws. Recall from Part I, Lemma 4.13, that \mathbf{G}_t^1 , applied to the ring product $\mathbb{K} \times \mathbb{K} \rightarrow \mathbb{K}$, gives the ring $\mathbb{K}[X]/(X^2 - tX)$, together with its two projections to \mathbb{K} . Likewise, \mathbf{G}_t^1 applying it to the (right) action map $V \times \mathbb{K} \rightarrow V$ gives the $\mathbb{K}[X]/(X^2 - tX)$ -module structure of the scalar extended module.

Theorem 4.10. *Fix $\mathbf{t} \in \mathbb{K}^n$. Applying \mathbf{Gsy}_t^n to the ring product $\mathbb{K} \times \mathbb{K} \rightarrow \mathbb{K}$ yields a ring of type $\underline{\mathbf{Goid}}_n$, that is, an n -fold groupoid $\mathbf{G}_t^n \mathbb{K}$, such that each vertex set $\mathbf{Gsy}_t^{\alpha;n} \mathbb{K}$ carries a ring structure, and both structures are compatible with each other: the structure maps of the n -fold groupoid are morphisms of rings, and the collection of ring structures defines a morphism of n -fold groupoids $\mathbf{Gsy}_t^n(\mathbb{K} \times \mathbb{K}) \rightarrow \mathbf{Gsy}_t^n \mathbb{K}$.*

Proof. For $n = 1$, see the result from Part I quoted above; for $n \in \mathbb{N}$ the claim now follows by induction. \square

Theorem 4.11. *An algebraic model for the structures from the preceding theorem can be constructed as follows: for $\alpha \in \mathcal{P}(n)$, consider the ring*

$$\begin{aligned} \mathbb{A}_t^{\alpha;n} &:= \mathbb{K}[X]/(X^2 - t_{\alpha_1}X) \otimes \dots \otimes \mathbb{K}[X]/(X^2 - t_{\alpha_\ell}X) \\ &= \mathbb{K}[X_{\alpha_1}, \dots, X_{\alpha_n}]/(X_{\alpha_1}^2 - t_{\alpha_1}X_{\alpha_1}, \dots, X_{\alpha_\ell}^2 - t_{\alpha_\ell}X_{\alpha_\ell}). \end{aligned}$$

For every edge $(\beta, \alpha; n)$, there are edge projections (source and target) and injections, such that the whole structure forms a ring of type $\underline{\mathbf{Goid}}_n$, isomorphic to $\mathbf{Gsy}_t^n \mathbb{K}$.

Proof. Again, for $n = 1$ this is shown in Part I, 4.4 and 4.5, and for $n \in \mathbb{N}$ it follows by induction. \square

Theorem 4.12. *Assume $P : \underline{V} \rightarrow \underline{W}$ is a polynomial law. Then P defines a homogeneous symmetric $C^{\infty, \text{sym}}$ -law between V and W .*

Proof. By induction it follows from Theorem 4.13 of Part I that P defines a homogeneous $C^{\infty, \text{sym}}$ -law between V and W . It remains to prove that this law is *symmetric* (note that the “density argument”, proving the corresponding fact in topological differential calculus, is not available here!). Now, by definition these laws are constructed by scalar extension with the rings $\mathbb{A}_t^\alpha = \mathbf{Gsy}_t^\alpha \mathbb{K}$, and the explicit formula shows that the group S_n acts by automorphisms on this ring of type $\underline{\mathbf{Goid}}_n$ (it permutes the X_{α_i} and at the same time the t_{α_i}), and hence the laws defined by scalar extension also commute with this action. \square

5. FULL CUBIC CALCULUS

The structure of the full cubic higher order groupoids is more complicated than the structure of the symmetric ones – the main reason is that in symmetric calculus *scaleoids* $\mathbf{Gsy}^n 0$ have a trivial n -fold groupoid structure, whereas in full calculus, complications are concentrated in the rather intricate structure of the scaleoids $\mathbf{G}^n 0$. In this chapter, we initiate the study of these higher order structures; but by no means we pretend that this study were complete. Let us start by describing in more detail the algorithm allowing to compute vertex sets and vertex maps. To fix notation for this chapter, let $N = \{a_1, \dots, a_n\} \subset \mathbb{N}$ be a finite subset of cardinality n , and we assume that $a_1 < \dots < a_n$; for a vertex $\alpha \in \mathcal{P}(N)$ of cardinality $k = \ell(\alpha)$, we write $\alpha = \{\alpha_1, \dots, \alpha_k\}$, and assume $\alpha_1 < \dots < \alpha_k$.

5.1. Top and bottom vertex sets. First of all, recall from Theorem 3.9 the description of the top and bottom vertex sets

$$\begin{aligned}\mathbf{G}^{N;N}U &= (\dots (U^{\{a_1\}})^{\{a_2\}} \dots)^{\{a_n\}} = U^{\{a_1, \dots, a_n\}} = U^N, \\ \mathbf{G}^{\emptyset;N}U &= U \times \mathbb{K}^n = U \times 0^{\{a_1\}} \times \dots \times 0^{\{a_n\}}.\end{aligned}$$

For $N = 1$, $U^{\{1\}} = \{(v_0, v_1, t_1) \mid v_0 \in U, t_1 \in \mathbb{K}, v_0 + t_1 v_1 \in U\}$, and for $N = 2$, see Equation (3.4). By induction, U^N is a subset of $V^{2^n} \times \mathbb{K}^{2^n-1}$. Its elements will be described by their “affine coordinates”

$$(5.1) \quad (\mathbf{v}, \mathbf{t}) = ((v_\alpha)_{\alpha \in \mathcal{P}(N)}, (t_\alpha)_{\alpha \in \mathcal{P}(N) \setminus \emptyset}),$$

where the elements $v_\alpha \in V, t_\alpha \in \mathbb{K}$ satisfy 2^n inductively defined conditions (the first of which is $v_\emptyset \in U$).

5.2. Description of vertex sets and vertex maps for \mathbf{G}^N . To define general vertex sets $\mathbf{G}^{\alpha;N}U$, the inductive procedure from Theorem 3.8 (and of Theorem B.2) leads us to distinguish, at each step of the induction procedure, the cases of “new” and “old” vertices. The following definition is designed to deal with this distinction:

Definition 5.1. *With N and $\alpha \in \mathcal{P}(N)$ as above, for all $j \in N$, we define*

$$\langle \alpha, j \rangle := \begin{cases} \{j\} & \text{if } j \in \alpha \\ \emptyset & \text{if } j \notin \alpha. \end{cases}$$

Theorem 5.2 (Computing $\mathbf{G}^{\alpha;N}$). *With notation as in the preceding definition,*

$$\mathbf{G}^{\alpha;N} = \mathbf{G}^{\langle \alpha, a_n \rangle, \{a_n\}} \circ \mathbf{G}^{\langle \alpha, a_{n-1} \rangle, \{a_{n-1}\}} \circ \dots \circ \mathbf{G}^{\langle \alpha, a_1 \rangle, \{a_1\}}.$$

This applies both to the computation of $\mathbf{G}^{\alpha;N}U$ and of $\mathbf{G}^{\alpha;N}f$: under the assumptions of topological differential calculus, or for polynomial laws, the procedure permits to compute the vertex maps $\mathbf{G}^{\alpha;N}f$ (Definition 3.17) from the base map f . In particular, top vertex and bottom vertex maps are given by

$$\begin{aligned}\mathbf{G}^{N;N}f &= (\dots (f^{\{a_1\}})^{\{a_2\}} \dots)^{\{a_n\}} =: f^{\{a_1, \dots, a_n\}}, \\ \mathbf{G}^{\emptyset;N}f &= f \times \text{id}_{\mathbb{K}}^n.\end{aligned}$$

Proof. Written out in more detail, to compute $\mathbf{G}^{\alpha;N}U$, the procedure from Theorem 3.8 says that, with $N' = N \setminus \{a_n\}$ and $\alpha' = \alpha \setminus \{a_k\}$,

(1) if the top element a_n belongs to α (“new vertex”) then we “derive”:

$$\mathbf{G}^{\alpha;N}U = \mathbf{G}^{\{a_n\}, \{a_n\}}(\mathbf{G}^{\alpha';N'}U),$$

(2) if $a_n \notin \alpha$ (“old vertex”), then we “take a product with \mathbb{K} ”

$$\mathbf{G}^{\alpha;N}U = \mathbf{G}^{\alpha;N'}U \times \mathbb{K} = \mathbf{G}^{\alpha;N'}U \times_0 0^{\{a_n\}},$$

and so on. This is exactly what the algorithm described above does. \square

Remark 5.1. When “deriving pullbacks”, i.e., when computing the composition of symbols $\mathbf{G}^{\{i\}, \{i\}} \circ \mathbf{G}^{\emptyset, \{i-1\}}$, one has to take account of Relation (E.2). For instance, the vertex $\{2\}$ of the 2-cube $\{1, 2\}$ gives rise to the vertex set

$$\mathbf{G}^{\{2\}, \{1,2\}}U = \mathbf{G}^{\{2\}, \{2\}} \circ \mathbf{G}^{\emptyset, \{1\}}U = \mathbf{G}^{\{2\}, \{2\}}(U \times 0^{\{1\}}) = U^{\{2\}} \times_{0^{\{2\}}} 0^{\{1,2\}},$$

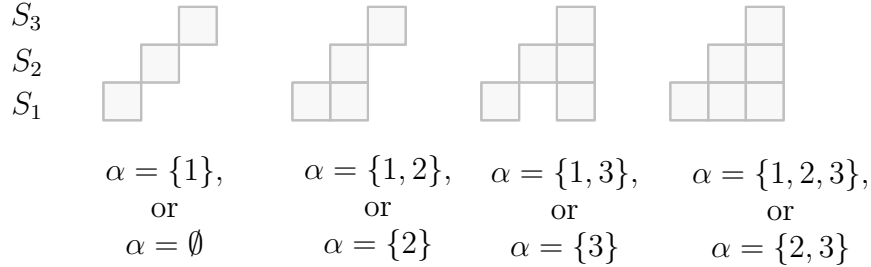
and since “affine coordinates” of $0^{\{1,2\}}$ are $(t_1, t_2, t_{12}) \in \mathbb{K}^3$, we get those of $\mathbf{G}^{\{2\},\{1,2\}}U$, namely $(v_0, v_2, t_1, t_2, t_{12})$ (where t_2 appears only once because we take the quotient by $0^{\{2\}}$). In the same way, we get the following tables for $N = 1$ and $N = 2$:

N	α	vertex set $\mathbf{G}^{\alpha;N}U$	affine coordinates of elements of $\mathbf{G}^{\alpha;N}U$
1	\emptyset	$U \times 0^{\{1\}}$	(v_0, t_1)
	$\{1\}$	$U^{\{1\}}$	(v_0, v_1, t_1)
2	\emptyset	$U \times 0^{\{1\}} \times 0^{\{2\}}$	(v_0, t_1, t_2)
	$\{1\}$	$U^{\{1\}} \times 0^{\{2\}}$	(v_0, v_1, t_1, t_2)
	$\{2\}$	$U^{\{2\}} \times_{0^{\{2\}}} 0^{\{1,2\}}$	$(v_0, v_2, t_1, t_2, t_{12})$
	$\{1, 2\}$	$U^{\{1,2\}}$	$(v_0, v_1, v_2, v_{12}, t_1, t_2, t_{12})$

To get the corresponding table of vertex sets for $N = 3$, for the “old” vertices, just copy and paste the table for $N = 2$ and add everywhere $\times 0^{\{3\}}$; for the “new” vertices, we “derive by $\{3\}$ ”: everwhere 3 is added in the superscripts, and \times_{0^A} is replaced by $\times_{0^{A \cup \{3\}}}$. This gives:

N	α	vertex set $\mathbf{G}^{\alpha;N}U$	affine coordinates of elements of $\mathbf{G}^{\alpha;N}U$
3	\emptyset	$U \times 0^{\{1\}} \times 0^{\{2\}} \times 0^{\{3\}}$	(v_0, t_1, t_2, t_3)
	$\{1\}$	$U^{\{1\}} \times 0^{\{2\}} \times 0^{\{3\}}$	$(v_0, v_1, t_1, t_2, t_3)$
	$\{2\}$	$U^{\{2\}} \times_{0^{\{2\}}} 0^{\{1,2\}} \times 0^{\{3\}}$	$(v_0, v_2, t_1, t_2, t_{12}, t_3)$
	$\{1, 2\}$	$U^{\{1,2\}} \times 0^{\{3\}}$	$(v_0, v_1, v_2, v_{12}, t_1, t_2, t_{12}, t_3)$
	$\{3\}$	$U^{\{3\}} \times_{0^{\{3\}}} 0^{\{13\}} \times_{0^{\{3\}}} 0^{\{2,3\}}$	$(v_0, v_3, t_1, t_3, t_{13}, t_2, t_{23})$
	$\{1, 3\}$	$U^{\{1,3\}} \times_{0^{\{3\}}} 0^{\{2,3\}}$	$(v_0, v_1, v_3, v_{13}, t_1, t_3, t_2, t_{13}, t_{23})$
	$\{2, 3\}$	$U^{\{2,3\}} \times_{0^{\{2,3\}}} 0^{\{1,2,3\}}$	$(v_0, v_2, v_3, v_{23}, t_1, t_2, t_3, t_{12}, t_{13}, t_{23}, t_{123})$
	$\{1, 2, 3\}$	$U^{\{1,2,3\}}$	$(v_0, v_1, v_2, v_3, v_{23}, v_{13}, v_{12}, v_{123},$ $t_1, t_2, t_3, t_{12}, t_{13}, t_{23}, t_{123})$

The underlying combinatorial scheme is what we call the α -stair, e.g., for $N = 3$:



Theorem 5.3 (Computing $\mathbf{G}^{\alpha;n}$, bis). *Let $N = \{a_1, \dots, a_n\} \subset \mathbb{N}$ (ordered), $\alpha \subset N$ and define the α -stair $S = S(\alpha)$, given by the following n subsets $S_1, \dots, S_n \subset N$:*

$$S_i = \{a_i\} \cup (\alpha \setminus \{a_1, \dots, a_{i-1}\}) = \{a_j \mid j = i \text{ or } (j < i \text{ and } a_j \in \alpha)\}.$$

Then the vertex set $\mathbf{G}^{\alpha;N}U$ is given by

$$\mathbf{G}^{\alpha;N}U = U^\alpha \times_{0^{\alpha \setminus \{a_1\}}} 0^{S_1} \times_{0^{S_1 \cap S_2}} 0^{S_2} \times \dots \times_{0^{S_{n-1} \cap S_n}} 0^{S_n}.$$

In particular, for $U = 0$, we get the vertex sets $\mathbf{G}^{\alpha;N}0$ of a point:

$$\mathbf{G}^{\alpha;N}0 = 0^\alpha \times_{0^{\alpha \setminus \{a_1\}}} 0^{S_1} \times_{0^{S_1 \cap S_2}} 0^{S_2} \times \dots \times_{0^{S_{n-1} \cap S_n}} 0^{S_n}$$

Proof. This description results in analyzing the construction from Theorem 5.2. First of all, note that sets α, α' that differ only by the element 1 give rise to the same α -stair. For $\alpha = \emptyset$, the α -stair is given by the singletons $S_i = \{a_i\}$, and the theorem describes correctly the construction of the bottom vertex set. Now, for general α , we have to take account of the occurrences of “derivation into α_i -direction”. When $\alpha_1 = a_1$, derivation concerns only the base set, so replace U by $U^{\{a_1\}}$; but if $\alpha_1 > a_1$, then it concerns all preceding indices. (In our graphical scheme, this means that we fill up all boxes over α_2 until you reach the diagonal.) We continue this way for all indices up to α_k (fill up all boxes over the the α_i until you reach the diagonal). \square

Remark 5.2. Of course, in the end, factors may cancel out: whenever $S_{k+1} \subset S_k$, the corresponding term will cancel. For instance, if $\alpha = N$ (the “solid α -stair”), then only the term U^N remains (top vertex).

Remark 5.3. The theorem shows that for scaleoids ($U = 0$), the vertex sets are uniquely determined by the α -stair. For $N = 1, 2, 3$, the following table is obtained from the preceding ones by taking $U = 0$; for $N = 4$, the reader may compute it herself, by drawing the 8 possible α -stairs :

N	α	vertex set $\mathbf{G}^{\alpha;N}0$	affine coordinates of elements of $\mathbf{G}^{\alpha;N}0$
1	$\{1\}$	$0^{\{1\}}$	(t_1)
2	$\{1\}$ $\{1, 2\}$	$0^{\{1\}} \times 0^{\{2\}}$ $0^{\{1,2\}}$	(t_1, t_2) (t_1, t_2, t_{12})
3	$\{1\}$ $\{1, 2\}$ $\{1, 3\}$ $\{1, 2, 3\}$	$0^{\{1\}} \times 0^{\{2\}} \times 0^{\{3\}}$ $0^{\{1,2\}} \times 0^{\{3\}}$ $0^{\{1,3\}} \times_{0^{\{3\}}} 0^{\{2,3\}}$ $0^{\{1,2,3\}}$	(t_1, t_2, t_3) (t_1, t_2, t_{12}, t_3) $(t_1, t_3, t_2, t_{13}, t_{23})$ $(t_1, t_2, t_3, t_{12}, t_{13}, t_{23}, t_{123})$
4	$\{1\}$ $\{1, 2\}$ $\{1, 3\}$ $\{1, 2, 3\}$ $\{1, 4\}$ $\{1, 2, 4\}$ $\{1, 3, 4\}$ $\{1, 2, 3, 4\}$	$0^{\{1\}} \times 0^{\{2\}} \times 0^{\{3\}} \times 0^{\{4\}}$ $0^{\{1,2\}} \times 0^{\{3\}} \times 0^{\{4\}}$ $0^{\{1,3\}} \times_{0^{\{3\}}} 0^{\{2,3\}} \times 0^{\{4\}}$ $0^{\{1,2,3\}} \times 0^{\{4\}}$ $0^{\{1,4\}} \times_{0^{\{4\}}} 0^{\{2,4\}} \times_{0^{\{4\}}} 0^{\{3,4\}}$ $0^{\{1,2,4\}} \times_{0^{\{4\}}} 0^{\{3,4\}}$ $0^{\{1,3,4\}} \times_{0^{\{3,4\}}} 0^{\{2,3,4\}}$ $0^{\{1,2,3,4\}}$	(t_1, t_2, t_3, t_4) $(t_1, t_2, t_{12}, t_3, t_4)$ $(t_1, t_3, t_2, t_{13}, t_{23}, t_4)$ $(t_1, t_2, t_3, t_{12}, t_{13}, t_{23}, t_{123}, t_4)$ $(t_1, t_2, t_3, t_4, t_{14}, t_{24}, t_{34})$ $(t_1, t_2, t_3, t_4, t_{12}, t_{24}, t_{14}, t_{124}, t_{34})$ $(t_1, t_2, t_3, t_4, t_{13}, t_{14}, t_{34}, t_{134}, t_{23}, t_{24}, t_{234})$ $(t_\alpha, \alpha \subset 4, \alpha \neq \emptyset)$

Remark 5.4. For a C^n -map $f : U \rightarrow U'$, the vertex maps $\mathbf{G}^n f$ can be expressed, with respect to affine coordinates, via higher order difference factorizers $f^{[k]}$: the analog of the formula from Theorem 4.9 is

$$\mathbf{G}^{\alpha;n} f(\mathbf{v}_\alpha, \mathbf{t}_\alpha) = \left((f^{[\ell_\beta]}(\mathbf{v}_\beta, \mathbf{t}_\beta))_{\beta \subset \alpha}, \mathbf{t}_\alpha \right),$$

where $\mathbf{v}_\lambda = (v_\mu)_{\mu \subset \lambda}$, and now $\mathbf{t}_\lambda = (t_\mu)_{\mu \subset \lambda}$ has also components for $|\mu| > 1$.

5.3. Source projections, unit sections, and composition law. These structural data of $\mathbf{G}^n U$ are easy to describe: fixing an edge $(\beta, \alpha; N)$ of the n -cube $\mathcal{P}(N)$, the vertex sets $\mathbf{G}^{\beta;N} U$ and $\mathbf{G}^{\alpha;N} U$ have been described above. In affine coordinates,

there is a canonical inclusion $\mathbf{G}^{\beta;N}U \subset \mathbf{G}^{\alpha;N}U$, which describes the unit (or zero) section $z^{\beta,\alpha;N}$ (by induction from the inclusion $U \times \mathbb{K} \rightarrow U^{\{1\}}$, $(v_0, t_1) \mapsto (v_0, 0, t_1)$), and there are natural projections, which describe the source projection $\pi_0^{\beta,\alpha;N}$ (by induction from $\pi_0(v_0, v_1, t_1) = (v_0, t_1)$). The inductive proofs are almost trivial since all maps in question are \mathbb{K} -linear, and their derivative is linear and of the same form (cf. Part I, Section 4.3). For the same reason, since at $n = 1$, the composition law

$$(v'_0, v'_1, t_1) * (v_0, v_1, t_1) = (v_0, v'_1 + v_1, t_1)$$

is linear, the composition law corresponding to $(\beta, \alpha; N)$ will again be linear, and basically given by the same formula: addition of certain components. More precisely, we get, in affine coordinates, if $\alpha = \beta \cup \{i\}$,

$$(5.2) \quad (\mathbf{v}', \mathbf{t}) *_{\beta,\alpha;N} (\mathbf{v}, \mathbf{t}) = (\mathbf{w}, \mathbf{t}),$$

$$w_\gamma = v'_\gamma + v_\gamma \text{ if } \gamma \subset \alpha, i \in \gamma, \quad w_\gamma = v_\gamma \text{ else.}$$

The condition describing for which \mathbf{v}', \mathbf{v} this composition is actually defined, depends of course on the target projections. These are much harder to describe, since for $n = 1$, the target projection $\pi_1(v_0, v_1, t_1) = (v_0 + t_1 v_1, t_1)$ is quadratic, and by deriving we get higher and higher degree polynomials (cf. Part I, Theorem 4.10: the formula for $f^{\{1\}}$ given there contains a trilinear term).

5.4. Target projections corresponding to an edge $(\beta, \alpha; N)$. Recall the “way of talking” from Appendix A: in induction procedures invoking edges, we distinguish 3 kinds of edges, called “old”, “copy of old” and “new”.

Theorem 5.4 (Edge projections). *Let $N \subset \mathbb{N}$ be a finite set, $\alpha \in \mathcal{P}(N)$ be a vertex, $\alpha = \{\alpha_1, \dots, \alpha_k\}$ with $\alpha_1 < \dots < \alpha_k$, and let (β, α) be an edge, so $\beta = \alpha \setminus \{\alpha_i\}$ for some $i \in \{1, \dots, k\}$, and let $U \subset V$ be a linear set. The target edge projections $\pi_U^{\beta,\alpha;N} : \mathbf{G}^{\alpha;N}U \rightarrow \mathbf{G}^{\beta;N}U$ are polynomial and can be computed inductively as follows: for $n = 1$, see Section 2, and for $n > 1$,*

- (1) (“old edge”) if $\alpha_i \leq \alpha_k < a_n$ (the top element belongs neither to α nor to β), then

$$\pi_U^{\beta,\alpha;N} = \pi_U^{\beta,\alpha;N'} \times \text{id}_{\mathbb{K}} = \pi_U^{\beta,\alpha;N'} \times \text{id}_{0^{\{a_k\}}},$$

- (2) (“copy of old edge”) if $\alpha_i < \alpha_k = a_n$ (the top element belongs to β and α), then

$$\pi_U^{\beta,\alpha;N} = \mathbf{G}^{\{a_k\};\{a_k\}} \pi_U^{\beta',\alpha';N'}$$

where $\alpha' = \alpha \setminus \{\alpha_k\}$, $\beta' = \beta \setminus \{\alpha_k\}$, $N' = N \setminus \{\alpha_k\}$,

- (3) (“new edge”) if $\alpha_i = \alpha_k = a_n$ (the top element belongs to α , but not to β), then $\pi_U^{\beta,\alpha;N}$ is the base projection of the groupoid with morphism set $G^{\alpha;N}U$ and object set $G^{\alpha',N'}U$, ie.

$$\pi_U^{\beta,\alpha;N} = \pi_{G^{\alpha',N'}U}^{\emptyset,\{a_n\};\{a_n\}}.$$

Proof. The proof amounts to analyzing the inductive construction from Theorem 3.8: with $N = \mathbf{n}$, Cases (1) and (2) come from deriving the family of projections $\mathbf{G}^{n-1}U \rightarrow \mathbf{G}^{n-2}U$ with respect to the new top element a_n , according to the procedure described above (distinguishing two cases, at each step), and Case (3) comes from the very definition of the groupoid $\mathbf{G}^n U \Downarrow \mathbf{G}^{n-1}U \times \mathbb{K}$. \square

Here is the list of the target projections in affine coordinates for $N = 1$ and $N = 2$:

N	edge (β, α)	coordinate formula of the edge projection $\pi = \pi_1^{\beta, \alpha; N}$
1	$(\emptyset, \{1\})$	$\pi(v_0, v_1, t_1) = (v_0 + t_1 v_1, t_1)$
2	$(\emptyset, \{1\})$	$\pi(v_0, v_1, t_1, t_2) = (v_0 + t_1 v_1, t_1, t_2)$
	$(\{2\}, \{1, 2\})$	$\pi(v_0, v_1, v_2, v_{12}, t_1, t_2, t_{12}) =$ $(v_0 + t_1 v_1, v_2 + t_{12} v_1 + t_1 v_{12} + t_2 t_{12} v_{12}, t_1, t_2, t_{12})$
	$(\emptyset, \{2\})$	$\pi(v_0, v_2, t_1, t_2, t_{12}) = (v_0 + t_2 v_2, t_1 + t_{12} t_2, t_2)$
	$(\{1\}, \{1, 2\})$	$\pi(v_0, v_1, v_2, v_{12}, t_1, t_2, t_{12}) = (v_0 + t_2 v_2, v_1 + t_2 v_{12}, t_1 + t_2 t_{12}, t_2)$

For $N = 2$, the first line is Case (1), the second Case (2), and the following two Case (3). Observe that the projection in Case (2) is a polynomial of degree 3, since it arises by “deriving” the projection from $N = 1$; the other projections are quadratic. For $N = 3$, the projection belonging to the edge $(\{2, 3\}, 3)$ is the “copy of the copy”, and will be of degree 5. However, in case of a scaleoid ($U = 0$, so all $v_\alpha = 0$), the formulae simplify, and deriving once more we get:

N	edge (β, α)	coordinate formula of the edge projection $\pi = \pi_1^{\beta, \alpha; N}$, for $U = 0$
1	none	trivial groupoid
2	$(\{1\}, \{1, 2\})$	$\pi(t_1, t_2, t_{12}) = (t_1 + t_2 t_{12}, t_2)$
3	$(\{1\}, \{1, 2\})$	$\pi(t_1, t_2, t_3, t_{12}) = (t_1 + t_2 t_{12}, t_2, t_3)$
	$(\{1, 3\}, \{1, 2, 3\})$	$\pi(t_1, t_2, t_3, t_{12}, t_{13}, t_{23}, t_{123}) =$ $(t_1 + t_2 t_{12}, t_2, t_3, t_{13} + t_2 t_{123} + t_{23} t_{12} + t_{32} t_{123} t_{12}, t_{23})$
	$(\{1, 2\}, \{1, 2, 3\})$	$\pi(t_1, t_2, t_3, t_{12}, t_{13}, t_{23}, t_{123}) = (t_1 + t_3 t_{13}, t_2 + t_3 t_{23}, t_3, t_{12} + t_3 t_{123})$
	$(\{1\}, \{1, 3\})$	$\pi(t_1, t_2, t_3, t_{13}, t_{23}) = (t_1 + t_3 t_{13}, t_2 + t_3 t_{23}, t_3)$

It is certainly possible to get a general combinatorial formula for general target projections, but it will be very complicated since the number of terms explodes.

Theorem 5.5 (Imbedding of \mathbf{Gsy}^n into \mathbf{G}^n). *The set $\mathbb{K}^n = 0^{\{1\}} \times \dots \times 0^{\{n\}}$ is a subgroupoid of $\mathbf{G}^n 0$, and it carries a trivial n -fold groupoid structure. The inverse image of \mathbb{K}^n under the projection $\mathbf{G}^n U \rightarrow \mathbf{G}^n 0$ is the sub- n -fold groupoid $\mathbf{Gsy}^n U$.*

Proof. For $n = 1, 2$, it can be read from the preceding table that the set defined by the conditions “ $t_\gamma = 0$ whenever $|\gamma| > 1$ ” defines a trivial subgroupoid of $\mathbf{G}^n 0$. For general n , the claim follows by induction; and more precisely, the induction procedure consists at each step by applying the functor $\mathbf{G}_{t_k}^{\{k\}}$, that is, exactly by the procedure defining the symbol \mathbf{Gsy}^n . On the level of $U = 0$, this just amounts to take a cartesian product with \mathbb{K} , at each step. \square

5.5. **The $2n$ -fold category $\mathbf{G}^{\overline{N}} U$.** To describe the $2n$ -fold cat $\mathbf{G}^{\overline{N}} U$, several of the preceding results generalize, since the formal rules are similar: from Definition E.2,

$$\mathbf{G}^{\overline{1}}(A \times B) = \mathbf{G}^{\overline{1}} A \times_{\mathbf{G}^{\overline{1}} 0} \mathbf{G}^{\overline{1}} B.$$

To compute vertex sets, recall from Definition 3.2 the vertex sets of $\mathbf{G}^{\overline{1}} U$. By induction, we see that the bottom vertex set $\mathbf{G}^{\overline{N}} U$ is the same as of $\mathbf{G}^N U$, namely $U \times \mathbb{K}^n$. By induction, the top vertex set $U^{\overline{N}} = \mathbf{G}^{\overline{N}; \overline{N}} U$ has affine coordinates

$$(v_\alpha, \mathbf{s}_\beta, \mathbf{t}_\beta)_{\alpha \in \mathcal{P}(N), \beta \in \mathcal{P}(N), \beta \neq \emptyset} \in V^{2^n} \times \mathbb{K}^{2^n - 1} \times \mathbb{K}^{2^n - 1},$$

that is, the \mathbf{t} -variables are “doubled”. Next, consider the types of vertices from Definition A.2:

Theorem 5.6. *The small $2n$ -fold cat $\mathbf{G}^{\overline{N}}U$ has the following n -fold subcats:*

- (1) *the N -vertices define an n -fold subgroupoid isomorphic to $\mathbf{G}^N U$,*
- (2) *the n -fold subcat of $\mathbf{G}^{\overline{N}}U$ having N' -vertices is naturally isomorphic to the product of the n -fold scaled action category of \mathbb{K} on \mathbb{K} (Theorem D.2, where $S = K = \mathbb{K}$) with the trivial cat on U .*

Proof. (1) Since $\mathbf{G}^N U$ is injected via a zero section into $\mathbf{G}^{\overline{N}}U$ as a subgroupoid, the vertex sets $\mathbf{G}^{\alpha; \overline{N}}U$ and $\mathbf{G}^{\alpha; N}U$ corresponding to N -vertices can be identified.

(2) For $n = 1$, the cat $U \times \mathbb{K}^2 \rightrightarrows U \times \mathbb{K}$ (corresponding to the edge $(\emptyset, \{1'\})$) is a direct product of the trivial cat on U with the left action cat of \mathbb{K} on itself (Part I, Section 3.2). For $n = 2$, we get a double cat $\mathbf{G}^{2', 2}U$, with all projections of type “ ∂ ”:

$$\begin{array}{ccc} U \times \mathbb{K}^4 & \rightrightarrows & U \times \mathbb{K}^3 \\ \Downarrow & & \Downarrow \\ U \times \mathbb{K}^3 & \rightrightarrows & U \times \mathbb{K}^2. \end{array}$$

Here, the target projections are the usual projections on cartesian products, but, for instance, the two source projections starting at the top left vertex are $\partial(v_0; s_1, t_1, s_2, t_2) = (v_0; s_1 t_1, s_2, t_2)$ and $(v_0; s_1, t_1, s_2 t_2)$. For general n , the claim follows by induction, comparing the iteration procedure defining $\mathbf{G}^{\overline{N}}$ with the one from Theorem D.2. \square

Theorem 5.7. *For saturated vertices $\overline{\alpha} \subset \overline{N}$ (so $\alpha \subset N$), vertex sets can be computed by the algorithm described in Theorem 5.2: letting, for all $j \in N$,*

$$\langle \overline{\alpha}, j \rangle := \begin{cases} \{j, \overline{j}\} & \text{if } j \in \alpha \\ \emptyset & \text{if } j \notin \alpha, \end{cases}$$

we have

$$\mathbf{G}^{\overline{\alpha}; \overline{N}} = \mathbf{G}^{\langle \alpha, a_n \rangle, \{a_n, \overline{a}_n\}} \circ \mathbf{G}^{\langle \alpha, a_{n-1} \rangle, \{a_{n-1}, \overline{a}_{n-1}\}} \circ \dots \circ \mathbf{G}^{\langle \alpha, a_1 \rangle, \{a_1, \overline{a}_1\}}.$$

Likewise, the description from Theorem 5.3 applies, with α and all S_i overlined.

Proof. One uses the same arguments, where now the cat-rule $\mathbf{G}^{\{k\}}$ is replaced by the doublecat-rule $\mathbf{G}^{\{\overline{k}\}}$. \square

For generic vertices, the description of the vertex sets is more involved. Note also that, in contrast to the \ast -laws, the description of the iterated \bullet -laws in affine coordinates will become as complicated as the one of target projections, for the same reason: for $n = 1$, \bullet is bilinear, hence by deriving (applying copies of $\mathbf{G}^{\{k\}}$), we get polynomial maps of higher and higher degree. In a certain sense, it seems that the $2n$ -fold cat rule $\mathbf{G}^{\overline{n}}$ is “assembled” from the two n -fold cat rules \mathbf{G}^n and \mathbf{A}^n , but at present it is not yet clear to me what kind of “ n -dimensional assembly construction” is at work here.

APPENDIX A. NATURAL HYPERCUBES

A.1. Hypercubes. Let N be a finite set of cardinality $n := |N| \in \mathbb{N}$. By definition, the *hypercube of type N* is given by its *vertex set*, the power set $\mathcal{P}(N)$, together with its natural *partial order* given by inclusion. The vertex \emptyset is called the *bottom vertex*, and N is called the *top vertex*. An (*oriented*) *edge of $\mathcal{P}(N)$* is a pair (β, α) of vertices such that α contains exactly one element more than β , so $\beta \subset \alpha$ and $|\alpha \setminus \beta| = 1$. If $\alpha = \beta \cup \{i\}$, we also say that (β, α) is *the edge from the vertex β in direction i* . More generally, for $k \leq n$, a *k -cube in $\mathcal{P}(N)$* is given by a pair (β, α) of vertices such that $\beta \subset \alpha$ and $|\alpha \setminus \beta| = k$. For $k = 2$, it is also called a *face*, for $k = 3$ a *cube* and for $k = 4$ a *tesseract*.

A.2. Boolean ring. Recall that $\mathcal{P}(N)$ with the two operations

$$(A.1) \quad \alpha \cdot \beta = \alpha \cap \beta, \quad \alpha + \beta = (\alpha \cup \beta) \setminus (\alpha \cap \beta)$$

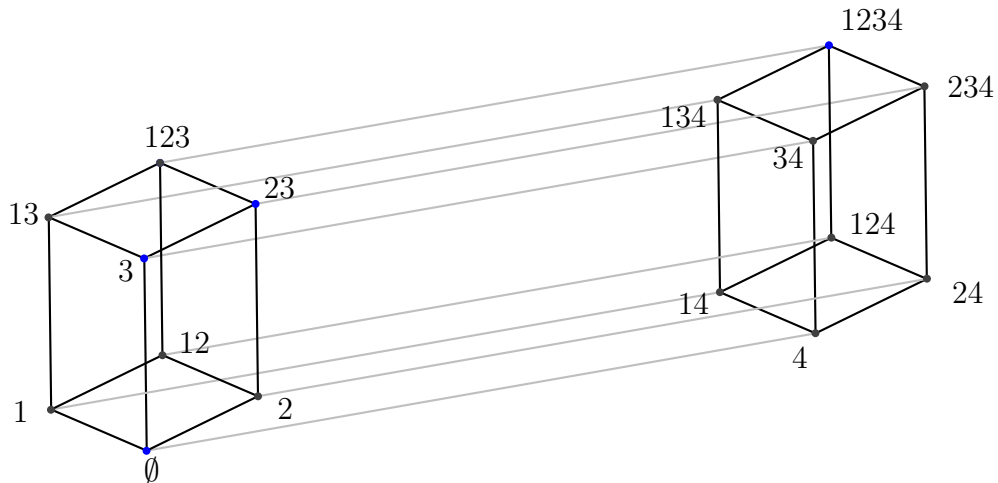
forms a commutative ring such that $\alpha + \alpha = \emptyset$ and $\alpha \cdot \alpha = \alpha$ for all α (Boolean ring). It also carries a metric defined by $d(\alpha, \beta) = |\alpha + \beta|$.

A.3. The natural n -hypercube. By definition, the *natural n -hypercube* is the hypercube $\mathcal{P}(n) = \mathcal{P}(\{1, \dots, n\})$. It has, as additional feature to the ones described above, a *total order on the vertex set*, given by *lexicographic order* denoted by \leq :

$$(A.2) \quad \mathcal{P}(n) = \left(\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \dots, n \right)$$

(which is the natural order of the “binary codes”). This total order is compatible with the partial order: $\beta \subset \alpha \Rightarrow \beta \leq \alpha$. More generally, such total order exists whenever N is a finite subset of \mathbb{N} , with its order induced by the natural order of \mathbb{N} . We then use the notation $(\alpha; N)$ for vertices and $(\beta, \alpha; N)$ for oriented edges of $\mathcal{P}(N)$. When writing $\alpha = \{\alpha_1, \dots, \alpha_\ell\}$, we assume that $\alpha_1 < \dots < \alpha_\ell$, and the cardinality $\ell = \ell_\alpha$ will also be called the *length of α* .

Remark A.1. By elementary combinatorics it is seen that the number of k -cubes in an n -cube is $\binom{n}{k} 2^{n-k}$. Thus there are 2^n vertices, $n 2^{n-1}$ edges and $n(n-1) 2^{n-3}$ faces. For instance, a tesseract has 16 vertices, 32 edges, 24 faces, and 8 cubes:



In the figure, vertices are labelled by ij , to abbreviate $\{i, j\}$, etc.

A.4. **A way of talking.** For induction procedures, a useful way of talking is

Definition A.1. A vertex α of the $n + 1$ -hypercube is called

- (1) an old vertex if it belongs to $\mathcal{P}(n)$, i.e., if $n + 1 \notin \alpha$,
- (2) a new vertex if $n + 1 \in \alpha$.

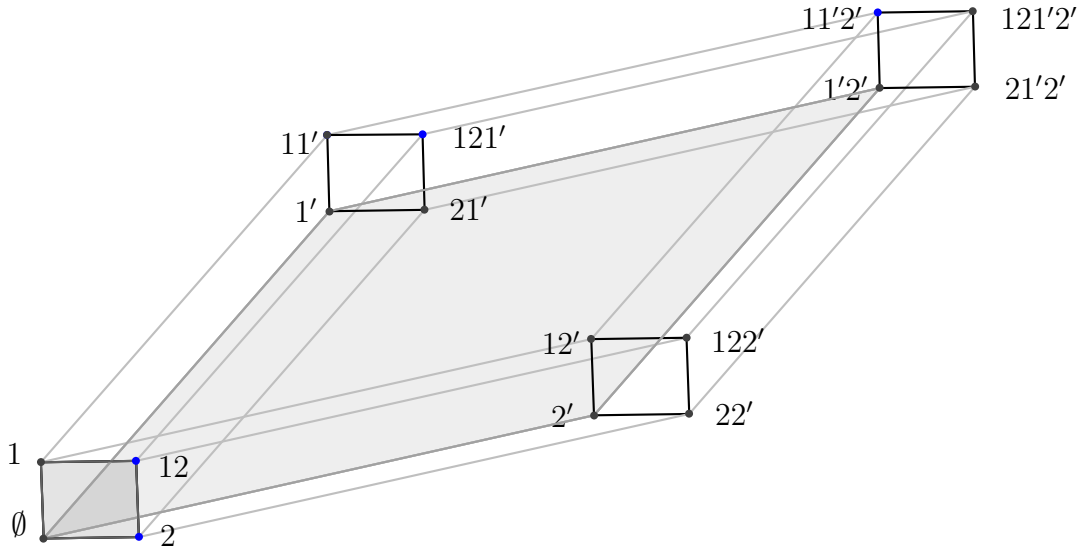
We distinguish three kinds of edges of $\mathcal{P}(n + 1)$: an (oriented) edge (β, α) is called

- (1) an old edge if both α and β are old vertices;
- (2) a copy of an old edge if both α and β are new vertices (namely, it is a copy of the “old” edge $(\alpha \setminus \{n + 1\}, \beta \setminus \{n + 1\})$), and
- (3) a new edge if β is an old vertex and α is a new one (so that $\alpha = \beta \cup \{n + 1\}$).

Similarly, we distinguish three types of faces of $\mathcal{P}(n + 1)$: “old” ones, “copies of old” ones, and “new” ones.

The figure of a tesseract given above illustrates this for $n = 3$: the “old” cube is the lower right cube $\mathcal{P}(3)$; edges are represented by segments joining the corresponding vertices (but the figure is just meant to represent the combinatorial structure): new edges are represented by grey diagonal segments; old edges are edges of the lower left cube and copies of old edges, of the upper right cube.

A.5. **The two-typed hypercube.** This is another way of seeing the $2n$ -hypercube. for $n = 2$ (tesseract), it is illustrated like this:



For any finite subset $N \subset \mathbb{N}$, let $N' := \{k' \mid k \in N\}$ be another copy of N , and let

$$(A.3) \quad \mathcal{P}(\overline{N}) := \mathcal{P}(N \cup N')$$

be the $2n$ -hypercube defined by $\overline{N} := N \cup N'$. We distinguish several types of vertices, edges and faces:

Definition A.2. Let $\alpha \subset N$, with preceding notation. A vertex of the form (α, \overline{N}) is called an N -vertex, a vertex of the form (α', \overline{N}) an N' -vertex, and a vertex of the form $(\overline{\alpha}, \overline{N})$ is called a saturated vertex. A vertex that is of neither of these forms will be called generic.

Definition A.3. An edge $(\beta, \alpha) \in \mathcal{P}(\overline{N})$ will be called of first kind if the unique element of $\alpha \setminus \beta$ belongs to N , and of second kind if it belongs to N' . We distinguish three kinds of faces, namely faces such that

- (a) all of its edges are of first kind,
- (b) all of its edges are of second kind,
- (c) it has two edges of the first and two of the second kind.

In the preceding illustration of a tesseract ($N = 2 = \{1, 2\}$), edges of the first kind are horizontal or vertical, and edges of the second kind are “diagonal” (grey). Vertices are labelled with the same convention as before. There are 16 edges of first kind and 16 edges of second kind, and 4 faces of type (a), 4 of type (b), and 16 of type (c). Moreover, we have indicated in grey the face formed by the 2-vertices and the face formed by the 2'-vertices.

APPENDIX B. n -FOLD CATEGORIES AND GROUPOIDS

B.1. The concrete category of small n -fold cats. As in Appendices A and C of Part I, the term *small (double) category* is abbreviated by “small (double) cat”, and *concrete category* by ccat. Ccats are denoted by underlined roman letters. In this appendix, we define (*strict*) *small n -fold categories* by induction (following the pattern from general category theory, see, e.g., the corresponding pages in the n -lab), and then establish a more direct and algebraic description (Theorem B.2):

Definition B.1. For $n = 1$, a *small 1-fold cat* is a *small cat*, and this defines the concrete category $\underline{\text{Cat}}_1$ of small cats. A *small n -fold cat* is a *small cat* C of type $\underline{\text{Cat}}_{n-1}$, and a *morphism of small n -fold cats* is a functor $f : C \rightarrow C'$ which is also a morphism with respect to the $\underline{\text{Cat}}_{n-1}$ -structures on C and C' . This defines a concrete category $\underline{\text{Cat}}_n$. In the same way, the concrete category $\underline{\text{Goid}}_n$ of n -fold groupoids is defined.

Theorem B.2. A *small n -fold cat* C is equivalent to the following data:

- (1) a family of sets $(C_\alpha)_{\alpha \in \mathcal{P}(n)}$, indexed by vertices of the natural hypercube $\mathcal{P}(n)$,
- (2) a structure of small cat $(\pi^e : C_\alpha \Downarrow C_\beta, *_e, z_e)$, called *edge-cat*, for every (oriented) edge $e = (\beta, \alpha; n)$ of the hypercube,

such that the following face condition holds: for every face $(\gamma, \alpha; n)$ of the hypercube, the edge categories form a *small double cat*:

$$\begin{array}{ccc} \alpha & \rightarrow & \beta & & C_\alpha & \rightrightarrows & C_\beta \\ \downarrow & & \downarrow & \text{gives a small double cat} & \Downarrow & & \Downarrow \\ \delta & \rightarrow & \gamma & & C_\delta & \rightrightarrows & C_\gamma \end{array}$$

To shorten notation, we often write $(C_\alpha)_{\alpha \in \mathcal{P}(n)}$ to denote the whole structure. Morphisms of small n -fold cats are equivalent to families of maps $f_{\alpha; n} : C_{\alpha; n} \rightarrow C'_{\alpha; n}$ such that, for each edge (β, α) , the pair (f_β, f_α) , is a morphism of small cats.

Proof. By induction on n . For $n = 1$, the equivalence is trivial (and the face condition is empty), and for $n = 2$, the equivalence is also trivial (since there is just one face, it says that, by definition, a 2-fold cat is the same as a double cat. Note that in Theorem C.1 from Part I an “explicit” description of double cats has been given). Assume now that the claim holds for some $n \in \mathbb{N}$.

Assume C is a small $n+1$ -fold cat and let us show that it gives rise to a structure as described in the theorem. By assumption, $C = C_0 \dot{\cup} C_1$, and C_0, C_1 are small n -fold cats. By induction, $C_0 = (C_\alpha)_{\alpha \in \mathcal{P}(n)}$ and $C_1 = (D_\alpha)_{\alpha \in \mathcal{P}(n)}$ are families of sets, projections and products having the properties listed in the theorem. Moreover, $\pi_\sigma : C_1 \rightrightarrows C_0$ defines two morphisms of n -fold cats, hence, again by induction, is given by two families $(\pi_\sigma^\alpha : D_\alpha \rightrightarrows C_\alpha)_{\alpha \in \mathcal{P}(n)}$ which for each α define a morphism of small cats. Letting for each $\alpha \in \mathcal{P}(n)$,

$$C_{\alpha \cup \{n+1\}} := D_\alpha, \quad \pi^{\alpha, \alpha \cup \{n+1\}; n+1} := \pi^\alpha : D_\alpha = C_{\alpha \cup \{n+1\}} \rightrightarrows C_\alpha,$$

and by distinguishing several cases (the three cases of “old”, “copies of old” and “new” edges, and similar for faces, see section A.4), one checks that the data $(C_\alpha)_{\alpha \in \mathcal{P}(n+1)}$ satisfy the properties from the theorem.

Conversely, assume data $(C_\alpha)_{\alpha \in \mathcal{P}(n+1)}$ with properties as in the theorem are given. Then $C_0 := (C_\alpha)_{\alpha \in \mathcal{P}(n)}$ and $C_1 := (C_\alpha)_{\alpha \in \mathcal{P}(n+1) \setminus \mathcal{P}(n)}$ both satisfy the induction hypothesis, hence define small n -fold cats. The family of projections $\pi^{\alpha, \alpha \cup \{n+1\}; n+1}$ is, by induction, a morphism between these n -fold cats, and so is the family of unit sections. To conclude that $(C_0, C_1, *)$ is a small cat of type $\underline{\text{Cat}}_n$, it only remains to show that $*$ is a morphism of small n -fold cats. By induction, this is equivalent to showing that $*$ defines, on the level of each “old edge” (γ, β) coming from $\mathcal{P}(n)$, a morphism of small cats. This, in turn, is equivalent to saying that the “new face” $(\gamma, \beta \cup \{n+1\})$ defines a small double cat. But this is ensured by the face condition, whence the claim. \square

Example B.1. If C_1, \dots, C_n are small cats (resp. groupoids), then the *external product* $C_1 \square \dots \square C_n$ is an n -fold cat (resp. n -fold groupoid) (see [FP10], def. 2.9).

Theorem B.3. *For a small n -fold cat $C = (C_\alpha)_{\alpha \in \mathcal{P}(n)}$, the following are equivalent:*

- (1) C is an n -fold groupoid,
- (2) for each edge $(\beta, \alpha; n)$ of $\mathcal{P}(n)$, the edge cat (C_β, C_α) is a groupoid.

Proof. Straightforward by induction. \square

See also [FP10] for a concise proof of the essentially same results. Note that the description given in the theorems does not refer to the total order of $\mathcal{P}(n)$, but only to its partial order. Therefore every permutation of n defines again an n -fold cat:

Definition B.4. *Let $\tau \in S_n$ be a permutation of n . For a small n -fold cat $C = (C_\alpha)_{\alpha \in \mathcal{P}(n)}$, the τ -transposed n -fold cat is given by $C^\tau := (C_{\tau(\alpha)})_{\alpha \in \mathcal{P}(n)}$. The n -fold cat $(C_\alpha)_{\alpha \in \mathcal{P}(n)}$ is called *edge-symmetric* if, for each $\tau \in S_n$, there is an isomorphism of n -fold cats $f_\tau : C \rightarrow C^\tau$ such that $f_\tau \circ f_\nu = f_{\tau\nu}$ and $f_{\text{id}} = \text{id}_C$. In other words, we have an action of S_n by isomorphisms.²*

Definition B.5. *Let $C = (C_0, C_1, *)$ be a small n -fold cat. Then the *opposite cat* C^{opp} is again a small n -fold cat. Iterating this procedure, for each subset $\gamma \subset n$ we get the γ -opposite n -fold cat $C^{\gamma\text{-opp}}$ defined by taking at level i the opposite cat structure if $i \in \gamma$ and the original one else.*

² In work of Brown and Higgins [BH81], and in [BHS11], it is always assumed that n -fold cats and groupoids are edge symmetric, and notation is then chosen such that, whenever α and α' are conjugate under S_n (i.e., have same cardinality), C_α and $C_{\alpha'}$ are denoted by the same symbols.

Definition B.6. In a small n -fold cat, the n -fold source map is defined by

$$\xi := \pi_0^{\emptyset,1} \circ \pi_0^{1,2} \dots \circ \pi_0^{n-1,n} : C_n \rightarrow C_\emptyset.$$

More generally, for any subset $\gamma \subset n$, we define the γ -top-down projection

$$\xi_\gamma := \pi_{k_1}^{\emptyset,1} \circ \pi_{k_2}^{1,2} \dots \circ \pi_{k_n}^{n-1,n} : C_n \rightarrow C_\emptyset$$

where $k_i = 1$ (target) if $i \in \gamma$ and $k_i = 0$ (source) if $i \notin \gamma$. In other words, this is the n -fold source map of $C^{\gamma\text{-opp}}$.

B.2. Two-typed $2n$ -fold cats.

Lemma B.7. Let $k \in \{1, \dots, n-1\}$. The following are equivalent:

- (1) C is a small n -fold cat,
- (2) C is a small k -fold cat of type $\underline{\text{Cat}}_{n-k}$.

Proof. By definition, (2) means that C is a small k -fold cat and a small $n-k$ -fold cat such that all structure maps of the k -fold cat are morphisms of the $n-k$ -fold cat structure. Applying the description from the theorem, this amounts to (1). \square

Note, however, that (2) defines a kind of additional structure on C , when the value of k is considered to be fixed.

Definition B.8. A two-typed $2n$ -fold cat is a small $2n$ -fold cat, together with a fixed structure of small n -fold cat of type $\underline{\text{Cat}}_n$. As index set, we then use the two-typed hypercube $\mathcal{P}(\bar{n})$, and we often denote projections and composition laws corresponding to edges of the second kind by the symbols ∂ and \bullet , whereas for those corresponding to edges of the first kind we use π and $*$.

B.3. Cat-rules. The n -fold cats considered in this work arise in the following way:

Definition B.9. Fix some ccat \underline{C} . We assume that one can form pullbacks in \underline{C} (cf. Appendix E). A cat-rule \mathbb{Q} on \underline{C} is a pullback-preserving functor from \underline{C} to $\underline{\text{C-Cat}}$. That is, we have a functor $\mathbb{Q} = \mathbb{Q}^1 = \mathbb{Q}^{\{1\}}$ from \underline{C} to itself, such that each set $\mathbb{Q}^1 U$ carries a structure of small cat of type \underline{C} (with object set $\mathbb{Q}^{\{0,1\}} U$, morphism set $\mathbb{Q}^{\{1,1\}} U$ and composition $*_1$), and each $\mathbb{Q} f : \mathbb{Q} U \rightarrow \mathbb{Q} U'$ is a morphism of small cats. Moreover, we require that \mathbb{Q} be compatible with (set-theoretic) pullbacks, i.e.,

$$(B.1) \quad \mathbb{Q}(A \times_B C) = \mathbb{Q} A \times_{\mathbb{Q} B} \mathbb{Q} C.$$

Theorem B.10. Assume \mathbb{Q} is a cat-rule on \underline{C} . Applying a second copy $\mathbb{Q}^{\{2\}}$ of $\mathbb{Q}^{\{1\}}$ to the data of the small cat $\mathbb{Q}^1 U$, we get a small double cat

$$\mathbb{Q}^{\{1,2\}} U := \mathbb{Q}^{\{2\}}(\mathbb{Q}^{\{1\}} U),$$

and iterating this procedure, we get a small n -fold cat $\mathbb{Q}^n U = \mathbb{Q}^{\{n\}}(\mathbb{Q}^{\{1,\dots,n-1\}} U)$.

Proof. Since $\mathbb{Q}^{\{2\}}$ is a functor, applying it to the small cat structure with composition $*_1$ gives another small cat with “derived” composition $\mathbb{Q}^{\{2\}}(*_1)$ and “derived” projections and unit section. To make this statement formally correct, we have, however, to pay attention to the domains of definition of composition laws: composition $*_1$ is defined on a domain $D = A \times_B C \subset A \times C$ with $A = C = \mathbb{Q}^{\{1,1\}} U$ and $B = \mathbb{Q}^{\{0,1\}} U$, and thus $\mathbb{Q}^{\{2\}}(*_1)$ is, *a priori*, defined on $\mathbb{Q}^{\{2\}} D$. But the very definition

of a small cat requires that this product be defined on the set $Q^{\{2\}}A \times_{Q^{\{2\}}B} Q^{\{2\}}C$ (pullback using projections of $Q^{\{2\}}(Q^{\{1\}}U)$). Therefore, a sufficient condition for the construction to work is that Q be compatible with pullbacks, i.e., (B.1).

On the other hand, $Q^{\{2\}}(Q^{\{1\}}U)$ arises from applying $Q^{\{2\}}$ to a set, hence is a small cat with composition $*_2$. Since “derived” maps $Q^{\{2\}}f$ are compatible with such structure, all structure maps defined above are morphisms of the small cat with composition $*_2$, and so $Q^{\{1,2\}}U$ is a small doublecat. Applying another copy $Q^{\{3\}}$ to this structure, we get a small triplecat, and so on. \square

Example B.2. Our examples of cat-rules are:

- (1) the *pair groupoid rule* PG , defined on $\underline{\mathbb{C}} = \underline{\text{Set}}$ (Appendix C),
- (2) the *scaled action groupoid rule* A , defined on the ccat of S -modules for a commutative monoid S (Appendix D),
- (3) the *derived groupoid rule* G (main object of this work), defined on the ccat of linear sets, with morphisms initially the polynomial laws (but then extended to all C^∞ -laws), and pullbacks as in Appendix E,
- (4) the *finite part derived groupoid rule* Gfi , defined on the ccat of linear sets, with morphisms arbitrary set-maps,
- (5) the *symmetric derived groupoid rule* Gsy , with morphisms as for G ,
- (6) for each $t \in \mathbb{K}$, the rule G_t (in particular, for $t = 0$, the *tangent functor* T).

For all of these examples, we check Property (B.1) “by hand” (see Appendix E for the most important case $Q = \text{G}$).

Definition B.11. *An n -fold cat rule on a ccat $\underline{\mathbb{C}}$ is given by functors, for each subset $N \subset \mathbb{N}$ of cardinality n , Q^N , from $\underline{\mathbb{C}}$ to the ccat $\underline{\mathbb{C}} - \underline{\text{Cat}}_N$ of small n -fold cats of type \mathbb{C} , and preserving pullbacks.*

Thus any cat-rule gives rise, by iteration, to an n -fold cat rule. But it is also possible to compose different cat-rules with each other:

Theorem B.12. *Assume Q and P are cat-rules defined on a common ccat $\underline{\mathbb{C}}$. Then $Q \circ P$ and $P \circ Q$ are doublecat rules on $\underline{\mathbb{C}}$ (different from each other, in general).*

Proof. Same arguments as in the proof of Theorem B.10. \square

Example B.3. The cat rules $Q = \text{G}$ and $P = \text{A}$ (for $S = K = \mathbb{K}$) cannot be composed since G does not preserve the required equivariance properties (we cannot define the common ccat $\underline{\mathbb{C}}$). On the other hand, $\text{G}^{\bar{1}}$ is a doublecat rule, which in effect does what a hypothetical composition of G and A should do.

APPENDIX C. THE n -FOLD PAIR GROUPOID

Let M be a set. Recall that the *pair groupoid* PG^1M of M is given by the two projections $M \times M \rightrightarrows M$, unit section the diagonal map $M \rightarrow M \times M$ and product $(x, y) * (y, z) = (x, z)$ and inverse $(x, y)^{-1} = (y, x)$. Every set-map $f : M \rightarrow M'$ induces a groupoid morphism $\text{PG}^1f = (f \times f, f)$, and PG^1 is a cat-rule (even, a groupoid-rule). Let $\text{PG}^{\{k\}}$ be a copy of PG^1 , “of k -th generation”. It follows that $\text{PG}^2M := \text{PG}^{\{2\}}(\text{PG}^1M)$ is a double groupoid (see also [Ko87]), etc.:

$$(C.1) \quad \text{PG}^n M := \text{PG}^{\{n\}}(\text{PG}^{\{1, \dots, n-1\}} M)$$

is an n -fold groupoid, called the n -fold pair groupoid. Let us describe it in terms of hypercubes, as in Theorem B.2:

Theorem C.1. *Let M be a set. For every $\beta \in \mathcal{P}(n)$, let M_β be a formal copy of M , and define the vertex sets, for $\alpha \in \mathcal{P}(N)$,*

$$(C.2) \quad \mathbf{PG}^{\alpha,n}M := \prod_{\beta \in \mathcal{P}(\alpha)} M_\beta \cong M^{2^{\ell\alpha}}.$$

Let $(\beta, \alpha) = (\beta, \beta \cup \{i\})$ be an edge, where $i \notin \beta$. The two projections corresponding to this edge are defined by

$$(C.3) \quad \pi_0^{\beta,\alpha}((x_\gamma)_{\gamma \subset \alpha}) := (x_\gamma)_{\gamma \subset \beta}, \quad \pi_1^{\beta,\alpha}((x_\gamma)_{\gamma \subset \alpha}) := (x_{\gamma \cup \{i\}})_{\gamma \subset \beta}$$

the zero sections are the “diagonal imbedding” $(x_\gamma) \mapsto (x_\gamma, x_\gamma)$, and the product corresponding to this edge is defined for x, y such that $x_{\gamma \cup \{i\}} = y_\gamma$ (for $\gamma \subset \beta$), by

$$(C.4) \quad (x_\gamma) *_{\alpha,\beta} (y_\gamma) = (z_\gamma), \quad z_\gamma = \begin{cases} x_\gamma & \text{if } i \notin \gamma \\ y_\gamma & \text{if } i \in \gamma. \end{cases}$$

These data define an edge-symmetric n -fold groupoid, naturally isomorphic to the inductively defined $\mathbf{PG}^n M$.

Proof. For $n = 1$, $\pi_0^{\emptyset,1}(x_\emptyset, x_1) = x_\emptyset$, $\pi_1^{\emptyset,1}(x_\emptyset, x_1) = x_1$, $(x_\emptyset, x_1) * (y_\emptyset, y_1) = (x_\emptyset, y_1)$ whenever $y_\emptyset = x_1$, describing the pair groupoid. For $n = 2$, we have the following source and target projections,

$$\begin{array}{ccccccc} M^4 & \rightrightarrows & M^2 & & (x_0, x_1, x_2, x_{12}) & \mapsto & (x_0, x_2) & & (x_0, x_1, x_2, x_{12}) & \mapsto & (x_1, x_{12}) \\ \Downarrow & & \Downarrow & , & \downarrow & & \pi_0 & \downarrow & \downarrow & & \pi_1 & \downarrow \\ M^2 & \rightrightarrows & M & & (x_0, x_1) & \mapsto & x_0 & & (x_2, x_{12}) & \mapsto & x_{12} \end{array}$$

which match precisely the data of $\mathbf{PG}^{\{2\}}\mathbf{PG}^1 M$. Similarly for the general induction step. Edge-symmetry is directly seen for $n = 2$ from the formulae above; more formally, whenever α and α' have same cardinality and $\tau : \alpha \rightarrow \alpha'$ is a bijection, there is an induced bijection $f_\tau : M_\alpha \rightarrow M_{\alpha'}$, which together assemble to define an isomorphism. \square

Note also that the 2^n top-down projections of $\mathbf{PG}^n M$ are precisely the 2^n projections of M^{2^n} to M , and that every map $f : M \rightarrow M'$ induces a morphism of n -fold groupoids $f^n : \mathbf{PG}^n(M) \rightarrow \mathbf{PG}^n(M')$, such that the symbol \mathbf{PG}^n can be seen as an n -fold groupoid rule on Set.

Theorem C.2. *For every small n -fold cat $C = (C_\alpha)_{\alpha \in \mathcal{P}(n)}$, the iterated anchor map*

$$C \rightarrow \mathbf{PG}^n(C_\emptyset)$$

is morphism of small n -fold cats. On C_α , this map is given by associating to $x \in C_\alpha$ the family $(\xi_\gamma(x))_{\gamma \in \mathcal{P}(\alpha)}$ where ξ_γ is the γ -top-down projection (Definition B.6).

Proof. For $n = 1$, this map is the usual anchor map $x \mapsto (\pi_0(x), \pi_1(x))$, which is a morphism $C \rightarrow \mathbf{PG}^1(C_\emptyset)$. For $n > 1$, the claim follows by induction. \square

Definition C.3. *An n -fold equivalence relation on M is a subgroupoid of $\mathbf{PG}^n M$.*

For $n = 1$, we get a usual equivalence relation, and for $n = 2$, we get a double equivalence relation as defined in [Ko87].

APPENDIX D. THE n -FOLD SCALED ACTION CATEGORY

Let S be a commutative monoid, acting (from the left) on a set K (“the scales”, later we will take for S and K a copy of \mathbb{K}). Assume S acts (from the right) on a set V , and let $U \subset V$ a non-empty subset. We define a small cat \mathbf{A}^1U (see Part I, Lemma B.1) by sets

$$(D.1) \quad \begin{aligned} \mathbf{A}^{\theta,1}U &:= U \times K \\ \mathbf{A}^{1,1}U &:= \{(v, s, t) \in V \times S \times K \mid v \in U, vs \in U\}, \end{aligned}$$

with injection $(v, t) \mapsto (v, 1, t)$ and and projections $\partial : \mathbf{A}^{1,1}U \rightrightarrows \mathbf{A}^{\theta,1}U$ given by

$$(D.2) \quad \partial_1(v, s, t) = (vs, t), \quad \partial_0(v, s, t) = (v, st)$$

and composition law $(v', s', t') \bullet (v, s, t) = (v, ss', t')$.

Lemma D.1. *With notation as above,*

(1) *every S -equivariant map $f : V \rightarrow V'$ defines a morphism*

$$\mathbf{A}^1f = (f \times \text{id}_S \times \text{id}_K, f \times \text{id}_K),$$

(2) *let S act on $V \times S \times K$ and on $V \times K$ by:*

$$(v, s, t).\lambda := (v\lambda, s, t), \quad (v, t).\lambda := (v\lambda, t),$$

then both projections ∂_σ and composition \bullet are equivariant maps.

Proof. Direct check. As to (1), equivariance of f is needed to ensure that $\partial_1 \circ \mathbf{A}^1f = (f \times \text{id}) \circ \partial_1$, and as to (2), equivariance of ∂_1 , amounts to $v\lambda s = vs\lambda$ for all s and λ , so here commutativity of S enters. \square

Theorem D.2. *Let $\mathbf{A}^{\{k\}}$ be a formal copy of the functor \mathbf{A}^1 defined above. Then, for any non-empty subset $U \subset V$, we get a small edge symmetric n -fold cat, defined by induction*

$$\mathbf{A}^nU := \mathbf{A}^{\{n\}}(\mathbf{A}^{n-1}U) = \mathbf{A}^{\{n\}} \dots \mathbf{A}^{\{1\}}(U).$$

Its vertex sets, edge projections and injections and composition laws are described as follows: For any $i \in \mathbb{N}$ let S_i be a formal copy of S and K_i one of K . For $\alpha = \{\alpha_1, \dots, \alpha_\ell\} \in \mathcal{P}(n)$ the vertex sets are

$$\begin{aligned} \mathbf{A}^{\alpha,n}U &= \{(x, \mathbf{s}, \mathbf{t}) \in U \times S_{\alpha_1} \times \dots \times S_{\alpha_\ell} \times K_1 \times \dots \times K_n \mid \\ &\quad \forall \beta \in \mathcal{P}(\alpha) : xs_{\beta_1} \cdots s_{\beta_\ell} \in U\}, \end{aligned}$$

and for an edge (β, α) with $\alpha = \beta \cup \{i\}$, the edge projections are

$$\partial_1(x, \mathbf{s}, \mathbf{t}) = (xs_i, (s_k)_{k \in \beta}, \mathbf{t}), \quad \partial_0(x, \mathbf{s}, \mathbf{t}) = (x, (s_k)_{k \in \beta}, s_i t_i, (t_k)_{k \in \beta});$$

in other words, each s_i acts on V but only on the i -th component of K^n . The zero section is the natural inclusion, and the composition law is

$$(x, \mathbf{s}', \mathbf{t}') \bullet (x, \mathbf{s}, \mathbf{t}) = (x, s_i s'_i, (s_j)_{j \neq i}, \mathbf{t}').$$

Moreover, any map $f : U \rightarrow U'$ satisfying $f(vs) = f(v)s$ whenever $v \in U$, $s \in S$ and $vs \in U$ gives rise to a morphism $\mathbf{A}^n f : \mathbf{A}^nU \rightarrow \mathbf{A}^nU'$. In particular, there is a canonical morphism $\mathbf{A}^nU \rightarrow \mathbf{A}^n0$, where 0 is the zero S -module.

Proof. From the preceding lemma, it follows that \mathbf{A} is a cat-rule (by direct check, \mathbf{A} is compatible with pullbacks), and hence by iterating we get an n -fold cat. The explicit formulae for vertex sets, edge projections and composition laws are proved by induction. These formulae show an edge-symmetric structure, when we let act the symmetric group S_n on the indices and define the maps on the level of vertex sets by the corresponding identification maps. For instance, when $n = 2$ and $U = V$, we have a diagram of projections

$$\begin{array}{ccc} V \times S_1 \times S_2 \times K_1 \times K_2 & \rightrightarrows & V \times S_1 \times K_1 \times K_2 \\ & \Downarrow & \Downarrow \\ V \times S_2 \times K_1 \times K_2 & \rightrightarrows & V \times K_1 \times K_2, \end{array}$$

having the natural symmetry properties. \square

Example D.1. If S acts trivially on V , then $\mathbf{A}^n U = U \times \mathbf{A}^n 0$ is just a direct product of U with the iterated “usual” n -fold action cat of S on K .

APPENDIX E. PULLBACKS IN FIRST ORDER CALCULUS

In general category theory, *pullbacks* are defined by a universal property. In the concrete cat $\underline{\text{Set}}$, they are given by the following construction (see [MM92], I.2), which equally applies to the concrete cat of linear sets with set-maps:

Definition E.1. *Assume $A \subset V_1$, $B \subset V_2$, $C \subset W$ are linear sets and $f : A \rightarrow C$ and $g : B \rightarrow C$ maps. The pullback (in the concrete category of linear sets with set maps), $(A \times_C B, A, B, p_A, p_B, f, g)$, of these data is the linear set defined by*

$$P := A \times_C B := \{(a, b) \in A \times B \mid f(a) = g(b)\} \subset V_1 \times V_2,$$

together with its two projections $p_A : P \rightarrow A$ and $p_B : P \rightarrow B$ fitting into the commutative diagram of mappings

$$(E.1) \quad \begin{array}{ccc} P & \rightarrow & B \\ \downarrow & & \downarrow \\ A & \rightarrow & C. \end{array}$$

In the following, we distinguish the set P and the whole structure of pullback, denoted by $A \times_C B$. Note that, if $C = 0$ is a point, then P is the usual cartesian product $A \times B$ of sets.

However, this definition is not appropriate for the concrete cat of linear sets with C^1 -laws. The reason for this is that base maps need not determine the whole law. Morally, they do if the domain of definition is “open”; but, for instance, diagonals Δ_V in vector spaces V are typically not open in $V \times V$, and thus the set $(\Delta_V)^{\{1\}}$ is “too big”: we want it to be equal to $\Delta_{V^{\{1\}}}$. In other words, we have “to derive maps first and then take set-theoretic pullbacks”, and not the other way round:

Definition E.2. *Recall from Definition 3.6 the ccat of linear sets with C^1 -laws. With notation as above, we define the pullback in the concrete cat of linear sets with C^1 -laws to be the set-theoretic pullback of $\mathbf{G}^1 A$ and $\mathbf{G}^1 C$ via $\mathbf{G}^1 f$ and $\mathbf{G}^1 g$, i.e., it is given by the base set $A \times_C B$ and the extended sets*

$$(E.2) \quad \mathbf{G}^1(A \times_C B) := \mathbf{G}^1 A \times_{\mathbf{G}^1 C} \mathbf{G}^1 B.$$

Similar definitions are given for $\mathbf{G}^{\bar{1}}$ and $\mathbf{G}^{\{\bar{k}\}}$. In particular, with $\mathbb{K}_1 := 0^{\{1\}}$,

$$\mathbf{G}^1(A \times B) = \mathbf{G}^1 A \times_{\mathbb{K}_1} \mathbf{G}^1 B, \quad \mathbf{G}^{\bar{1}}(A \times B) = \mathbf{G}^{\bar{1}} A \times_{\mathbb{K}_1 \times \mathbb{K}_1} \mathbf{G}^{\bar{1}} B.$$

The following theorem ensures that the pullbacks thus defined are again groupoids (resp. small double cats).

Theorem E.3. *With notation as in Definition E.1, assume \mathbf{f}, \mathbf{g} are C^1 -laws with base maps f, g . Then the following pair of linear sets is a subgroupoid of the groupoid $\mathbf{G}^1 P = (P \times \mathbb{K}, \mathbf{G}^{1;1} P)$:*

$$\mathbf{G}^1 A \times_{\mathbf{G}^1 C} \mathbf{G}^1 B = (\mathbf{G}^{0;1} A \times_{\mathbf{G}^{0;1} C} \mathbf{G}^{0;1} B, \mathbf{G}^{1;1} A \times_{\mathbf{G}^{1;1} C} \mathbf{G}^{1;1} B)$$

A similar statement holds for \mathbf{G}^1 replaced by $\mathbf{G}^{\bar{1}}$.

Proof. On the level of object sets, both groupoids from the claim coincide, by using the natural identification $(A \times \mathbb{K}) \times_{(C \times \mathbb{K})} (B \times \mathbb{K}) = (A \times_C B) \times \mathbb{K}$. On the level of morphism sets, let us show that $(\mathbf{G}^{1;1} A \times_{\mathbf{G}^{1;1} C} \mathbf{G}^{1;1} B) \subset \mathbf{G}^{1;1} P$. On the one hand,

$$\begin{aligned} P^{\{1\}} &= \left\{ ((a, b), (u, v), t) \mid \begin{array}{l} a \in A, b \in B, u \in V_1, v \in V_2, t \in \mathbb{K} : a + ut \in A, b + tv \in B, \\ f(a) = g(b), \quad f(a + ut) = g(b + tv) \end{array} \right\} \\ &= \left\{ ((a, b), (u, v), t) \mid \begin{array}{l} a \in A, b \in B, u \in V_1, v \in V_2, t \in \mathbb{K} : a + ut \in A, b + tv \in B, \\ f(a) = g(b), \quad f^{[1]}(a, u, t) t = g^{[1]}(b, v, t) t \end{array} \right\}. \end{aligned}$$

(For the last equality, note that $f(a + ut) = f(a) + f^{[1]}(a, u, t) t$. Note also that the set $P^{\{1\}}$ depends only on the base maps of \mathbf{f} and \mathbf{g} .) On the other hand, using the maps $f^{\{1\}} : A^{\{1\}} \rightarrow C^{\{1\}}$ and $g^{\{1\}} : B^{\{1\}} \rightarrow C^{\{1\}}$:

$$\begin{aligned} Q &:= A^{\{1\}} \times_{C^{\{1\}}} B^{\{1\}} \\ &= \left\{ ((a, u, t), (b, v, s)) \mid \begin{array}{l} (a, u, t) \in A^{\{1\}}, (b, v, s) \in B^{\{1\}} : \\ f^{\{1\}}(a, u, t) = g^{\{1\}}(b, v, s) \end{array} \right\} \\ &= \left\{ ((a, u, t), (b, v, t)) \mid \begin{array}{l} a \in A, b \in B, u \in V_1, v \in V_2, t \in \mathbb{K} : a + tu \in A, b + tv \in B, \\ f(a) = g(b), \quad f^{[1]}(a, u, t) = g^{[1]}(b, v, t) \end{array} \right\} \end{aligned}$$

(and this set does not only depend on the base maps). Since, obviously, $f^{[1]}(a, u, t) = g^{[1]}(b, v, t) \Rightarrow f^{[1]}(a, u, t) t = g^{[1]}(b, v, t) t$, we have $Q \subset P^{\{1\}}$. Now let us show that Q is stable under the law $*$ of $P^{\{1\}}$. Write $(a, u; b, v; t)$ for the element $((a, u, t), (b, v, t)) \in Q$, so $f^{[1]}(a, u, t) = g^{[1]}(a, u, t)$, etc.; then, for $a' = a + ut$, $b' = b + vt$,

$$(a', u'; b', v'; t) * (a, u; b, v; t) = (a, u' + u; b, v' + v; t)$$

satisfies, since $f^{\{1\}}$ and $g^{\{1\}}$ are groupoid morphisms,

$$\begin{aligned} f^{[1]}(a, u' + u, t) &= f^{[1]}(a', u', t) + f^{[1]}(a, u, t) \\ &= g^{[1]}(a', u', t) + g^{[1]}(a, u, t) = g^{[1]}(a, u' + u, t), \end{aligned}$$

and similarly for $(b, v' + v', t)$. The description of Q given above shows now that $(a', u'; b', v'; t) * (a, u; b, v; t) \in Q$. Obviously, Q contains the units and is stable under inversion, hence Q is a subgroupoid. This proves the first claim, and the final claim follows from this by a straightforward computation, using that, by definition, $U^{\{\bar{1}\}} = \{(x, v, s, t) \in V^2 \times \mathbb{K}^2 \mid (x, v, st) \in U^{\{1\}}\}$. \square

Example E.1. In general, the inclusion is strict. However, if $C = 0$ is a point, so $f : A \rightarrow 0$ and $g : B \rightarrow 0$ are constant and $A \times_C B = A \times B$, we get with $C^{\{1\}} \cong \mathbb{K}$,

$$(A \times B)^{\{1\}} = (A \times_C B)^{\{1\}} \supset A^{\{1\}} \times_{\mathbb{K}} B^{\{1\}},$$

and in this case the inclusion is an equality (see Theorem 2.10 in Part I). One may interpret this by saying that, “when $C = 0$, the diagonal is open in $C \times C$ ”.

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