

SOME REMARKS ON TEACHING MATHS: AFFINE SPACES AND AFFINE ALGEBRA

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The purpose of the following short note is to propose an axiomatic approach, called *affine algebra*, to affine spaces over a field \mathbb{K} : I will explain that *affine spaces over \mathbb{K} are the same thing as vector spaces equipped with the new laws $S(a, b, c) = a - b + c$ and $P_r(b, c) = (1 - r)b + rc$* . To start an axiomatic theory, we characterize these two laws by certain algebraic identities (in the sense of universal algebra), implying that, for b fixed, we recover a vector space with origin b . The main difference with the “usual” definitions is of logical order: whereas the “usual” definition takes vector spaces as logically prior to affine spaces, we opt for two independent definitions – you can start with affine spaces and go to vector spaces, or the other way round, as you wish. This has the advantage that all categorical notions (morphisms, subspaces, etc.) are simpler and more natural than in the usual approach. The presentation given here is a further development of ideas from my paper [13] (see [paper list on my homepage](#)), but, certainly, all this essentially is “folklore” – see, e.g., the informal description of affine spaces in the [wikipedia article](#): all I do here is to turn this “informal” description into a “formal” one. For teaching affine spaces on an elementary level, it is not necessary to discuss the axioms in full length: it suffices to know that S and P contain the whole thing.

Since there is no extra cost, in the following we allow \mathbb{K} to be any unital ring, and we define the affine space analog of left \mathbb{K} -modules. Of course, the reader may think of a commutative field and vector spaces.

1. AFFINE SPACES

Definition. An affine space over \mathbb{K} is given by a set A with two structure maps

$$\begin{aligned} S : A \times A \times A &\rightarrow A, & (a, b, c) &\mapsto S(a, b, c), \\ P : \mathbb{K} \times A \times A &\rightarrow A, & (r, a, b) &\mapsto P_r(a, b) := r_a(b) \end{aligned}$$

satisfying, for all $a, b, c, u, v \in A$,

- (S1) $S(a, a, c) = c = S(c, a, a)$
- (S2) $S(a, b, c) = S(c, b, a)$
- (S3) $S(S(a, b, c), u, v) = S(a, S(b, c, u), v)$,

and, for all $r, s \in \mathbb{K}$ and $a, b, c, d \in A$,

- (P1) $P_1(b, c) = c$
- (P2) $P_r(b, P_s(b, c)) = P_{rs}(b, c)$
- (P3) $P_r(b, S(a, b, c)) = S(P_r(b, a), b, P_r(b, c))$
- (P4) $P_{r+s}(b, c) = S(P_r(b, c), b, P_s(b, c))$
- (P5) $S(a, b, P_r(c, d)) = P_r(S(a, b, c), S(a, b, d))$.

A morphism of affine spaces $(A, S, P), (A', S', P')$ is a map $f : A \rightarrow A'$ commuting with structure maps: $f(S(a, b, c)) = S'(fa, fb, fc)$, $f(P_r(b, c)) = P'_r(fb, fc)$.

Lemma 1.1. *If (A, S, P) is an affine space over \mathbb{K} , then, for any $b \in A$, the set A with vector addition and multiplication by scalars*

$$\begin{aligned} a + c &:= a +_b c := S(a, b, c) \\ rc &:= r \cdot_b c := P_r(b, c) \end{aligned}$$

becomes a left \mathbb{K} -module (A, b) with origin b . Moreover, an affine map $f : A \rightarrow A'$ gives rise to a linear map $f : (A, b) \rightarrow (A', f(b))$.

Proof. Properties (S2), (S3) say that $(A, +_b)$ is commutative and associative, and (S1) that b is a neutral element. The element $S(b, a, b)$ is a negative element of a :

$$S(S(b, a, b), b, a) = S(b, S(a, b, b), a) = S(b, a, a) = b = S(a, b, S(b, a, b)).$$

Thus $(V, +_b, b)$ is an abelian group. Now, (P1) – (P4) are precisely the properties of scalar action: $1c = c$, $r(sc) = (rs)c$, $r(a+c) = ra+rc$, $(r+s)c = rc+sc$, proving that we get \mathbb{K} -modules. (Property (P5) is not needed to prove the lemma.) In the same way, it is immediate that $f : (A, b) \rightarrow (A', f(b))$ is linear. \square

Lemma 1.2. *Let V be a left \mathbb{K} -module. Then (V, S, P) , with structure maps*

$$\begin{aligned} S(a, b, c) &:= a - b + c \\ P_r(b, c) &:= (1 - r)b + rc \end{aligned}$$

satisfies Properties (S1) – (P5), i.e., it is an affine space over \mathbb{K} .

Proof. This is checked by straightforward computation which we leave to reader. \square

Theorem 1.3. *The constructions from the preceding two Lemmas are inverse to each other.*

Proof. Obviously, starting with a linear space $(V, 0)$, constructing (V, S, P) by Lemma 1.2, we get back the linear space we started with, by using Lemma 1.1 with $b = 0$. Conversely, assume given an affine space (A, S, P) and fix some base point $o \in A$ and define $a + c := S(a, o, c)$ and $rc := P_r(o, c)$. We have to show that

$$S(a, b, c) = a - b + c, \quad P_r(b, c) = (1 - r)b + rc.$$

The first of these conditions follows from (S1) - (S3), recalling that $-b = S(o, b, o)$,

$$(a + (-b)) + c = S(S(a, o, S(o, b, o)), o, c) = S(S(a, o, o), b, S(o, o, c)) = S(a, b, c).$$

In order to prove the second condition we use (for the first time) condition (P5):

$$\begin{aligned} P_r(b, c) &= P_r(b + o, b + (c - b)) = P_r(S(b, o, o), S(b, o, c - b)) \\ &= S(b, o, P_r(o, c - b)) = b + r(c - b) = (1 - r)b + rc \end{aligned}$$

(for the last equality, recall that by Lemma 1.1, (V, o) is a linear space). \square

Remark and example. Condition (P5) is needed only for the last step. One may wonder whether it is a consequence of the other properties. In fact, it is not: for

instance, let $V \neq 0$ be a complex vector space, so $\mathbb{K} = C$, and define new laws: $\tilde{S} := S$, and \tilde{P} is defined as follows:

$$\tilde{P}_r(0, c) = rc, \quad \forall b \neq 0 : \tilde{P}_r(b, c) := (1 - \bar{r})b + \bar{r}c.$$

Then, for any fixed b , the “local” laws $+_b, \cdot_b$ define complex vector spaces (the usual space, for $b = 0$, and the complex conjugate spaces, for $b \neq 0$), hence (S1) - (P4) are satisfied, but (P5) is not: translations are not isomorphisms from one local space to another. On the other hand, for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$, condition (P5) does indeed follow from the other ones, since these fields do not admit non-trivial automorphisms.

Theorem 1.4. *Affine spaces are the same thing as linear spaces equipped with the laws $S(a, b, c) = a - b + c$ and $P_r(b, c) = (1 - r)b + rc$. Stated more formally: there is an equivalence of categories between affine spaces as defined above and affine spaces, defined in the usual way. The category of linear spaces (i.e., \mathbb{K} -modules) is equivalent to the category of affine spaces with a fixed base point.*

Proof. This is a fairly direct consequence of the preceding theorem: if (A, S, P) is an affine space as defined above, define the *translations* and *translation group* by

$$T_{a,b} : A \rightarrow A, \quad x \mapsto T_{a,b}(x) := S(a, b, x),$$

$$V := \text{Tran}(A) := \{f : A \rightarrow A \mid \exists u, v \in A : f = T_{u,v}\}.$$

Fixing an origin $o \in A$, it follows from the preceding theorem that V is an abelian group carrying a natural \mathbb{K} -module structure and acting simply transitively on A , so we have the usual properties of an affine space. Conversely, in Lemma 1.2 we have seen that an affine space in the usual sense defines one in our sense.

In order to see that morphisms also correspond to each other, it remains to show that an affine map $f : A \rightarrow A'$ in our sense induces a linear map $F : \text{Tran}(A) \rightarrow \text{Tran}(A')$. This can be proved in the usual way, by choosing an origin $o \in A$ and using arguments in linear spaces. (The reader who wishes so may give a more conceptual proof, avoiding the choice of base point.) \square

Remark. In the context of the preceding proof, property (P5) enters to the effect that the translation group is not only an abelian group, but also carries a well-defined scalar action, and thus is a module.

2. AFFINE COMBINATIONS AND BARYCENTRIC CALCULUS

As a consequence of the preceding theorem, affine combinations of the form $a - b + c$ and $(1 - r)b + rc$ are independent of the base point o chosen to define addition $+$ and multiplication by scalars \cdot . In fact, any sum of the form $\sum_i \lambda_i a_i$ with $\sum_i \lambda_i = 1$ is independent of the choice of base point p used in its definition: the notion of *affine combinations* is intrinsic to affine geometry. Indeed, choosing some other base point p instead of o , we claim that

$$\lambda_1 a_1 + \dots + \lambda_k a_k = \lambda_1 \cdot_p a_1 +_p \dots +_p \lambda_k \cdot_p a_k.$$

To prove this, note the sum on the right hand side has k terms, and hence there are $k - 1$ signs $+_p$; since $x +_p y = x - p + y$, we have to subtract $(k - 1)p$. But, $\lambda_i \cdot_p a_i = (1 - \lambda_i)p + \lambda_i a_i$, thus we have to add the term $\sum_{i=1}^k (1 - \lambda_i)p = kp - p = (k - 1)p$, so the terms cancel out, and we get the sum on the left hand side.

3. CATEGORICAL NOTIONS

Since affine spaces are defined by structure maps satisfying certain identities, they form a variety in the sense of universal algebra, and thus all “categorical” notions are completely natural (see also [13]):

- (1) *morphisms (=affine maps)* $f : A \rightarrow A'$ have been defined above;
- (2) *subspaces* are subsets $B \subset A$ closed under P and S ; equivalently, they are stable under *all* affine combinations,
- (3) *direct products* $A \times A'$ are defined by

$$S((a, a'), (b, b'), (c, c')) := (S(a, b, c), S'(a', b', c')),$$

$$P_r((b, b'), (c, c')) = (P_r(b, c), P'_r(b', c')),$$
- (4) the *space* $\text{Map}(A, A')$ of *all maps* between affine spaces A, A' is again an affine space, by taking the *pointwise structure*

$$(S(f, g, h))a := S(fa, ga, ha), (P_r(g, h))(a) := P_r(ga, ha);$$
- (5) if, in the preceding item, $A = A'$, then $\text{Map}(A, A)$ carries a *canonical structure of \mathbb{K} -module*, by taking id_A as base point in the affine space $\text{Map}(A, A)$,
- (6) if \mathbb{K} is commutative, then the space $\text{Aff}(A, B)$ of affine maps between A and B is an affine subspace of $\text{Map}(A, B)$; thus $\text{Aff}(A, A)$ then carries a canonical \mathbb{K} -module structure.

Using this, we get a very natural construction of the universal space \hat{A} of the affine space A , and certain facts and constructions easily generalize for torsors.

4. IDENTITIES

The maps S and P satisfy many other identities, apart from those defining them. Indeed, one may replace the defining identities by other ones from the following list – and maybe the reader may find shorter, or more convincing lists of defining axioms. We leave the proof of the following identities as an exercise, as well as finding some geometric or categorical interpretation for them:

- (1) $P_r(a, a) = a$
- (2) $P_{1-r}(a, b) = P_r(b, a)$
- (3) if $rs = sr$, then $P_r(a, P_s(b, c)) = P_s(P_r(a, b), P_r(a, c))$
- (4) if $rs = sr$, then $P_r(a, P_s(a, b), P_s(c, d)) = P_s(P_r(a, c), P_r(b, d))$
- (5) $P_r(S(a, b, c), S(u, v, w)) = S(P_r(a, u), P_r(b, v), P_r(c, w))$
- (6) $P_r(a, S(u, v, w)) = S(P_r(a, u), P_r(a, v), P_r(a, w))$
- (7) $P_r(a, P_s(b, c)) = S(P_r(a, b), b, P_{rs}(b, c))$
- (8) $S(S(a, b, c), S(u, v, w), S(x, y, z)) = S(S(a, u, x), S(b, v, y), S(c, w, z))$
- (9) $S(a, b, a) = P_{-1}(a, b) = P_2(b, a)$

If the scalar 2 is invertible in \mathbb{K} , the last relation (9) can be used to recover the map S from the map P : namely, rewriting $a + c = 2\frac{a+c}{2}$, with new origin b , reads

$$S(a, b, c) = P_2(b, P_{\frac{1}{2}}(a, c)).$$

This permits to give an axiomatic definition of affine spaces based on the map P only – see [13], where the approach is carried further to define *projective algebra*.