CONCEPTS GÉOMÉTRIQUES

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Abstract. The following notes – in English –, available at the address http://iecl.univ-lorraine.fr/~Wolfgang.Bertram/WB-coursED.pdf, summarize the contents of the lecture series "Concepts géométriques", given (in French) spring 2016. I will also provide, throughout the text, some hopefully useful hyperlinks to books, papers and other references on the web.

Contents

Présentation du cours en français. Ce cours d'un volume de 20 heures, destiné à un public large de doctorants et de chercheurs dans les domaines scientifiques, traîtra de thématiques qui m'ont occupées – et qui m'occupent toujours – dans ma recherche : la *géométrie* était, depuis plus que 2500 ans, non seulement un domaine des mathématiques, mais aussi un langage, une façon de "voir" le monde matériel ou immatériel. Plusieurs mathématiciens, suivant en cette matière Platon et Euclide, seraient prêts à défendre que *toute* mathématique est géométrique ; d'autres, issus de la grande école mathématique de l'ex-URSS, voient la géométrie plutôt comme une manifestation de l'activité, synthétique et intuitive, de notre hémisphère droite du cerveau, en constante interaction dialectique avec l'action du cerveau gauche, analytique et formelle. La physique fondamentale du 20e siècle reflète cette dialectique : théorie géométrique pure, pour l'une des grandes théories (relativité) ; théorie linéaire, analytique et non-géométrique, pour l'autre (théorie quantique).

Bien qu'issus de recherches géométriques au sens premier, les sujets du cours feront plutôt partie d'une étude du "langage géométrique", au sens second : le titre du cours est une petite allusion à celui du livre *Conceptual Mathematics – A first introduction to categories* par Bill Lawvere et Stephen Schanuel. En effet, le langage des *catégories* est aujourd'hui un aspect important des mathématiques fondamentales, et j'essaierai, dans une partie du cours, d'y donner une sorte d'introduction. Comme je suis autodidacte dans ce domaine, mon approche sera quelque peu différente de ce qu'on trouve dans la littérature, et elle devait plaîre à des informaticiens et autres "usagers" des maths – j'essaierai de "déclarer" toutes les variables et leurs domaines pour arriver à une description algorithmique la plus complète possible. Cependant, le but de cette approche sera d'aller plus loin et de parler de *catégories supérieures* : ces bêtes sont devenues populaires en physique théorique, grâce aux blogs et articles de John Baez, `http://math.ucr.edu/home/baez/week73.html`, et d'une activité assez importante autour du n-lab, `http://ncatlab.org/nlab/show/HomePage`. Mon intérêt en ces théories est d'origine géométrique, suivant ici de près Charles Ehresmann qui est le premier à avoir parlé de *catégories doubles*. En me basant sur un travail récent dont la première partie est disponible sur arxiv, je montrerai que les idées fondamentales d'Ehresmann s'appliquent déjà aux *fondations du calcul différentiel* : on parlera donc, dans une partie du cours, de calcul différentiel. Des connaissances de base de calcul différentiel en plusieurs variables, niveau Licence, suffisent ; à partir de là, on reconstruira tout ce qui mène plus loin.

Selon le temps et les intérêts des auditeurs, je compte aborder dans une autre partie du cours une thématique, à première vue, différente : les mathématiques modernes, dont la théorie des catégories, ainsi que la façon d'appliquer les maths dans les autres sciences, sont dominées par la notion de *fonction* ou *application*. Or, les fonctions sont des cas particuliers de *relations*, et une grande partie de mathématiques devait se généraliser, et ce parfois de façon très naturelle, au cadre des relations : on pourrait parler de "mathématiques relationnelles", pour les distinguer des "mathématiques fonctionnelles". A ma grande surprise, quand j'ai fait des recherches bibliographiques en lien avec cette thématique, je n'ai trouvé que très peu de références écrites par des mathématiciens, la quasi totalité étant écrite par des informaticiens. Plus précisément, je me suis intéressé à la *loi d'associativité* (cf. ici), qui est la loi la plus fondamentale en mathématiques : d'où vient-elle, qu'est-ce qu'on peut faire avec ? Pour bien apprécier l'intérêt de cette question, on pourra, selon temps et envie, faire une excursion dans le "monde non-associative" : il existe bel et bien des structures mathématiques qui ne sont pas associatives ; on les qualifie parfois comme "exceptionnelles" ou "exotiques", ce qui les rend encore plus intriguantes...

Objectifs. Cet enseignement sera donné sous forme de cours, mais j'espère qu'il donnera lieu à des discussions et à des interactions diverses. Le but n'est pas de présenter le contenu d'un livre ou une théorie classique et finalisée, mais de donner un aperçu de thématiques qui ne sont pas encore figées et qu'on peut rencontrer dans la recherche actuelle.

Public visé et prérequis. Tout public ayant une formation scientifique incluant les notions de base mathématiques : *fonction, application, ensembles, calcul différentiel de plusieurs variables*, et qui souhaite revenir sur ces notions en y portant un nouveau regard : mathématiciens de tout bord ; informaticiens ; physiciens...

1. First lecture: Introduction

1.1. **Some personal motivation.** I started research in the domain of Lie theory. The main actors of this theory are *Lie groups*: they arise from a marriage of two things, namely *groups*, and *differentiable spaces*. Thus a Lie group has two aspects: (1) it represents a certain *mathematical concept* (group), that is, a piece of general mathematical language (here: expressing the idea of *symmetry*), and (2) it is a *geometric object* in its own right – a “space”, as one says in geometry, that is, a set carrying structures having a “geometric flavor”. For instance, the following matrix groups (with group law the usual matrix product XY) are Lie groups:

$$\begin{aligned} \mathrm{Gl}(n,\mathbb{R}) &:= \{X \in M(n,n;\mathbb{R}) \mid \det(X) \neq 0\}, \\ \mathrm{O}(n) &:= \{X \in M(n,n;\mathbb{R}) \mid X^t X = I_n\}, \\ \mathrm{U}(n) &:= \{X \in M(n,n;\mathbb{C}) \mid \overline{X}^t X = I_n\}. \end{aligned}$$

The above mentioned double aspect (“concept” and “space”) makes these groups so important in mathematics: they are present everywhere since they are “part of the general mathematical language”.

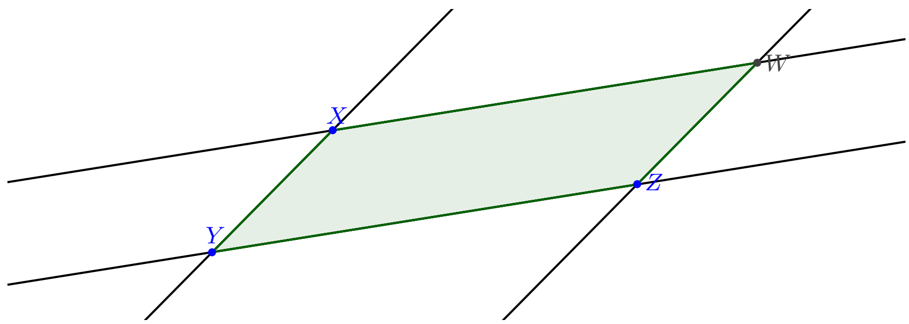
The characteristic feature of Lie theory is the so-called “Lie dictionary”: there is a 1:1-correspondence between Lie groups and algebraic structures called *Lie algebras*. Geometric properties of the groups can be described in terms of these algebras, which are sort of “differential of the group structure at the origin”. For instance, for our three examples above, the algebras are vector spaces of matrices, with algebra structure being the *commutator* $[X,Y] := XY - YX$ of two matrices X, Y,

$$\begin{aligned} \mathfrak{gl}(n,\mathbb{R}) &:= M(n,n;\mathbb{R}), \\ \mathfrak{o}(n) &:= \{X \in M(n,n;\mathbb{R}) \mid X^t + X = 0\} \\ \mathfrak{u}(n) &:= \{X \in M(n,n;\mathbb{C}) \mid \overline{X}^t + X = 0\}. \end{aligned}$$

Algebra is much easier than geometry! Indeed, to put it with the words of Michael Atiyah (see [At02], p.7, talking on “Geometry and Algebra”): *Algebra is the offer made by the devil to the mathematician. The devil says: ‘I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine.’ (Nowadays you can think of it as a computer!) Of course we like to have things both ways; we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless, the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.* However, Atiyah himself certainly is aware that doing mathematics already means to accept the devil’s offer... (and some pages later he emphasizes the importance of Lie’s theory).

Once we have accepted the devil’s offer, we may ask if the “Lie dictionary” generalizes to other contexts, that is, to other classes of algebras. I call this the “coquecigrue problem”, en hommage à Jean-Louis Loday – on my homepage here (Section 2.2) one may find some more remarks about this. Indeed, this question really is close to my own research interests: I have been interested for a long time in the class of *Jordan algebras* (examples include matrix algebras, with symmetric product $X \bullet Y := \frac{1}{2}(XY + YX)$, instead of the commutator product), and later (with M. Kinyon) in the case of *associative algebras* (e.g., the matrix space $M(n, n, \mathbb{R})$ with its "usual" product XY). We call "associative geometry" the geometric object corresponding to an associative algebra. At this place, a remark (cf. loc. cit., Section 2.4) on the relation with *Non-Commutative Geometry* is in order. But let's turn to a concrete example:

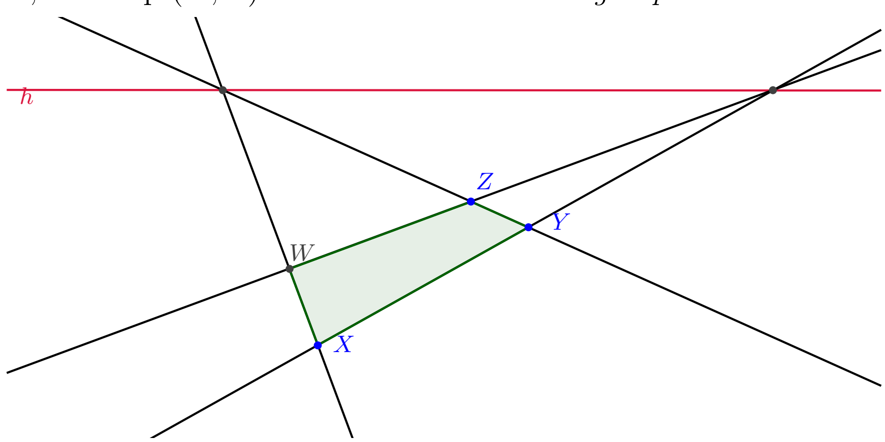
1.2. **Example: the associative plane.** The simplest example of an associative geometry can be nicely represented by images – the following four illustrations are taken from a "travaux pratiques" session made at the occasion of "Journées d'immersion" 2013 – 15, and the reader should do the exercices proposed there: first image – construct the fourth point W in a the parallelogram spanned by X, Y, Z, as follows,



The "geometric formula" for W is $W = \mathrm{Par}\big(Y \vee Z, X\big) \wedge \mathrm{Par}\big(Y \vee X, Z\big)$. There is also an "analytic" formula (seeing X, Y, Z, W as elements of the plane $\mathbb{R}^2$):

$$W = X - Y + Z.$$

This formula shows that W is a *continuous function of three variables* X, Y, Z, and, for Y fixed, the map $(X, Z) \mapsto W$ is a *commutative group law*. Second image

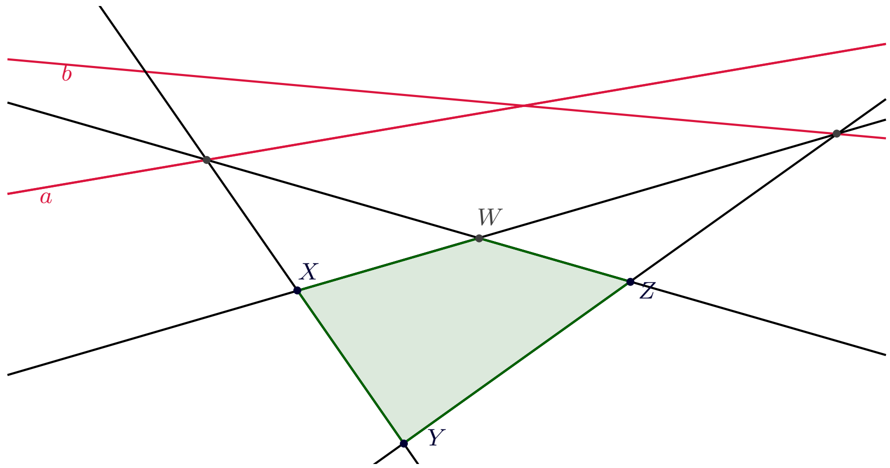


– in fact, this is the *same* image as before, but seen differently, with some line h considered as "horizon" of our drawing plane. (The sense of this phrase is made precise in projective geometry.) The geometric formula is

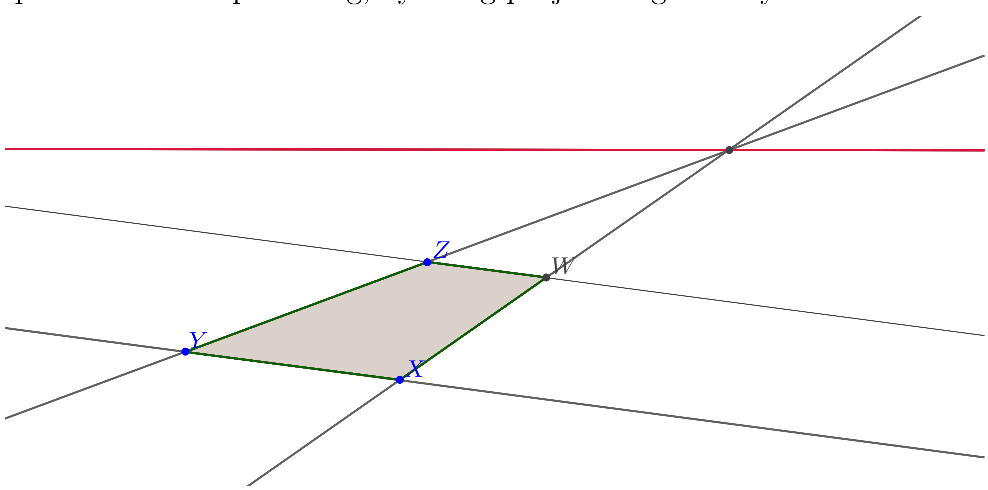
$$W = \Big(\big((X \vee Y) \wedge h \big) \vee Z \Big) \wedge \Big(\big((Z \vee Y) \wedge h \big) \vee X \Big).$$

It follows that $(X,Z) \mapsto W$ still is a commutative group law. (Exercise: check this by an analytic computation in $\mathbb{R}^2$.) The third picture really is something new: do the same construction with *two different horizons*, a and b,

$$W = \Big(\big((X \vee Y) \wedge a \big) \vee Z \Big) \wedge \Big(\big((Z \vee Y) \wedge b \big) \vee X \Big).$$



Theorem. *Fix a, b and Y. Then the map $(X,Z) \mapsto W$ defines a group law (commutative if $a = b$, but non-commutative else).* This result is by no means obvious; we asked several geometers if they had seen it before, but it seems that they haven't. The figure represents what one might call the *associative plane*. Before explaining where it comes from, here is a last image, most useful for a more detailed study, and equivalent to the preceding, by using projective geometry:



So, here W is obtained by the geometric formula

$$W = \Big(\mathrm{Par}\big((X \vee Y), Z \big) \Big) \wedge \Big(\big((Z \vee Y) \wedge a \big) \vee X \Big)$$

1.3. **Torsors.** To explain the theorem stated above, we need some algebra. First recall the fundamental definition of a group. We recall also that the inverse of an element a is unique, justifying the notation a^{-1}. But note that, in a group such as above, the "origin" Y is a point like any other, and has no reason to be preferred to others. Therefore, in geometry we often need a concept describing something like a group, but with no point distinguished among the others. Here it is:

Theorem 1.1 (Forgetting the origin in a group)**.**

(1) Assume $(G, \cdot, e)$ *is a group, and let* $\boxed{(xyz) := x \cdot y^{-1} \cdot z}$*. Then the following* para-associativity *(PA), and* idempotent law *(IP) hold:* $\forall x, y, z, u, v, w \in G$*,*

(PA) $\quad (xy(zuv)) = (x(uzy)v) = ((xyz)uv),$

(IP) $\quad (xxy) = y, \quad (wzz) = w.$

(2) Conversely, if G *is a set with a ternary law denoted by* $G^3 \to G$*,* $(x, y, z) \mapsto (xyz)$ *and satisfying (IP) and (PA), then, for any fixed element* $y \in G$*, we get a group law on* G*, with neutral element* y*, given by* $\boxed{x \cdot z := x \cdot_y z := (xyz)}$*.*

Proof. (1): By direct computation, (IP) and (PA) follow from the group properties; (2): Associativity of $\cdot$ is direct from (PA), y is neutral by (IP), and for existence of an inverse of x, we check that $u := (yxy)$ satisfies $u \cdot x = (uyx) = ((yxy)yx) = (yx(yyx)) = (yxx) = y = (xyu)$ by applying twice (IP), so $x^{-1} = (yxy)$ is an inverse of x. □

Definition 1.2. *A* torsor *is a set* G *together with a ternary map* $G^3 \to G$ *satisfying (IP) and (PA). A torsor is called* commutative *if* $(xyz) = (zyx)$*, for all* $x, y, z \in G$*.*

Remark. Unfortunately, there is no universally accepted terminology: other terms such as *heap, flock, herd, pregroup, groud,....* are also used for what we call here "torsor" – see here for some remarks.

Examples. (1) If the group is commutative and written additively, such as for $G = \mathbb{R}^n$ or any other vector space, then its torsor law is $\boxed{(xyz) = x - y + z}$.

(2) If A and B are two sets, then the set $G := \mathrm{Bij}(A, B)$ of bijective maps $f : A \to B$ is a torsor with respect to the ternary map $\boxed{(fgh) := f \circ g^{-1} \circ h}$ (proof: as above). When $A \neq B$, then G has no "natural origin"! However, note that G may be empty (and the empty set is a torsor).

Exercise. In the situation of point (2) of the theorem, show that moreover $(uvw) = u \cdot_y v^{-1} \cdot_y w$, so we really recover (1) from (2). Observe that the proof uses only

(PA') $\quad (xy(zuv)) = ((xyz)uv),$

so that a torsor could also be defined by requiring (PA') and (IP).

Lemma 1.3. *The torsor axioms (PA)* $\wedge$ *(IP) are equivalent to (Ch)* $\wedge$ *(IP):*

(Ch) left Chasles relation: $(xy(yuv)) = (xuv)$*, and*
right Chasles relation: $((xyz)zv) = (xyv)$*;*

(IP) idempotency: $(xxy) = y = (yxx)$*.*

Proof. (IP) $\wedge$ (PA) implies (Ch), directly by taking $y = z$. Conversely, (IP) $\wedge$ (Ch) implies (PA'): let $k := (uvw)$ and $m := (xyu)$, so
$(xy(uvw)) = (xyk) = ((xyu)uk) = ((xyu)u(uvw)) = (mu(uvw)) = (mvw) = ((xyu)vw)$. By the preceding exercise, (IP) $\wedge$ (PA') implies (IP) $\wedge$ (PA). □

Remark. The same approach can be used to define *affine spaces*: see here.

1.4. **Back to the associative plane.** Fix two lines (“horizons”) a and b, as in the third image above. We want to show that the map $(X,Y,Z) \mapsto (XYZ) := W$ defines a torsor law. By the preceding lemma, it suffices to prove that (IP) and (Ch) hold. The idempotency law (IP) does not follow from the construction, but experience (use geogebra!) shows that it makes good sense, so you may be willing to accept that it holds. To prove (Ch), then, you may simplify the situation by taking the line a “at infinity”, that is, by switching to the fourth image. By “exercice 6” in the travaux pratiques-session (link above), (Ch) amounts to the assertion that always $H = U$ (resp. $H' = U'$). This, in turn, can be proved in various ways: either, by a brute-force computation in the coordinate plane (see here, Section 2), or by noticing that the geometric configuration (the non-commutative prisms shown in “exercice 6”) are equivalent to another classical geometric configuration, namely the one from Desargues’s theorem. Indeed, this confirms what is well-known since David Hilbert’s famous Grundlagen der Geometrie, where it is explained that Desargues’s theorem is the key for understanding *associativity* in the foundations of geometry. Summing up, we may state:

Theorem 1.4. *For any two lines a, b in a Desarguesian projective plane, the set of points not lying on either of these lines is a torsor with torsor law $(XYZ)_{ab} := W$ defined as above.*

Now, as Hilbert shows in his text, there do exist *exceptional geometries* (that is, that do not satisfy Desargues’s theorem), so one may ask: *What can we say about $(XYZ)_{ab}$ in an exceptional geometry?* This is an open problem (but see here for a partial answer in the important special case of *Moufang planes*).

1.5. **Outline of plan of the lectures.** The contents of the lectures is not yet fixed. Roughly, I would like to explain in more detail how to describe and understand the above mentioned *double aspect* of notions such as Lie groups, torsors and things like that: they are part of mathematical language, and they are geometric objects in their own right. This is most clearly seen, and best explained, by studying *groupoids* and *categories*, which generalize groups in a certain way.

In a further step, one observes that the double aspect makes it possible to *apply such gadgets to themselves*: we may apply the gadget, seen as a concept, to study it as an object. On a mathematical level, this leads to notions such as *double groupoids* and *double categories*, and can be iterated n times, leading to *n-fold groupoids* and *n-fold categories*. On a more philosophical level, such a situation deals with self-reference, which often leads to difficult and profound problems (and also shows up in theoretic informatics). Surprisingly, such difficulties also re-appear in *differential calculus*: taking iterated derivatives $f, f', f'', \ldots, f^{(n)}$ of a function of *one* variable looks rather innocent; but already taking higher derivatives $f, Df, D^2 f = D(Df), \ldots$ of a function of several variables is much more complicated, and quickly leads to rather delicate questions. I intend to talk about such things in some of the lectures.

2. Second lecture: Small cats and groupoids

Mathematics, in its present shape, is based on *set theory*. But one can base it also on the richer notion of *category*. As said above, this notion also has the double aspect of being a *concept* and a *mathematical object* (a "space"). To separate these two aspects, I will first of all treat them as concept ("concrete categories") and then as mathematical object ("small categories"). The general notion of category, containing both of these special cases, bears a danger, often named the "size problem", related to any situation where self-reference is possible. Therefore, in these lectures we shall work either with concrete or small categories.

2.1. **Concrete cats.**

Whenever one encounters some kind of mathematical "objects", as a rule, one should also look at the "morphisms", that is, the mappings which preserve the structure of this kind of objects. For instance, if "objects" are vector spaces, then "morphisms" are linear maps:

Definition 2.1. *A* concrete cat *(concrete cat, or: ccat, henceforth) is given by its* objects *which are sets equipped with a structure of a certain "type" $\mathcal{T}$, and by its* morphisms*, which are certain maps between objects. It is required that*

1. *the identity map* id_A *of an object* A *is a morphism, and*
2. *if* $g : A \to B$, $f : B \to C$ *are morphisms, then so is* $f \circ g : A \to C$.

A morphism $f : V \to W$ *is called an* endomorphism *if* $V = W$*; it is called an* isomorphism *if it is bijective and if* $f^{-1} : W \to V$ *is also a morphism; it is called an* automorphism *if it as an endo- and an isomorphism.*

We won't try define formally here what we mean by "type"; rather, we proceed by giving a list of examples. As a matter of mathematical habit, the reader should ask himself, whenever he meets some kind of stuff (mathematical objects): *what are the morphisms of this kind of stuff, and are (1) and (2) satisfied?* For instance, the reader may have met (or not)

- the ccat $\underline{\mathrm{Vect}_{\mathbb{R}}}$; objects: real vector spaces, morphisms: linear maps,
- the ccat $\underline{\mathrm{Grp}}$; objects: groups, morphisms: group homomorphisms,
- the ccat $\underline{\mathrm{Tor}}$ of all torsors (define the morphisms!),
- the ccat $\underline{\mathrm{Ring}}$ of all rings, its subcat $\underline{\mathrm{Field}}$ of all fields,
- the ccat $\underline{\mathrm{Top}}$ of all topological spaces (with continuous maps as morphisms),
- the ccat $\underline{\mathrm{Met}}$ of all metric spaces (who are the morphisms?)
- the ccat $\underline{\mathrm{Man}_{\mathbb{R}}}$ of real, smooth manifolds (with smooth maps as morphisms),
- the ccat $\underline{\mathrm{Alg}_{\mathbb{K}}}$ of all (associative) $\mathbb{K}$-algebras,
- the ccat $\underline{\mathrm{Cat}}$ of all small cats (see later) and its subcat $\underline{\mathrm{Goid}}$ of all groupoids,
- the ccat $\underline{\mathrm{Set}}$; objects; sets, morphisms: (arbitrary) maps,
- ... add here your own favorite ccats !

Remark/exercise. The automorphisms of an object A always form a *group* $\mathrm{Aut}(A)$. Describe it for the examples! Often it is automatic that, if f is bijective, the inverse f^{-1} is a morphism. For which examples is this not automatic?

For the moment, that's all we need to know about ccats (see [BM], for a similar approach; if you want to know more on ccats, have a look at [AHS04]).

2.2. **General cats: size problems!** Here is the general definition:

Definition 2.2. *A* category *(short: cat) C is given by a class* Ob(C) *of* objects *and a class* Mor(C) *of* morphisms (or: arrows)*. Each morphism f has a* source object $a = s(f)$ *and a* targent object $b = t(f)$*; we then write $f : a \to b$. It is required:*

(1) *for morphisms $f : a \to b$ and $g : b \to c$, there is defined a morphism $g * f : a \to c$, and the* composition law $*$ is associative*: for all $f : a \to b$, $g : b \to c$, $h : c \to d$,*

$$(h * g) * f = h * (g * f),$$

(2) *and for each object x, there is a morphism $1_x : x \to x$, which is a* unit *for composition: for any $f : a \to x$ and $g : x \to b$, we have $1_x \cdot f = f$ and $f \cdot 1_x = 1_x$.*

A morphism $f : x \to y$ is called an endomorphism *if $x = y$; it is called an* isomorphism *if there is a morphism $g : y \to x$ such that $g * f = 1_x$ and $f * g = 1_y$, and it is called an* automorphism *if it as an endo- and an isomorphism.*

Remark. If f is an isomorphism, then, as for groups (see above!), it is seen that g is *unique*, so we may speak of "the" inverse of f, and denote it by $g = f^{-1}$.

Example. Every concrete cat is a category, where morphisms are usual maps and the law $*$ is usual composition of maps. But there are categories that are not of this kind (e.g., homotopy category in topology, see later).

Warning. We speak of a "class" Ob(C) of objects, and not of a "set of objects" (and likewise for morphisms), because the collection of all objects is in general "too big" and does not form a set. For instance, in the ccat Set of all sets and set-maps, the class of objects is the "class of all sets". Now, if one is not careful enough, defining a *set of all sets* may lead to a contradiction: this is the famous Russuell's paradox – which is indeed the best known example of dangers related to "self-reference". Today, there are several "solutions" how to circumvent these dangers; but each of them would need careful explanations on axiomatic set theory, which is not the topic of this course. The interested reader may look here, and there, or pages 21-24 in [CWM]. A simple way to avoid size problems is to restrict attention to *small categories.*

2.3. **Small cats, and groupoids.** Now we view a category as a "space":

Definition 2.3. *A* small category *(small cat) is a category such that objects and morphisms form a set, in the usual sense. In other terms (and with some change of notation: π_0, π_1 instead of s, t, etc.): A* small cat *C is given by two sets M (or C_1), the* set of morphisms*, and B (or C_0), the* set of objects*, together with*

- *maps $\pi_0 : M \to B$ ("source map") and $\pi_1 : M \to B$ ("target map"),*
- *a map $\delta : B \to M$ called the* unit section*, such that $\pi_i \circ \delta = \mathrm{id}_B$ for $i = 0, 1$,*
- *and a* partially defined binary product map

$$\boxed{* : M \times_B M \to M, \quad (g, f) \mapsto g * f},$$

defined on the domain given by

$$\boxed{M \times_B M := \{(g,f) \in M \times M \mid \pi_0(g) = \pi_1(f)\}},$$

such that the following holds:

(1) $\forall (g,f) \in M \times_B M$: $\quad \pi_1(g * f) = \pi_1(g)$, $\pi_0(g * f) = \pi_0(f)$,

(2) *the law* $*$ *is* associative*: whenever* (h,g) *and* (g,f) *are in* $M \times_B M$, *then* $(h * g) * f = h * (g * f)$ *(henceforth denoted by* $h * g * f$*),*

(3) units: $\forall f \in M$: $\delta(\pi_1(f)) * f = f$ *and* $\forall g \in M$: $g * \delta(\pi_0(g)) = g$.

So, the whole thing is $(M, B, \pi_0, \pi_1, \delta, *)$, but we will often just write (M, B).

Definition 2.4. *Given a small cat* $(M, B, \ldots)$, *an element* $f \in M$ *is called* invertible*, or an* isomorphism *if there is* $g \in M$, *such that* $\pi_0(g) = \pi_1(f)$ *and* $\pi_1(g) = \pi_0(f)$ *and* $g * f = \delta(\pi_0(f))$ *and* $f * g = \delta(\pi_0(g))$.

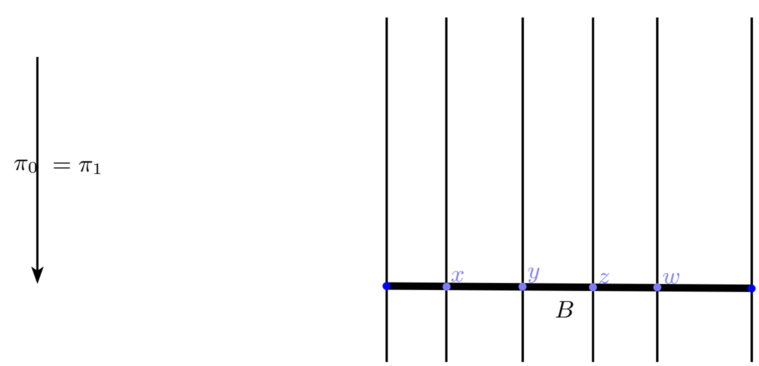
Lemma 2.5. *In a small cat, the inverse element* g *of* f*, if it exists, is unique (and then is denoted by* f^{-1}*). If* h *is also invertible and* $(h,f) \in M \times_B M$, *then* $h * f$ *is invertible, and* $(h * f)^{-1} = f^{-1} * h^{-1}$, *and* $(f^{-1})^{-1} = f$. *For each* $x \in B$, *the set* G_x *of all automorphisms of* x *forms a group.*

Proof. If there is another inverse, g', then $g = g*e = g*f*g' = g'$ where $e = \delta(\pi_0(f)$. Next, $(f^{-1} * h^{-1}) * (h * f) = f^{-1} * (h^{-1} * h) * f = \delta(\pi_0(f))$, and similarly the other way round, showing that $h * f$ is invertible and that $(h * f)^{-1} = f^{-1} * h^{-1}$. $\square$

Definition 2.6. *A* groupoid *is a small cat in which* every *element* $f \in M$ *is invertible. The* inversion map of a groupoid *is the map* $i : M \to M$, $f \mapsto f^{-1}$.

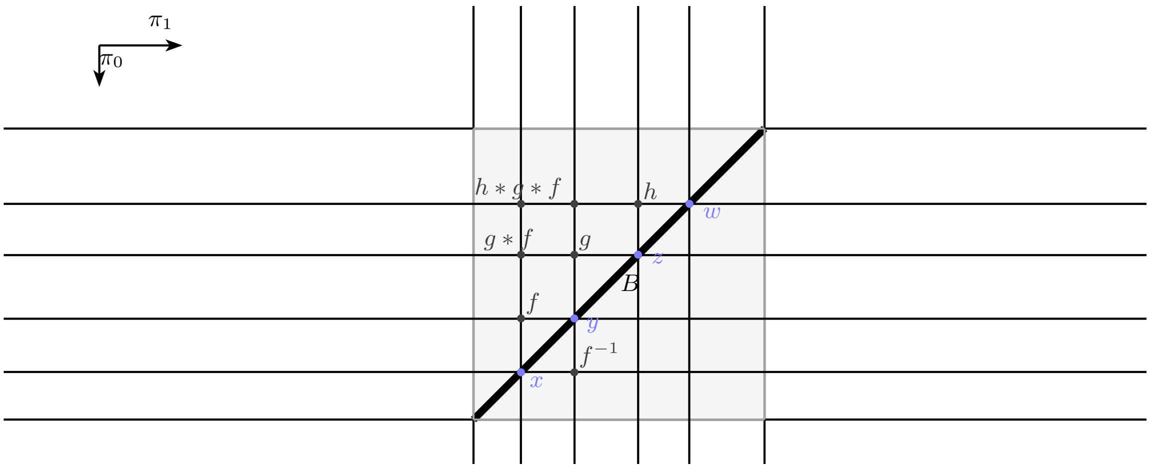
Example 1. If (M, B) is a groupoid and B has exactly one element e, then M is just a group (with unit element $1 = \delta(\pi_0(e))$). If (M, B) is a small cat and B has exactly one element, then M is like a group, but elements need not have inverses: this is what one calls a *monoid* (fr.: monoïde ; example: $\mathbb{Z}$ with its usual product).

Example 2. If $\pi_1 = \pi_0 =: \pi$, then each set $\pi^{-1}(x) = \{f \in M \mid \pi(f) = x\}$ is a group (resp. monoid), with unit $\delta(x)$, as in the preceding example, so (M, B) may be seen as a "disjoint union", or "bundle" of groups, resp. monoids. As a "space", M is then visualized as follows.

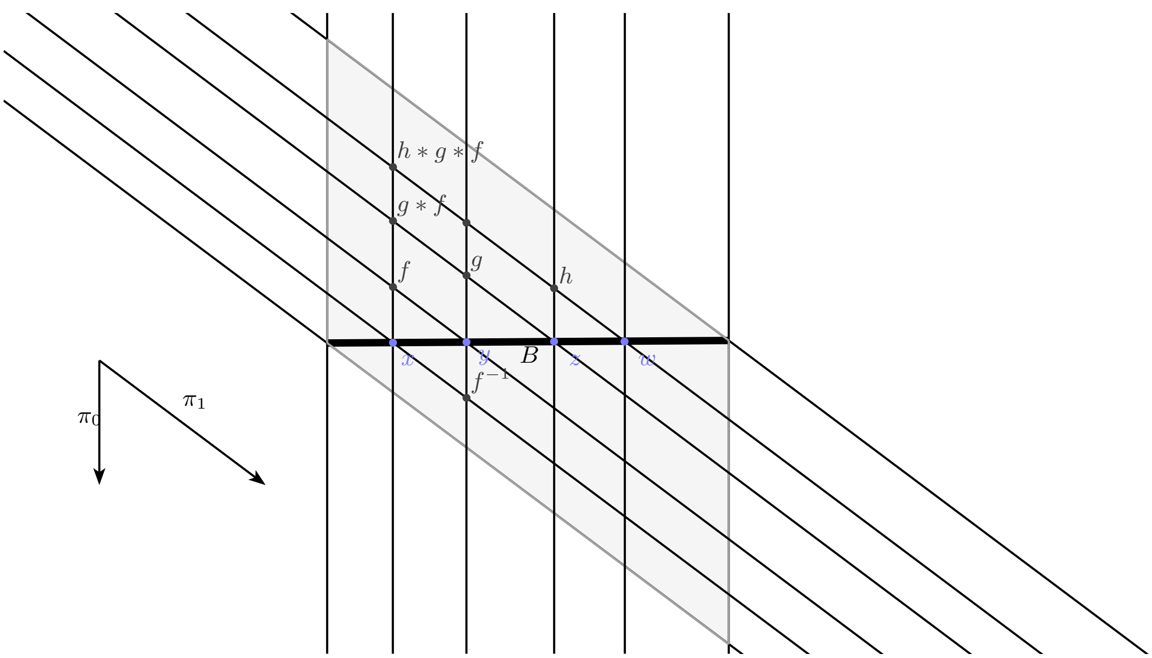


Each vertical line represents a group, and we may identify x with $\delta(x)$ (that is, we identify B with the set of units inside M), and we view π as a "projection" with direction indicated by the arrow. In the special case where each fiber has just one point (the unit), we speak of a *discrete cat* (which is automatically a groupoid). Note that then π and δ are inverse bijections, so one may then identify B with M.

Example 3 (The pair groupoid). If Ω is a set, then let $B := \Omega$, $M := \Omega \times \Omega$ $\pi_0(x, y) = y$, $\pi_1(x, y) = x$, $\delta(x) = (x, x)$ and $(x, y) * (y, z) = (x, z)$. Proposition: this defines a groupoid (note: the inverse of (x, y) is (y, x)), called the *pair groupoid of* Ω. Note: the pair groupoid is characterized by the fact that, for any two objects $x, y \in B$, there is *exactly one* morphism with source x and target y. Here are two visualizations of M as a “space”: the first is a “usual square”, with B its diagonal,



but I prefer this one, with π_0 “vertical” and π_1 oblique, and B a horizontal line:



We indicate also elements $f, g, h \in M$ and where to find $g * f$, f^{-1}, and so on.

Exercise. If (B, M) is a groupoid, show that the following defines an equivalence relation on B: $x \sim y$ iff there is $g \in M$ such that $x = \pi_0(g)$ and $y = \pi_1(g)$.

Show that each equivalence class defines a groupoid in its own right, which can be described in terms of a pair groupoid and of the automorphism group of one of its points. (See [Br1], top of p. 125, for a more precise statement of this exercise, and check these statements!)

Remark. The preceding exercise says that every groupoid can, in a certain way, be "decomposed" into pair groupoids, true groups and discrete groupoids. However, such a decomposition is not useful in pratice (cf. [Br1], loc. cit.) But one may retain that the preceding picture already illustrates quite well the situation of a general groupoid, if one interpretes the image as follows: elements of B are identified with points of the horizontal segment (the units); the vertical and oblique lines describe the fibers of π_0 (morphisms having same source), resp. of π_1 (morphisms having same target); the intersection of two such lines represents the whole set $\mathrm{Mor}(x,y)$ of morphisms between an object x and an object y (in case of the pair groupoid, there is just one such morphism, whereas in general there will be many of them). Note that $\mathrm{Mor}(x,x)$ always is a monoid (resp. a group), and it contains a specific element $\delta(x)$ (its unit element).

One may also note that every small cat has un *underlying (oriented) graph*: then B is the *vertex set*, M is the *set of edges*, where an edge f has vertices $\pi_0(f)$ (tail) and $\pi_1(f)$ (head). If C is moreover a groupoid, then the inversion map i is a *reflection* of the graph, that is, a map of order 2 reversing the orientation of the edges.

Example 4 (Locally defined maps). Let Ω be a set and $B := \mathcal{P}(\Omega) = \{A \mid A \subset \Omega\}$ be the *power set* of Ω (set of all subsets of Ω).

(a) Let $M := \mathcal{F}_{loc}(\Omega) := \{f : U \to U' \mid U, U' \in \mathcal{P}(\Omega), f \text{ function}\}$ be the set of functions with domain and range in Ω, let $\pi_0(f) := \mathrm{dom} f = U$ be the *domain* of f and $\pi_1(f) := \mathrm{codom} f = U'$ the *codomain* of f, and $g * f := g \circ f$ whenever $\mathrm{dom} g = \mathrm{codom} f$; and for $U \in \mathcal{P}(\Omega)$, we let $\delta_U = \mathrm{id}_U$. Obviously, this forms a small cat (denoted by $(\mathcal{F}_{loc}(\Omega), \mathcal{P}(\Omega))$.

(b) We can take for M also the sets $\mathcal{F}^s_{loc}(\Omega)$ of *surjective*, or $\mathcal{F}^i_{loc}(\Omega)$ of *injective*, or $\mathcal{F}^b_{loc}(\Omega)$ of *bijective* functions with range and image in Ω, and get subcats of the one defined in (a). Moreover, $\mathcal{F}^b_{loc}(\Omega)$ is a groupoid (the inverse of f is simply its inverse map), the *groupoid of local bijectijons of* Ω. The automorphism group of an object U is then nothing but the *group of bijections of* U.

2.4. **Example: binary relations.** This generalizes example 4. First, recall:

Definition 2.7. *A* binary relation *between two sets A and B is simply a subset R of $B \times A$. We denote the set of binary relations from A to B by*

$$\mathcal{R}(B, A) := \mathcal{P}(B \times A).$$

Instead of $(y, x) \in R$ one often writes yRx and says that x is in relation R with *y. (We consider x as "source element" and y as "target element", see* <u>note</u> *below!) We say that*

$$R^{op} := \{(x, y) \in A \times B \mid (y, x) \in R\} \in \mathcal{R}(A, B)$$

is the opposite*, or* reverse*, relation of* R*. The* identity relation *on a set* A *is*

$$\delta_A := \{(x,x) \mid x \in A\} \in \mathcal{P}(A \times A).$$

Theorem 2.8. *For* $R \in \mathcal{P}(C \times B)$ *and* $S \in \mathcal{P}(B \times A)$*, we define their* relational composition $R \circ S \in \mathcal{P}(C \times A)$ *by*

$$\boxed{R \circ S = \{(z,x) \in C \times A \mid \exists y \in B : (z,y) \in R, (y,x) \in S\}}.$$

Then the associative law *holds:* $(R \circ S) \circ T = R \circ (S \circ T)$*. Moreover, the identity relation is a neutral element:* $\delta_B \circ S = S$*,* $R \circ \delta_A = R$*. In particular, the set* $\mathcal{P}(A \times A)$ *of* endorelations on A *is a monoid. Moreover,*

$$(R \circ S)^{op} = S^{op} \circ R^{op}, \qquad (R^{op})^{op} = R.$$

Proof. The product is associative: both $(R \circ S) \circ T$ and $R \circ (S \circ T)$ are equal to

$$R \circ S \circ T = \{(z,w) \in \Omega^2 \mid \exists (y,x) \in S : (z,y) \in R, (x,w) \in T\}.$$

And $\delta_A \circ S = \{(z,x) \mid \exists y : z = y, (y,x) \in S\} = S$. □

Theorem 2.9. *For every set* Ω*, the pair of sets* $(M,B) = (\mathcal{P}(\Omega \times \Omega), \mathcal{P}(\Omega))$*, with projections*

$$\pi_0(R) = \{x \in \Omega \mid \exists y \in \Omega : (y,x) \in R\}, \quad \pi_1(R) = \{y \in \Omega \mid \exists x \in \Omega : (y,x) \in R\},$$

units $\delta(A)$*, and* $*$ *being relational composition* $\circ$*, is a small cat (which we call the* cat of endorelations of Ω*).*

Proof. The main point (associativity) has been proved above, and the rest follows by direct check. □

Note that this small cat is not a groupoid! This may seem surprising, since the relation R^{op} looks like an inverse of R. The reader is invited to find out why this does *not provide* an inverse of R...

Theorem 2.10. *Functions are special cases of relations: the small cat* $(\mathcal{F}_{loc}(\Omega), \mathcal{P}(\Omega))$ *from Example 4 is included in* $(\mathcal{P}(\Omega \times \Omega), \mathcal{P}(\Omega)$*, when we identify a function* $f : A \to B$ *with its* graph

$$\Gamma_f := \{(f(x),x) \mid x \in A\} = \{(y,x) \mid x \in A, y = f(x)\}.$$

Proof. Composition of functions corresponds to relational composition of their graphs:

$$\Gamma_g \circ \Gamma_f = \{(z,y) \mid z = g(y)\} \circ \{(y,x) \mid y = f(x)\} = \{(g(f(x)),x) \mid x \in A\} = \Gamma_{g \circ f}.$$

Note: we'll see below ("duality") that one could write products also the other way round, and then one should define graphs by $\{(x,y) \mid y = f(x)\}$, which might seem nicer. However, our writing habits are determined by writing the source element x on the right of the function symbol f, namely $f(x)$. (Some algebraists prefer to write xf instead of $f(x)$, so the source element comes "first", and formulae can be read "from left to right". Neither of these ways of writing can be defined mathematically; the choice is a purely cultural and conventional matter.) □

Using language to be introduced next, we may say that cats of functions are subcats of cats of binary relations. Moreover, if f is a bijective map, then $\Gamma_{f^{-1}} = (\Gamma_f)^{op}$. Hence in this case, the opposite relation really is an inverse element. Thus the local bijections form a subcat which is a subgroupoid.

3. Third lecture: more on groupoids

3.1. **Opposite cat, duality.** Every mathematician fears "sign errors": often there is one chance out of two to put the wrong sign... likewise, when talking about categories, there is one chance out of two to write composition "the wrong way":

Theorem 3.1. *For every (small) cat $C = (M, B, \pi_1, \pi_0, *, \delta)$, we get another (small) cat, the* opposite cat C^{op}*: its sets of objects and morphisms are the same, and so is δ, but we exchange source and target and the order in the composition:*

$$\pi_0^{op} := \pi_1, \qquad \pi_1^{op} := \pi_0, \qquad g *^{op} f := f * g.$$

Proof. The main point is to check associativity of $*^{op}$:

$$(f *^{op} g) *^{op} h = h * (g * f) = (h * g) * f = f *^{op} (g *^{op} h).$$

For the remaining properties: e.g., $\pi_1^{op}(g *^{op} f) = \pi_0(f * g) = \pi_0(g) = \pi_1^{op} g$, etc. □

Note that $(C^{op})^{op} = C$, so every statement about categories can be transformed into another statement about categories by exchanging words "source" and "target" and all terms related to these, and changing the order in all products. This is sometimes called the *duality principle for categories*. It is closely related to duality in projective geometry and is a manifestation of the general aspect of "dualities" in mathematics. To quote again Atiyah [At07]: *Duality in mathematics is not a theorem, but a "principle".*

3.2. **Functors.** Functors are the morphisms between categories. Again, there is a "general", a "concrete", and a "small" version:

Definition 3.2. *A* functor *between two (general) cats C and C' is a rule assigning to each object x of C an object $\phi(x)$ of C', and to each morphism $f : x \to y$ of C a morphism $\Phi(f)$ (from $\phi(x)$ to $\phi(y)$) such that always $\Phi(1_x) = 1_{\phi(x)}$ and $\Phi(g * f) = \Phi(g) * \Phi(f)$.*

Example. There is a functor between the concrete cat $C = \mathrm{Vect}_{\mathbb{R}}$ and its dual cat $C' = C^{opp}$, by assocating to a vector space V its *dual space* V^* and to every linear map $f : V \to W$ its *dual (or transposed) map* $f^* : W^* \to V^*$.

Definition 3.3. *A* morphism of small cats*, (M, B) and (M', B'), or* functor*, is a pair of maps $\Phi : M \to M'$, $\phi : B \to B'$ preserving all structures:*

$$\phi \circ \pi_0 = \pi_0' \circ \Phi, \qquad \phi \circ \pi_1 = \pi_1' \circ \Phi, \qquad \Phi \circ \delta = \delta' \circ \phi, \qquad \Phi(h) *' \Phi(g) = \Phi(h * g).$$

The following result is kind of automatic (cf. linear maps: the composition of linear maps is again linear, and so on), and the proof is by direct check:

Theorem 3.4. *If (Φ, ϕ) is a functor from C to C' and (Ψ, ψ) a functor from C' to C'', then $(\Psi \circ \Phi, \psi \circ \phi)$ is a functor from C to C''. Moreover, $(\mathrm{id}_M, \mathrm{id}_B)$ is a functor from C to itself. It follows that small cats with their morphisms form a concrete category that we shall denote by* $\underline{\mathrm{Cat}}$*.*

Remark 3.1. On a very formal level, in a concrete cat, every object is *one* "set with some structure", whereas a small cat is a *pair* (B, M) of "sets with structure". Therefore, to get things straight, let us say that the *underlying set* of the small cat

$(M, B, \ldots)$ is, by definition, the *disjoint union of sets* $C := M \cup B$, and (Φ, ϕ) then corresponds to a single map $C \to C'$ (which is Φ on M and ϕ on B). In informatics terms, instead of a pair (B, M) of "lists", we put lists B and M together to get a single list. With these definitions, the preceding theorem is formally correct: we get a concrete category.

Example. In a groupoid, the pair of maps $\mathrm{id} : B \to B$ and $i : M \to M$ (inversion) is a functor from C to its opposite category. More generally:

Definition 3.5. *An* involution *of a small cat* $C = (M, B)$ *is a map* $f : M \to M$ *such that*

$$f \circ f = \mathrm{id}_M, \quad \pi_1 \circ f = \pi_0, \qquad \forall a, b \in M \times_B M : \ f(a * b) = f(b) * f(a).$$

Then (id_B, f) *is a functor from* C *to* C^{op} *(which is an fact an* isomorphism of categories*, that is, there is an inverse functor).*

For instance, in the small cat of endorelations $(\mathcal{P}(\Omega), \mathcal{P}(\Omega \times \Omega))$, assigining to a relation R its reverse relation R^{op} defines an involution.

Exercise. Show: If $C = (C_0, C_1, \ldots)$ is a small cat, the pair of maps

$$\mathrm{id} : C_0 \to C_0, \quad C_1 \to C_0 \times C_0, \, a \mapsto (\pi_1(a), \pi_0(a))$$

defines a functor from C to the pair groupoid $(C_0, C_0 \times C_0)$. It is called the *anchor*.

3.3. Subcats, subgroupoids.

Just as subspaces of vector spaces and subgroups of groups, one defines:

Definition 3.6. *A* subcat *of a small cat* $(M, B, \ldots)$ *is a pair of subsets* (M', B') *such that*

$$\forall a \in M' : \ \pi_0(a) \in B', \pi_1(a) \in B', \quad \forall (a, b) \in M' \times_B M' : a * b \in M'.$$

A subgroupoid *of a groupoid is defined in the same way, by adding the condition that* $a^{-1} \in M'$ *if* $a \in M'$.

Theorem 3.7. *A subcat of a small cat is again a small cat, and a subgroupoid is again a groupoid.*

Proof. Again, the proof is kind of automatic (cf. vector spaces....) $\square$

As alreay said there are many examples: e.g., cats of locally defined functions are subcats of cats of endorelations, and so on.

Exercise. Caracterise the subcats of the pair groupoid $(\Omega \times \Omega, \Omega)$, and then caracterize the subgroupoids of the pair groupoid.

3.4. Examples from topology (homotopy category, fundamental groupoid).

This will not be needed in the sequel, but it is certainly helpful to have seen the main examples motivating category theory, in the middle of the 20-the century, coming from *topology*. We assume that that the reader has some working knowledge on *continuity* of maps (say, between M and N, supposed to be metric, or general topological spaces, or subsets of some $\mathbb{R}^n$).

Definition 3.8. *Two maps $f : M \to N$, $g : M \to N$ are called* homotopic *if there exists a* continuous map

$$H : [0,1] \times M \to N, \quad (t,x) \mapsto H(t,x) =: H_t(x)$$

such that $H(0,x) = f(x)$ and $H(1,x) = g(x)$ (so $H_0 = f, H_1 = g$).

We think of $(H_t)_{t\in[0,1]}$ as a "family of maps" interpolating continuously from f to g. In homotopy theory, homotopic maps are considered to be "essentially the same". See here for some dynamic illustrations! Formally:

Lemma 3.9. *Write $f \sim g$ if f and g are homotopic. Then $\sim$ is an equivalence relation on the space of all continuous maps from M to N.*

Proof. $f \sim g$ (via H_1) and $g \sim h$ (via H_2) implies $f \sim h$: use homotopy

$$H(t,x) := \begin{cases} H_1(2t,x) & \text{if } t \in [0,1/2], \\ H_2(2t-1,x) & \text{if } t \in [1/2,1], \end{cases}$$

$f \sim f$: use homotopy $H(t,x) = x$;
$f \sim g \Rightarrow g \sim f$: use homotopy $H'(t,x) := H(1-t,x)$. □

Example. The maps $f : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto 0$ and $g = \mathrm{id}_{\mathbb{R}^n}$ are homotopic: take $H(t,x) = tx$, whence $[f] = [g]$.

Lemma 3.10. *Homotopy is compatible with composition: let $f_1 \sim f_2$ (from M to N, via H_1) and $g_1 \sim g_2$ (from N to P, via H_2), then $g_1 \circ f_1 \sim g_2 \circ f_2$.*

Proof. Take $H(t,x) := H_2(t, H_1(t,x))$. Note that this is again continuous! □

Definition 3.11. *The lemmas imply that the following defines an (abstract) category* <u>HTop</u>*, the* homotopy category*: objects are topological (or metric...) spaces $M, N, \ldots$, and morphisms are equivalence classes (for $\sim$) $[f]$ of continuous maps $f : M \to N$, with composition law $[g] * [f] = [g \circ f]$.*

Remark 3.2. This category is neither small nor concrete! But recall the concrete cat <u>Top</u>. There is a functor from <u>Top</u> to <u>HTop</u>, assigning M to M and $[f]$ to f.

Theorem 3.12. *In the homotopy category, all spaces $\mathbb{R}^n$ are isomorphic to $\mathbb{R}^0 = \{0\}$ (and hence are isomorphic among each other).*

Proof. Let $f : \mathbb{R}^n \to \{0\}$, $x \mapsto 0$ and $g : \{0\} \to \mathbb{R}^n$, $0 \mapsto 0$. Then $[f] * [g] = [\mathrm{id}_0]$, and $[g] * [f] = [\mathrm{id}_{\mathbb{R}^n}]$ (by the example above!) □

One says that all real vector spaces *have same homotopy type*. However, in <u>Top</u> the situation is different: one can prove (but that's not easy!) that in <u>Top</u>, *a space $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^m$ if, and only if, $n = m$*. Turning now to more complicated topological spaces, for instance, the homotopy type of *spheres*

$$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

is much harder to understand. Here, an important tool is the following:

Definition 3.13. *A* path (from a to b) *in a (metric or topological) space M is a continuous map $\gamma : [0,1] \to M$ such that $\gamma(0) = a$ and $\gamma(1) = b$. If $a = b$, it is called a* loop.

Lemma 3.14. *If γ_1 is a path from a to b and γ_2 from b to c, then the following defines a path from a to c:*

$$\gamma_2 * \gamma_1(t) := \begin{cases} \gamma_1(2t) & \text{if } t \in [0, 1/2], \\ \gamma_2(2t-1) & \text{if } t \in [1/2, 1]. \end{cases}$$

Note that the “product” $*$ is not quite associative. Re-parametrizing will make it associative:

Lemma 3.15. *If $\gamma_1 \sim \gamma_1'$ and $\gamma_2 \sim \gamma_2'$ and $\gamma_1(1) = \gamma_2(0)$ and $\gamma_1'(1) = \gamma_2'(0)$, then*

$$\gamma_2 * \gamma_1 \sim \gamma_2' * \gamma_1'.$$

Theorem 3.16. *Assume M is a metric or topological space, and denote by $F(M)$ the set of all homotopy classes of paths in M (with respect to homotopies fixing end points: $\forall t : H(t,0) = a$, $H(t,1) = b$). Then $(F(M), M, *)$ is a groupoid, where $\pi_0([\gamma]) = a = \gamma(0)$, $\pi_1([\gamma]) = b = \gamma(1)$, and $\delta(x)$ is the class of the constant path $t \mapsto x$, and composition $[\alpha] * [\gamma] = [\alpha * \gamma]$ (well-defined according to the lemma).*

There are several things to check: for a very detailed proof of the theorem, see here. The theorem implies that, for any fixed point $a \in M$, the set $F_a(M)$ of loops at a is a group (the automorphism group of a), called the fundamental group of M at a. A very important result in this context is:

Theorem. *The fundamental group of the circle S^1, and the one of $\mathbb{R}^2 \setminus \{(0,0)\}$, at any of its points, is the group of integers $(\mathbb{Z}, +)$.*

Here are some ideas for further reading: Wikepedia page about general topololgy, and more mathematical (but still quite pedagogical and with many figures): book “Algebraic Topology” by A. Hatcher.

4. Fourth lecture: the Cat Family

Cats come in families: we have already seen that every cat has a twin (its dual cat) and that most cats bear interesting subcats. In particular, recall from the second lecture (Theorems 2.9 and 2.10) the *small cat of endorelations on a set* Ω, which contains as subcats the small cat of locally defined functions, and hence also the one of local bijections (which is a groupoid). We shall first give an important generalization of such examples, and second look at other relatives of the cat familiy.

4.1. **The power cat.** Recall from Lecture 2 the small cat of binary relations on a set Ω. We denote by $\mathcal{P}(A)$ the power set of a set A, and if $f : A \to B$ is a map and $C \subset A$, we denote the image of C under f by

$$(\mathcal{P}(f))(C) := f(C) := \{f(x) \mid x \in C\} = \{y \in B \mid \exists x \in C : y = f(x)\}.$$

Thus we have defined a map $\mathcal{P}(f) : \mathcal{P}(A) \to \mathcal{P}(B)$.

Lemma 4.1. *If g and f are composable maps, then*

$$\mathcal{P}(g \circ f) = \mathcal{P}(g) \circ \mathcal{P}(f), \qquad \mathcal{P}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{P}(A)}.$$

In order words, the symbol $\mathcal{P}$ is a functor *(from the ccat $\underline{\mathrm{Ens}}$ to itself).*

Proof. This is just another way of writing $(g \circ f)(C) = g(f(C))$ and $\mathrm{id}(C) = C$. $\square$

Theorem 4.2 (The power cat)**.** *Asssume $(M, B, \ldots)$ is a small cat. Then the pair of* power sets *of M, resp. of B, $(\mathcal{P}(M), \mathcal{P}(B), \ldots)$ is a small cat, with*

$$\pi_i(U) = \{\pi_i(x) \mid x \in U\} = \mathcal{P}(\pi_i)(U), \qquad \delta(K) = \{\delta(k) \mid k \in K\} = \mathcal{P}(\delta)(K),$$
$$U * V = \{a * b \mid a \in U, b \in V, \pi_1(b) = \pi_0(a)\} = \mathcal{P}(*)(U \times_B V).$$

We call this the power cat of (M, B).

Proof. All properties follow easily with the help of the preceding lemma. For instance, concerning associativity: computing $(U * V) * W$ and $U * (V * W)$, both sets agree with $\{a * b * c \mid a \in U, b \in V, c \in W, \pi_1(c) = \pi_0(b), \pi_1(b) = \pi_0(a)\}$. $\square$

Example. Taking for (M, B) the pair groupoid $(\Omega \times \Omega, \Omega)$, we get back the small cat of binary relations with relational composition! Thus the "power cat" generalizes binary relations. Likewise, we may generalize notions of *function, injective, surjective, bijective* to this framework:

Definition 4.3. *Let $(M, B, \ldots)$ be a small cat. A set $U \subset M$ is called*

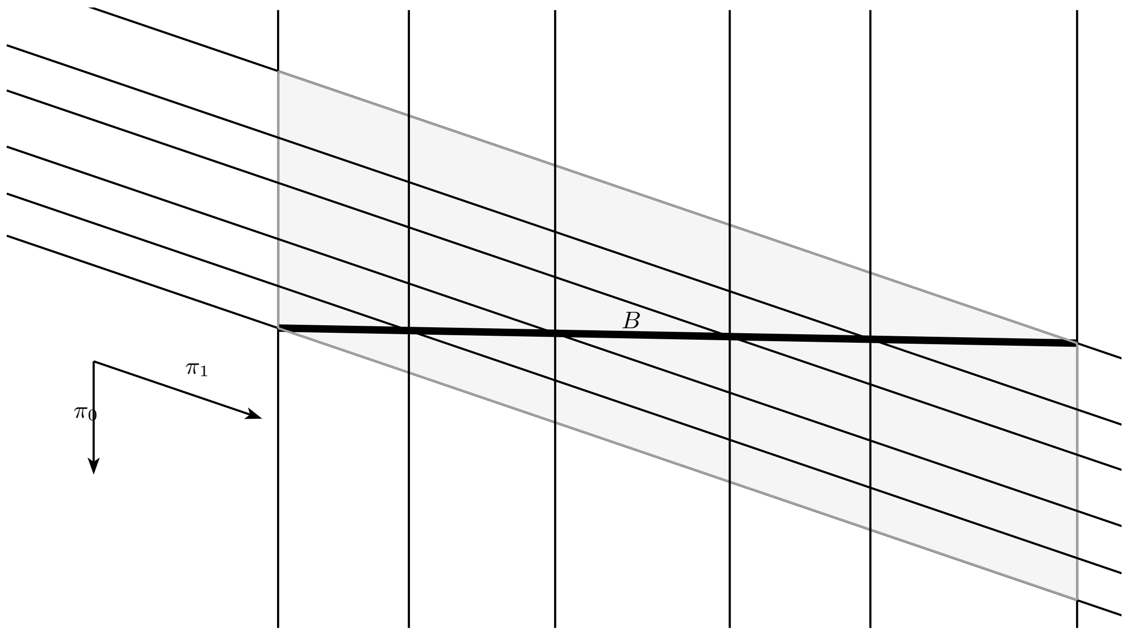
1. *local functional* if $\forall x, y \in U : \pi_0(x) = \pi_0(y) \Rightarrow x = y$,
2. *injective*, if $\forall x, y \in U : \pi_1(x) = \pi_1(y) \Rightarrow x = y$,
3. *everywhere defined* if $\pi_0(U) = B$,
4. *surjective* if $\pi_1(U) = B$,
5. *functional* if it is both everywhere defined and local functional,
6. *surinjective* if it is both injective and surjective,
7. a *local bisection* if it is local functional and injective,
8. a *bisection* if it is functional, injective, and surjective.

Here are "translations" how to read these definitions:

1. *local functional*: each π_0-fiber has *at most one* point in common with U,

(2) *injective*: each π_1-fiber has *at most one* point in common with U,
(3) *everywhere defined*: each π_0-fiber has *at least one* point in common with U,
(4) *surjective*: each π_1-fiber has *at least one* point in common with U,
(5) *functional*: each π_0-fiber has *exactly one* point in common with U,
(6) *surinjective*: each π_1-fiber has *exactly one* point in common with U,
(7) a *local bisection*: each π_0-fiber and each π_1-fiber has *at most one* point in common with U,
(8) a *bisection*: each π_0-fiber and each π_1-fiber has *exactly one* point in common with U.

(By definition, for $i = 0, 1$, by π_i-fiber are meant the sets $\{a \in M \mid \pi_i(a) = x\}$, for $x \in B$, represented by lines in the following image.) Using this, the reader is invited to inscribe into the following image sets U of the 8 types defined above.



Note that type (1) is *dual* to type (2) under exchanging source and targent, and so are [(3) and (4)], resp. [(5) and (6)], whereas (7) and (8) are “self-dual”. If fibers of π_0 are drawn vertically and those of π_1 horizontally, and B diagonally, then some of these images coincide again with the usual way of representing functions.

Theorem 4.4. *If $Q_1 \subset \mathcal{P}(M)$ is the set of all locally functional (respectively: injective; everywhere defined and surjective; functional; surinjective; local bisection; bisection) subsets of M, and $Q_0 = \mathcal{P}(B)$ (respectively, $Q_0 = \{B\}$ for the cases of surjective and everywhere defined sets, and of bisections), then (Q_1, Q_0) is a subcat of $(\mathcal{P}(M), \mathcal{P}(B))$, and hence is itself a small cat.*

Proof. Let us show that Q is stable under the $*$-product. Assume U, V are local functional and $\pi_1(V) = \pi_0(U)$, and let $\pi_0(u * v) = \pi_0(u' * v')$ with $u, u' \in U$ and $v, v' \in V$. Hence $\pi_0(v) = \pi_0(v')$, so $v = v'$ since V is local functional. It follows that $\pi_0(u) = \pi_1(v) = \pi_1(v') = \pi_0(u')$ and hence $u = u'$ since U is local functional. Summing up, $U * V$ is local functional. In the same way we see that $U * V$ is injective if so are U and V.

Aussume U, V are everywhere defined and $\pi_1(V) = \pi_0(U)$. Let $x \in B$. Since U is everywhere defined, there is $u \in U$ with $\pi_0(u) = x$. Since V is everywhere defined, there is $v \in V$ with $\pi_0(v) = \pi_1(u)$. Hence $\pi_1(u * v) = x$, and so $U * V$ is everywhere defined. In the same way we see that $U * V$ is surjective if so are U and V, and combining the preceding items, the same holds with respect to (local) bisections.

Concerning units, for $A \subset B$ let $\delta_A := \{\delta(x) \mid x \in A\}$. This is a locally functional and injective subset of M. It is surjective iff $A = B$, iff it is everywhere defined (and this explains why in this case we have only one object, $Q_0 = \{B\}$). □

Theorem 4.5. *Assume (M, B) is a groupoid and denote by $\mathcal{B}_{loc}(M)$ the set of local bisections and by $\mathcal{B}(M)$ its set of (global) bisections. Then $(\mathcal{B}_{loc}(M), \mathcal{P}(M))$ is a groupoid, and $\mathcal{B}(M), *$ is a group (called the* group of bisections *of (M, B)) with unit element δ_B, the unit bisection of the groupoid.*

Proof. If (M, B) is a groupoid, for any set $U \subset M$, we may define the set $U^{op} = \{u^{-1} \mid u \in U\}$. In general, this will not be an inverse element of U in the power cat (cf. the example of binary relations!). But note that U^{op} is [local functional, resp. injective, everywhere defined, surjective] iff U is [injective, resp. local functional, surjective, everywhere defined]. Therefore, if U is a [local] bisection, then so is U^{op}, and thus $U * U^{op}$ is also a [local] bisection. Moreover, it contains the bisection $\delta(\pi_1(U))$ and hence must agree with it (since it has *at most one* element in common with each π_0 and each π_1-fiber), so $U * U^{op} = \delta(\pi_1(U))$. Similarly we see that $U^{op} * U = \delta(\pi_0(U))$, and hence U^{op} indeed is an inverse of U. This proves the first claim. It follows that $\mathcal{B}(M)$ is a groupoid with one object, i.e., a group. □

4.2. **Marrying groupoids with torsors: pregroupoids.** Recall from the first lecture that, "forgetting" the unit element in a group, and replacing the binary product xy by a ternary product (xyz), we get a *torsor*. In the same way, we may "forget" the units of a groupoid, replacing $*$ by a ternary product. The resulting concept is called (following A. Kock) a *pregroupoid*.

Lemma 4.6. *In a groupoid $(C_0, C_1, \ldots)$, the ternary product*

$$[abc] := a * b^{-1} * c$$

is defined if, and only if, $\boxed{\pi_0(a) = \pi_0(b),\ \pi_1(c) = \pi_1(b)}$*, and it satisfies:*

(IP) $[aab] = b$, $[abb] = a$,

(PA) $[[abc]uv] = [a[ucb]v] = [ab[cuv]]$.

Proof. Direct computation using properties of the inverse (cf. Lemma 2.5). □

Definition 4.7. *Assume M is a set together with two projections (= surjective maps) $\pi_0 : M \to A$ and $\pi_1 : M \to B$ and a ternary product*

$$M \times_{\pi_0} M \times_{\pi_1} M \to M, \quad (a, b, c) \mapsto [abc]$$

such that $\pi_0([abc]) = \pi_0(c)$ and $\pi_1([abc]) = \pi_1(a)$ and defined on the set

$$M \times_{\pi_0} M \times_{\pi_1} M := \big\{(a, b, c) \in M^3 \mid \pi_0(a) = \pi_0(b),\ \pi_1(c) = \pi_1(b)\big\}.$$

We say that $(M, \pi_0, \pi_1, [\ \ ])$ is a pregroupoid *if the product satisfies the identities (IP) and (PA), and we say that it is a* semi-pregroupoid *if it satisfies only (PA).*

Example 1. Every groupoid gives rise to a pregroupoid, by the preceding lemma. We then say that *the pregroupoid comes from a groupoid*.

Example 2. Binary relations form an important example of *semi*-pregroupoids: let $M = \mathcal{R}(B, A) = \mathcal{P}(B \times A)$. For three relations $R, S, T \in \mathcal{R}(B, A)$, we define

$$[RST] := R \circ S^{op} \circ T.$$

Then $(M, \pi_0, \pi_1, [\quad])$, with the usual source and target maps, is a semi-pregroupoid (but not a pregroupoid!). Indeed, it is easy to check (PA),

$$[U[RST]V] = U \circ (R \circ S^{op} \circ T)^{op} \circ V) = (U \circ T^{op} \circ S) \circ R^{op} \circ V = [[UTS]RV].$$

It is not a pregroupoid since in general $[RRT]$ is much bigger than the set T (work out an example: e.g., take $R = B \times A$, the *total relation*).

Example 3. It is not true that every pregroupoid comes from a groupoid: let $M = B \times A$, $\pi_0(y, x) = x$, $\pi_1(y, x) = y$ and

$$\big[(w, z), (y, z), (y, x)\big] = (w, x).$$

Easy check: this defines a pregroupoid. (Indeed, it is a subcat of the preceding example: the subcat of "singleton relations". Here, (IP) works well since for $w = y$ we obtain the element (y, x).) Note: when $A = B$, this comes from the pair groupoid on A. But if A, B are sets of different cardinality, then it is not possible that this pregroupoid comes from a groupoid (because in a groupoid both A and B are identified with set of objects, or with the set of units). We call it the *pair pregroupoid (over A and B)*. Question: which pregroupoids come from groupoids?

Definition 4.8. *In a (semi-) pregroupoid,* (local) bisections*, and all other of the* 8 *types of subsets of M, are defined exactly as in Definition 4.3.*

Theorem 4.9. *A pregroupoid comes from a groupoid if, and only if, it admits a bisection. Then* every *bisection may serve as the set of units for a groupoid law. Summing up, "groupoids are the same as pregroupoids together with some distinguished bisection".*

Proof. One direction is trivial: if a pregroupoid comes from a groupoid, then its set of units is indeed a bisection. The other direction is more interesting: we leave the proof as an exercise – essentially, the arguments are the same as in the case of groups (Theorem 1.1), modulo the technical complications coming from the fact that instead of one unit there is now a whole set of units. To define the binary product, with respect to a distinguished bisection U, one has to define it by $a * c = [auc]$ where $u \in U$ is the unique element with $\pi_0(u) = \pi_0(a)$, $\pi_1(u) = \pi_1(c)$, and then check the groupoid properties. □

Exercise/further remarks. (1) If you are familiar with principal bundles: show that a pregroupoid with $B = \{e\}$ (singleton) is essentially (i.e., neglecting the topological conditions) the same as a (left) principal bundle.

(2) State and prove an analog of Theorems 4.2 and 4.3 for (semi-)pregroupoids: if $(M, [\quad], \ldots)$ is a pregroupoid, then $(\mathcal{P}(M), \ldots)$ is, in a natural way, a semi-pregroupoid, and the set of (local) bisections is a pregroupoid. See here for solutions.

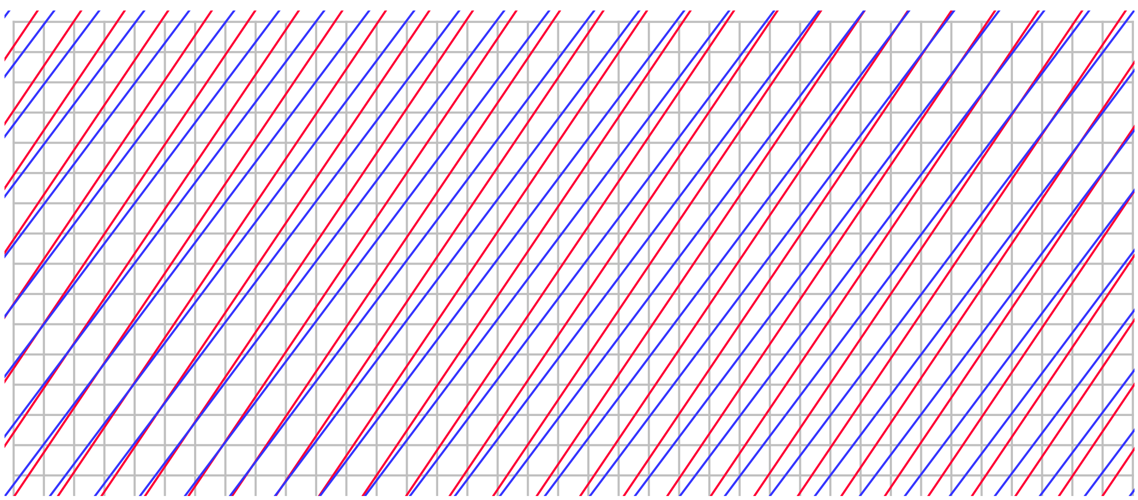
5. Fifth lecture: first order differential calculus

5.1. **Contraction.** Differential calculus can be seen as a manifestation of the the general idea of "contraction of mathematical structure": structures often appear in *families*, say $(S_\lambda)_{\lambda \in I}$, where λ is a *parameter*. When λ is "regular", or "generic", the structures S_λ are often also "regular", or "non-degenerate", whereas when λ approaches a "singular" value (often denoted by 0), then the structure also somehow becomes singular (it "contracts" to S_0). Let's look at an example:

Definition 5.1 (W. Blaschke: Geometrie der Gewebe)**.** *Let $d \geq 1$ be a natural number. A d-*web *in the plane (French: d-*tissu*) is given by d families of lines (which may be "straight", or "curved") in the plane $\mathbb{R}^2$, such that*

(1) each point of the plane belongs to exactly one line of each family,
(2) no two different lines of the same family intersect,
(3) any two lines of two different families intersect in exactly one *point.*

The following figure shows two illustrations of 2-nets: the grey lines form an "orthogonal" 2-net (two classes of straight lines, intersecting orthogonally), and the colored lines form an "oblique" 2-net (classes of straight red lines and classes of straight blue lines; the red lines are not parallel to the blue ones, hence each red line intersects each blue line in exactly one point). For simplicity, we have chosen "straight" lines for our figure, but one could also use families of "curved lines".



Now imagine that the red lines become "more and more parallel" to the blue lines: we will end up with a single family of parallel lines, i.e., a 1-web. Thus 2-webs (regular configuration of 2 families of lines) may "contract" to 1-webs (singular configuration). A general setting is the one of *equivalence relations*:

Equivalence relations. Fix a set Ω and recall the notion of an *equivalence relation*, say α, on Ω. Instead of $x\alpha y$ we also write $x \sim_\alpha y$, and denote by $[x]_\alpha = \{y \in \Omega \mid y \sim_\alpha x\}$ the *equivalence class* of x. Recall that the equivalence classes form a partition of Ω, and the *quotient set* Ω/α or $\Omega/\sim_\alpha$ is the set of all equivalence classes (a subset of $\mathcal{P}(\Omega)$).

Definition 5.2. *Two equivalence relations α, β on Ω are called* transversal *if each equivalence class of α intersects each equivalence class of β at exactly one point, i.e.,* $\mathrm{card}([x]_\alpha \cap [y]_\beta) = 1$ *for all $x, y \in \Omega$. We then write* $\boxed{\alpha \top \beta}$.

Remark/exercise. Using notation from the preceding chapter, $\alpha \top \beta$ if, and only if [$\alpha \circ \beta = \Omega \times \Omega$ (relational composition is the total relation) and $\alpha \cap \beta = \delta(\Omega)$ (set theoretic intersection is the diagonal)]. In particular, it follows that $\alpha \circ \beta = \beta \circ \alpha$ (we say that α and β *commute*).

Example 1. On $\Omega = \mathbb{R}^2$, each 1-net defines a partition, hence defines an equivalence relation. Two such equivalence relations are transversal if, and only if, they form a 2-net.

Example 2. If $\Omega = A \times B$, we define

$$(x, y) \sim_\alpha (x', y') \text{ iff } x = x', \text{ and } (x, y) \sim_\beta (x', y') \text{ iff } y = y'.$$

These are equivalence relations, and the equivalence classes are the "vertical", resp. "horizontal lines". Each horizontal "line" intersects each vertical "line" at exactly one element of Ω, so $\alpha \top \beta$. Indeed, every pair of transversal equivalence relations is obtained in this way:

Proposition 5.3. *Assume (α, β) is a pair of transversal equivalence relations on Ω. Then $\Omega = A \times B$, as in the preceding example, with $A = \Omega/\alpha$ and $B = \Omega/\beta$.*

Proof. Let $A = \Omega/\alpha$ and $B = \Omega/\beta$. The map

$$A \times B \to \Omega, \quad (a, b) \mapsto a \cap b$$

is well-defined since $a \cap b$ can be identified with the unique element it contains. This map is bijective: an inverse map is given by

$$\Omega \to A \times B, \quad x \mapsto ([x]_\alpha, [x]_\beta).$$

Identifying thus Ω with $A \times B$, (α, β) are as in the example. □

However, on a single set Ω we may define plenty of very different pairs of equivalence relations, and a regular (transversal) pair (α, β) may contract into a singular (non-transversal) pair. Question: do other structures contract as well?

Exercise. Let (α, β) be a pair of transversal equivalence relations on Ω. Identifying Ω with the direct product $\Omega/\alpha \times \Omega/\beta$, show that, for a triple of sets $(X, Y, Z) \in \mathcal{P}(\Omega)^3$ the ternary composition $W := [XYZ]_{\alpha,\beta} = X^{-1} \circ Y \circ Z \subset \Omega$ is given by the following expression:

$$[XYZ]_{\alpha,\beta} = \left\{ \omega \in \Omega \;\middle|\; \begin{array}{c} \exists \xi \in X, \exists \eta \in Y, \exists \zeta \in Z : \\ \omega \sim_\alpha \zeta, \quad \eta \sim_\alpha \xi, \quad \omega \sim_\beta \xi, \quad \eta \sim_\beta \zeta \end{array} \right\}.$$

Note that the result depends on α and on β. Next assume that (α, β) are no longer transversal but still commute. Show that the same formula $[XYZ]_{\alpha,\beta}$ still defines a para-associative ternay product on $\mathcal{P}(\Omega)$, and that $\mathcal{P}(\Omega)$ with this product still is a semi-pregroupoid. (See here (page 9) for solutions. This is closely related to the associative geometries mentioned in the first lecture. If time is permitting, we shall later in these lectures say more on *generalized* 3-*webs*, which are closely related to *loops and quasigroups*.)

5.2. **First order differential calculus.** Let V and W be finite-dimensional real vector spaces, U a non-empty open subset of V and $f : U \to W$ a map. The *difference quotient* or *slope* is defined for regular $t \in \mathbb{R}$ (i.e., $t \neq 0$), by

$$f^{]1[}(x,v,t) := \frac{f(x+tv) - f(x)}{t}.$$

We view $f^{]1[}$ as a *function of three variables* (x,v,t) where $x \in U, v \in V, t \in \mathbb{R}^\times$ such that $x + tv \in U$. Differential calculus defines a contraction of this expression for the singular value $t = 0$. (This idea has been developed in joint work with H. Glöckner and K.-H. Neeb. See the book "Calcul différentiel topologique élémentaire" for a detailed exposition of this approach, and here (Chapter 1) for some complementary information.)

Theorem 5.4. *The following statements are equivalent:*

(1) the map f is of class C^1 (i.e., continuously differentiable),
(2) the slope $f^{]1[}$ admits a continuous *extension to a map $f^{[1]} : U^{[1]} \to W$ defined on the set*

$$\boxed{U^{[1]} := \{(x,v,t) |\, x \in U, v \in V, x + tv \in U\}}.$$

If this holds, the total differential of f is given by $Df(x)v = f^{[1]}(x,v,0)$.

Proof. Assume (2) holds. Then $\lim_{\substack{t\to 0\\ t\neq 0}} \frac{f(x+tv)-f(x)}{t} = \lim_{\substack{t\to 0\\ t\neq 0}} f^{[1]}(x,v,t) = f^{[1]}(x,v,0)$, by continuity of $f^{[1]}$. In particular, $\frac{\partial}{\partial_i} f(x) = f^{[1]}(x, e_i, 0)$, so all partial derivatives exist and are continuous, hence f is C^1. Now assume that (1) holds and define a map

$$f^{[1]} : U^{[1]} \to F, \quad (x,v,t) \mapsto \begin{cases} \frac{f(x+tv)-f(x)}{t} & \text{if} \quad t \in \mathbb{R}^\times \\ Df(x)v & \text{if} \quad t = 0. \end{cases}$$

We must show that $f^{[1]}$ is continuous. Since f is continuous, it follows that $f^{[1]}$ is continuous on the set

$$U^{]1[} := \{(x,v,t) \in U^{[1]} |\, t \neq 0\}. \tag{5.1}$$

Thus it remains to prove that $f^{[1]}$ is continuous at all points of the form $(x,v,0)$. To show this, note first that, whenever the line segment $[x, x+tv]$ belongs to U, then

$$f^{[1]}(x,v,t) = \int_0^1 Df(x+stv)v\, ds. \tag{5.2}$$

Indeed, for $t = 0$, this is obviously true, and for $t \neq 0$, it follows from the fundamental relation between differential and integral calculus, along with a change of variable $r = st$,

$$f(x+tv) = f(x) + \int_0^t Df(x+rv)v\, dr = f(x) + t\int_0^1 Df(x+stv)v\, ds$$

which implies (5.2). Now, the right hand side is the integral of a continuous function, depending continuously on the parameter (x,v,t). Since integration carries over a compact interval, the continuity of the right hand side follows by elementary analysis (no need to invoke more powerful machinery like Lebesgue's theorem). □

Comment. This theorem is important because it permits to explain *differentiability of* f (1) in terms of *continuity of* $f^{[1]}$ (2). Condition (2) makes sense in any framework where continuity is defined, e.g. in general (infinite dimensional) topological vector spaces:

Definition 5.5. *Assume V, W are arbitrary topological vector spaces. Then we say that f* is of class C^1 *if it satisfies the condition (2) from the theorem. The same definition is given if V, W are topological vector spaces over the field $\mathbb{K} = \mathbb{C}$, or over any other (non-discrete) topological field (such as the p-adic numbers $\mathbb{Q}_p$), or even a* topological module over a topological ring.

See the above mentioned references for motivation of this definition: the "classical" definitions cannot be reasonably generalized to such general contexts (the main reason is that the integral representation 5.1 does not always carry over). – The proofs of the following classical results hold in this general framework:

Theorem 5.6. *If f is C^1, then the differential $Df(x) : V \to W$ at each point $x \in U$ is a linear map.*

Proof. Let us show that $Df(x)(v+w) = Df(x)v + Df(x)w$ for $v, w \in V$. For $t \neq 0$, we have

$$\begin{aligned} f^{[1]}(x, v+w, t) &= \frac{f(x + t(v+w)) - f(x)}{t} \\ &= \frac{f(x + tv + tw) - f(xx + tv)}{t} + \frac{f(x + tv) - f(x)}{t} \\ &= f^{[1]}(x + tv, w, t) + f^{[1]}(x, v, t). \end{aligned}$$

Thus $f^{[1]}(x, v+w, t) = f^{[1]}(x + tv, w, t) + f^{[1]}(x, v, t)$ for all non-zero t in a neighborhood of 0. Since, by Theorem 5.4, both sides are continuous functions of t, it follows that equality also holds for $t = 0$, which directly implies the claim. Now we prove that $Df(x)(rv)r = r\, Df(x)v$. For $t \neq 0$ and $r \neq 0$,

$$\begin{aligned} f^{[1]}(x, rv, t) &= \frac{f(x + t(rv)) - f(x)}{t} \\ &= r\frac{f(x + tr\, v) - f(x)}{tr} = r\, f^{[1]}(x, v, tr). \end{aligned}$$

Thus $f^{[1]}(x, rv, t) = rf^{[1]}(x, v, tr)$ for all non-zero t, r in a neighborhood of 0. Again by Theorem 5.4, equality still holds for $t = 0$: $Df(x)(rv) = rDf(x)v$. □

Comment. In the approach often chosen in France, linearity of the differential holds *by definition*, and therefore it is not realized that linearity is indeed a *consequence* of other properties. See again loc. cit. for more comments on this.

Theorem 5.7. *If g and f are composable mappings of class C^1 defined on open subsets of finite-dimensional vector spaces, then $g \circ f$ is again of class C^1, and its differential is given by*

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x).$$

Proof. We use again Theorem 5.4 and write $f(x+tv) = f(x) + tf^{[1]}(x,v,t)$. Then, for $t \neq 0$ the difference quotient is

$$\begin{aligned}(g \circ f)^{[1]}(x,v,t) &= \frac{g(f(x+tv)) - g(f(x))}{t} \\ &= \frac{g(f(x) + tf^{[1]}(x,v,t)) - g(f(x))}{t} = g^{[1]}(f(x), f^{[1]}(x,v,t), t).\end{aligned}$$

Our assumptions imply that the right hand side is a continuous function of (x,v,t), and hence a continuous difference factorizer for $g \circ f$ exists, given by this formula. Taking $t = 0$ now yields $D(g \circ f)(x)v = Dg(f(x))(Df(x)v)$, as claimed. □

Definition 5.8. *If $f : U \to W$ is of class C^1, then the* tangent map *is defined by*

$$Tf(x,v) := \big(f(x), Df(x)v\big).$$

Thus $Tf : TU \to TW$ is defined on the set $TU := U \times V$.

The chain rule can be re-written

$$T(g \circ f)(x,v) = \big(g(f(x)), Dg(f(x))(Df(x)v)\big) = Tg(Tf(x,v))$$

so $\boxed{T(g \circ f) = Tg \circ Tf}$, which means that the symbol T is a *functor*! Whenever we have such a functorial rule, we may be pretty sure that this has an "intrinsic", or "geometric", meaning. Indeed, in the present case, the notion of tangent map is intrinsically defined for any differentiable manifold. It is also true, but less classical, that already *before* taking the limit we get such a kind of "tangent map":

Definition 5.9. *If f is of class C^1, then the* extended tangent map *is defined by*

$$\hat{T}f(x,v,t) := \big(f(x), f^{[1]}(x,v,t), t\big),$$

on the set $\hat{T}U := U^{[1]}$, so we have a map $\hat{T}f : \hat{T}U \to \hat{T}W$.

The computation proving Theorem 5.6 can now be re-written

$$\begin{aligned}\hat{T}(g \circ f)(x,v) &= \big(g(f(x)), (g \circ f)^{[1]}(x,v,t), t\big) \\ &= \big(g(f(x), g^{[1]}(f(x), f^{[1]}(x,v,t), t) = \hat{T}g(\hat{T}f(x,v)),\end{aligned}$$

so again we have a functorial rule $\boxed{\hat{T}(g \circ f) = \hat{T}g \circ \hat{T}f}$. This supports the conjecture that the set $\hat{T}U$ carries some "intrinsic structure". Indeed, this is the case:

5.3. **The tangent groupoid.** In the following, one may read $\mathbb{K} = \mathbb{R}$, or $\mathbb{K} = \mathbb{C}$, or some other field.

Theorem 5.10. *The set $\hat{T}U = U^{[1]}$ carries the structure of a groupoid, over the object set $U \times \mathbb{K}$, with source and target projections, product and units defined by*

$$\begin{aligned}\pi_0 : U^{[1]} \to U \times \mathbb{K}, &\quad (x,v,t) \mapsto (x,t), \\ \pi_1 : U^{[1]} \to U \times \mathbb{K}, &\quad (x,v,t) \mapsto (x+tv,t), \\ (x,v,t) * (x',v',t') := (x', v+v', t) &\quad \text{ if } x = x' + tv', t = t', \\ \delta : U \times \mathbb{K} \to U^{[1]}, &\quad \delta(x,t) := (x,0,t).\end{aligned}$$

For every $t \in \mathbb{R}$, we have a groupoid $U_t = \big\{(x,v) \mid (x,v,t) \in U^{[1]}\big\}$ over the base U.

Proof. First note that $\pi_0(x', v+v', t) = x' = \pi_0(x', v', t')$ and $\pi_1(x', v+v', t) = x' + t(v+v') = x + tv = \pi_1(x,v,t)$. Associativity of $*$ now follows directly from associativity of $+$ in V. Next, let's check that $(x,0,t)$ is indeed a neutral element: $(x,0,t) * (x',v',t) = (x',v',t)$ and $(x,v,t)*(x',0,t) = (x',v,t) = (x,v,t)$ (since $x'=x$ here). An inverse of (x,v,t) is

$$(x,v,t)^{-1} = (x+tv, -v, t).$$

Indeed, $(x,v,t)*(x+tv,-v,t) = (x+tv,0,t)$, $(x+tv,v,t)*(x,v,t) = (x,0,t)$. Note that in the whole proof the variable t does not play an "active" role, hence all arguments remain valid for t fixed, defining a groupoid U_t over U. $\square$

We summarize the definition of $\hat{T}U$ by the diagram

$$\begin{array}{ccc} \hat{T}U & \rightrightarrows & U \times \mathbb{K}_1 \\ (x,v,t) & \overset{\pi_0}{\mapsto} & (x,t) \\ & \overset{\pi_1}{\mapsto} & (x+tv,t) \\ (x,0,t) & \overset{\delta}{\longleftarrow\!\shortmid} & (x,t) \end{array}$$

Remark. For $t=0$, the groupoid $U_0 = TU$ is the *tangent bundle*: in this case $\pi_1(x,v) = x + 0v = x = \pi_0(x,v)$, Ie., $\pi_0 = \pi_1$, so we have a *group bundle*. Indeed, each fiber of π then is a copy of V with its usual sum as group law (the *tangent space*). Nowadays, the tangent bundle is recognized, in mathematics and physics, as an important geometric object in its own right. For $t \neq 0$, we get a "true" groupoid:

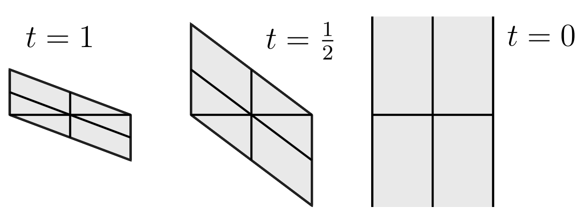
Proposition 5.11. *For $t \neq 0$, the groupoid U_t is isomorpic to the pair groupoid $U \times U$. More precisely,*

$$\Phi : U_t \to U \times U, \quad (x,v) \mapsto (x, x+tv)$$

is an isomorpism of groupoids over the base map id_U.

Proof. The map Φ is bijective: an inverse map is given by $(x,y) \mapsto (x, \frac{y-x}{t})$. By direct compution, (Φ, id_U) is a morphism of groupoids. (Indeed, it is the *anchor morphism*, cf. the exercise p. 16 !) Note that $(x,0)$ is mapped to (x,x). $\square$

Summing up, for invertible t, and in particular $t=1$, π_0 and π_1 have transversal fibers (they form a generalized 2-web), which for $t \to 0$ become "more and more parallel" and finally contract for $t=0$ to the singular situation $\pi_0 = \pi_1$ (1-web). Here a figure, where U is an interval, with same conventions as for the figure p. 19,



Remark. The *tangent groupoid* defined by Alain Connes in Section II.5 of his book Non-Commutative Geometry, essentially coincides with the one defined here: Connes defines it as disjoint union

$$TU \bigcup (U \times U \times]0,1]),$$

gluing together the tangent bundle TU with copies of the pair groupoid. This construction is somewhat artificial, and it does not reveal the purely algebraic nature of the construction: indeed, our Theorem 5.10 is valid for any field (even, every base ring), and topological properties are not used at this stage.

5.4. **Morphisms: C^1-maps revisted.** The groupoid structure on $\hat{T}U$ really is "natural", because of the following

Theorem 5.12. *If $f : U \to W$ is of class C^1, then the pair of maps*
$$(\hat{T}f, f \times \mathrm{id}_{\mathbb{K}}) : (\hat{T}U, U \times \mathbb{K}) \to (\hat{T}W, W \times \mathbb{K})$$
is a (continuous) groupoid morphism. We indicate this by the diagram
$$\begin{array}{ccc} \hat{T}U & \stackrel{\hat{T}f}{\to} & \hat{T}W \\ \downdownarrows & & \downdownarrows \\ U \times \mathbb{K} & \stackrel{f\times \mathrm{id}_{\mathbb{K}}}{\to} & W \times \mathbb{K}\,. \end{array}$$

Proof. The condition $\pi_0 \circ \hat{T}f = (f \times \mathrm{id}_{\mathbb{K}}) \circ \pi_0$ holds since
$$\pi_0 \circ \hat{T}f(x, v, t) = (f(x), t) = f \times \mathrm{id}_{\mathbb{K}}(x, t) = (f \times \mathrm{id}_{\mathbb{K}})(\pi_0(x, v, t)).$$
The condition $\pi_1 \circ \hat{T}f = (f \times \mathrm{id}_{\mathbb{K}}) \circ \pi_1$ holds since
$$\begin{aligned} \pi_1 \circ \hat{T}f(x, v, t) &= \pi_1(f(x), f^{[1]}(x, v, t), t) = (f(x) + t \cdot f^{[1]}(x, v, t), t) \\ &= (f(x + tv), t) = (f \times \mathrm{id}_{\mathbb{K}})(\pi_1(x, v, t)). \end{aligned}$$
The condition $\hat{T}f(x, 0, t) = (f(x), 0, t)$ holds since $f^{[1]}(x, 0, t) = 0$ for all t. Indeed, $\frac{f(x+0t)-f(x)}{t} = 0$ for all $t \neq 0$, and by continuity also for $t = 0$. It remains to prove that $\hat{T}f$ preserves $*$. We compute
$$\begin{aligned} \hat{T}f((x', v', t') * (x, v, t)) &= \hat{T}f(x, v' + v, t) = (f(x), f^{[1]}(x, v' + v, t), t) \\ \hat{T}f(x', v', t') * \hat{T}f(x, v, t) &= (f(x), f^{[1]}(x + vt, v', t), t) * (f(x), f^{[1]}(x, v, t), t) \\ &= \big(f(x), f^{[1]}(x + vt, v', t) + f^{[1]}(x, v, t), t\big)\,. \end{aligned}$$
Thus equality holds iff $f^{[1]}(x, v' + v, t) = f^{[1]}(x + vt, v', t) + f^{[1]}(x, v, t)$. But that is exactly what we have seen in the proof of Theorem 5.6, and so is true (moreover, we see that additivity of $Df(x)$ is the special case $t = 0$ of this property). □

Summing up: a map $f : U \to W$ is C^1 if, and only if, it "extends" continuously to an "extended tangent map" $\hat{T}f : \hat{T}U \to \hat{T}W$, which is a groupoid morphism. As seen above, the construction is functorial, and hence the symbol $\hat{T}$ is a functor from the ccat of open subsets of vector spaces into the ccat of (topological) groupoids with their (continuous) morphisms. Without going into details (that can be found here), let us mention that this statement carries over to the ccat of *differentiable manifolds*, instead of open subsets of vector spaces. The construction of $\hat{T}U$ pursues similar aims as synthethetic differential geometry, but by different (more elementary) methods.

Next, we have to look at *second, and higher order differential calculus.*

6. Sixth lecture: second order differential calculus

Second and higher order calculus consist in *iterating*, in one way or another, the procedure of deriving once. For functions of *one (scalar)* variable, iteration is “easy”: since f' is again a function of one variable, let $f'' = (f')'$, and so on: $f^{(n+1)} = (f^{(n)})'$. For functions of *vector variables*, this becomes more difficult: given $f : U \to W$, there are (at least) two variants of defining the differential:

(a) write the first differential as $Df : U \to \mathrm{Hom}(V, W)$, $x \mapsto (v \mapsto Df(x)v)$. Then $D^2 f := D(Df)$ is a map $U \to \mathrm{Hom}(V, \mathrm{Hom}(V, W))$, $x \mapsto D^2 f(x)$. But one may identify $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$ with the space $\mathrm{Bil}(V \times V, W)$ of bilinear maps. By *Schwarz’ theorem*, the bilinear map $D^2 f(x)$ is *symmetric*.

(b) writing the first differential as $df : U \times V \to W$, $(x, v) \mapsto Df(x)v$, the second differential will be a map $d^2 f : U \times V \times V \times V \to W$. This is a map of *four* (vector) variables!

Generally speaking, the number of variables increases, at each step of the iteration procedure. We will do even “worse”: iterating $f^{[1]}$, we will define $(f^{[1]})^{[1]}$, the “slope of the slope”, and look at a map of *seven* variables.

6.1. Second differential, and Schwarz’ theorem.

Definition 6.1. *With notation as in the preceding lecture, a map $f : U \to W$ is called* of class C^2, *if: f is C^1, and the map $f^{[1]} : U^{[1]} \to W$ is also C^1. In this case, we define $f^{[2]} := (f^{[1]})^{[1]}$. We say that f* is of class C^n *if f is of class C^{n-1}, and $f^{[n-1]}$ is of class C^1. We then define $f^{[n]} := (f^{[n-1]})^{[1]}$. We then also define, inductively, $\hat{T}^n U := \hat{T}(\hat{T}^{n-1} U)$, and*

$$\hat{T}^n f := \hat{T}(\hat{T}^{n-1} f) : \hat{T}^n U \to \hat{T}^n W.$$

Theorem 6.2. *If $V = \mathbb{R}^k$ and $W = \mathbb{R}^m$, then the preceding definition of class C^n coincides with all other usual definitions.*

Proof. Essentially, one uses the argument from the proof of Theorem 5.4 repeatedly. See references given there for details. The main argument uses that in (5.2), one can “derive under the integral”. □

Remark. The theorem also holds for Banach spaces. In more general situations, most classical definitions fail, and our definition turns out to be strictly stronger.

Formulae for $U^{[2]}$, for $f^{[2]}$, and for $\hat{T}^2 f$. Applying twice the definitions from the preceding lecture, we get the following explicit formulae:

$$\begin{aligned} f^{[2]}\big((x,v,t),(x',v',t'),t''\big) & \\ &= \frac{1}{t''}\Big(f^{[1]}\big((x,v,t)+t''(x',v',t')\big) - f^{[1]}(x,v,t)\Big) \\ &= \frac{f(x+t''x'+(t+t''t')(v+t''v')) - f(x+t''x')}{t''(t+t''t')} - \frac{f(x+tv)-f(x)}{t''t} \end{aligned}$$

$$U^{[2]} = \left\{ (x,v,t;x',v',t';t'') \,\middle|\, t,t',t'' \in \mathbb{K}, \substack{x\in U \\ x+tv\in U \\ x'+t''x'\in U \\ x+t''x'+(t+t''t')(v+t''v')\in U} \right\}$$

$$\hat{T}^2 f\big(x,v,t;x',v',t';t''\big) = \big(f(x), f^{[1]}(x,v,t), t; f^{[1]}(x,x',t''), f^{[2]}(x,\ldots,t''), t''\big)$$

Exercise: write up a formulae for $\hat{T}^3 U$, $f^{[3]}$ and $\hat{T}^3 f$ in a similar way! To be honest, I never tried to do this myself; it seems indeed fairly hard to "understand" the iterated slopes $f^{[n]}$. As far as I know, not much work has been done in this direction. Here is one classical result known as "Schwarz' theorem" – in the following, we use the *directional derivative* $\partial_v f(x) := Df(x)v = f^{[1]}(x,v,0)$ of a function f in direction v. Note that $\partial_v f$ is again a function $U \to W$.

Theorem 6.3. *Assume f is of class C^2. Then we have, for all $x \in U$ and $v, w \in V$,*

$$\partial_v(\partial_w f)(x) = \partial_w(\partial_v f)(x),$$

and the second order differential *at x is a symmetric bilinear map*

$$D^2 f(x) : V \times V \to W, \quad (v,w) \mapsto \partial_v(\partial_w f)(x).$$

Proof. The formula given above yields, for $t' = 0$, $v' = 0$ and $t = t'' \neq 0$,

$$f^{[2]}((x,v,t),(x',0,0),t) = \frac{f(x+tx'+tv) - f(x+tx') - f(x+tv) + f(x)}{t^2}.$$

Obviously, this is symmetric in v and x'. Thus, for all $t \neq 0$,

$$f^{[2]}((x,v,t),(x',0,0),t) = f^{[2]}((x,x',t),(v,0,0),t).$$

Since both sides are continuous functions of t, we have equality also for $t = 0$. But

$$\partial_v(\partial_w f)(x) = \partial_{(v,0,0)} f^{[1]}(x,w,0) = f^{[2]}((x,w,0),(v,0,0),0), \tag{6.1}$$

whence the first claim, i.e., symmetry of $D^2 f(x)$. Moreover, by Theorem 5.6, first differentials are linear maps, so this expression is linear with respect to v. By symmetry, it is then also linear with respect to w. □

6.2. **Notational conventions.** For understanding higher iterated structures, the choice of good *notation* is very important. First of all, the rôle of a variable shall be indicated by its *index*, and not by its *position* in a 7-tuple. For instance, instead of $\big(x,v,t;x',v',t';t''\big)$, we shall write $\big(v_0,v_1,v_2,v_{12},t_1,t_2,t_{12}\big)$. At each step, a new digit is adjoined to the indices:

Definition 6.4. *Fix an integer $j \in \mathbb{N}$. We denote by $U^{\{j\}}$ a copy ("of j-th generation") of $\hat{T}U$, and by $f^{\{j\}}$ a copy of $\hat{T}f$. Then we let $U^{\{1,2\}} := (U^{\{1\}})^{\{2\}}$, $f^{\{1,2\}} := (f^{\{1\}})^{\{2\}}$, and inductively (if f is of class C^n)*

$$U^{\{1,\ldots,n\}} := (U^{\{1,\ldots,n-1\}})^{\{n\}}, \qquad f^{\{1,\ldots,n\}} := (f^{\{1,\ldots,n-1\}})^{\{n\}}.$$

The notational principle is that, at the k-th step, new variables are added whose index is obtained from the preceding ones by joining everywhere the index k to the old index. Moreover, for sake of clarity, we first gather "space variables" v_α and then "time variables" t_β. To get even shorter notation, we use sans serif letters

$$\mathsf{n} := \{1,\ldots,n\}, \quad \mathsf{2} := \{1,2\}, \quad \mathsf{3} := \{1,2,3\}, \ldots$$

to denote finite sets, so f^{2} is another way of writing $\hat{T}^2 f$, and f^{n} for $\hat{T}^n f$.

For instance, instead of $\hat{T}^2 U$ we write

$$U^{\{1,2\}} = U^{\mathsf{2}} = \left\{ (v_0, v_1, v_2, v_{12}, t_1, t_2, t_{12}) \;\middle|\; \substack{v_0 \in U \\ v_0 + t_1 v_1 \in U \\ v_0 + t_2 v_2 \in U \\ v_0 + t_2 v_2 + (t_1 + t_2 t_{12})(v_1 + t_2 v_{12}) \in U} \right\}.$$

By "deriving a map $f : U \to W$ at level j" we mean passing from f to its corresponding groupoid morphism (where $\mathbb{K}_j$ is a copy of $\mathbb{K}$):

$$\begin{array}{ccc} U^{\{j\}} & \overset{f^{\{j\}}}{\to} & W^{\{j\}} \\ \downdownarrows & & \downdownarrows \\ U \times \mathbb{K}_j & \overset{f \times \mathrm{id}_{\mathbb{K}}}{\to} & W \times \mathbb{K}_j . \end{array} \tag{6.2}$$

Recall that this operation is functorial: $(g \circ f)^{\{j\}} = g^{\{j\}} \circ f^{\{j\}}$. By induction:

Theorem 6.5. *When g and f are composable and of class C^n, then $(g \circ f)^{\mathsf{n}} = g^{\mathsf{n}} \circ f^{\mathsf{n}}$.*

Example 1: linear maps. Assume $\alpha : V \to W$ is a (continuous) linear map. The slope of α is

$$\alpha^{[1]}(x, v, t) = \frac{\alpha(x + tv) - \alpha(x)}{t} = \frac{\alpha(x) + t\alpha(v) - \alpha(x)}{t} = \alpha(v),$$

whence is continous for $t = 0$ included, and hence $\alpha^{\{j\}}(v, v_j, t_j) = (\alpha(v), \alpha(v_j), t_j)$ is again a continuous linear map. By induction, α is C^n for all $n \in \mathbb{N}$.

Example 2: constant maps. Assume $f(x) = c$ is a constant map. Then $f^{[1]}(x, v, t) = 0$, hence $f^{\{j\}}(x, v, t) = (c, 0, t)$ (sum of a constant and a linear term), and hence f is C^∞.

Example 3: bilinear maps. Let $\beta : E \times F \to W$ be a (continuous) bilinear map. We compute the difference quotient map of β:

$$\frac{\beta(e + t_j e_j, f + t_j f_j) - \beta(e, f)}{t} = \beta(e_j, f) + \beta(e, f_j) + t_j \beta(e_j, f_j)$$

This is again continuous, for $t = 0$ included, and it follows that β is of class C^1 and

$$\beta^{\{j\}}((e, e_j), (f, f_j), t_j) = (\beta(e, f), \beta(e, f_j) + \beta(f, e_j) + t_j \beta(e_j, f_j), t_j).$$

For $t = 0$, we get the usual derivative. However, as a function of all variables, this is of degree 3, and computing higher slopes, we will get polynomials of higher and higher order. We don't need the explicit formulae for the moment.

Exercise. Show that, if g and f are C^n, then so is $f + g$, and that $(f + g)^{\mathsf{n}} = f^{\mathsf{n}} + g^{\mathsf{n}}$. Show that, if f is continuous polynomial (of degree k), then f is C^n for all n, and f^{n} is polynomial of degree $2k - 1$.

6.3. **The structure of $U^{\{1,2\}}$.** The set $U^{\{1,2\}}$ has a very significant structure: each derivation procedure $\{j\}$ produces a groupoid. If we apply it to another groupoid, the groupoid structure therefore is kind of doubled: we get a *double groupoid*. More precisely: the two projections $\pi_0, \pi_1 : U^{\{1\}} \rightrightarrows U \times \mathbb{K}_1$ are linear, resp., of degree 2, hence differentiable (by the preceding examples), and so we may "derive them at level 2". So we apply (6.2) to $f = \pi_0$ and $f = \pi_1$. This gives rise to two diagrams of the form (6.2) which together we denote by

$$\boxed{\begin{array}{ccc} U^{\{1,2\}} & \rightrightarrows & U^{\{2\}} \times \mathbb{K}_1 \times \mathbb{K}_{12} \\ \downarrow\downarrow & & \downarrow\downarrow \\ U^{\{1\}} \times \mathbb{K}_2 & \rightrightarrows & U \times \mathbb{K}_1 \times \mathbb{K}_2 \end{array}}$$

where $\mathbb{K}_{12}$ is another copy of $\mathbb{K}$. Here we have used

Lemma 6.6. *For a subset $A \subset V$, and $i < j$,*

$$(A \times \mathbb{K}_i)^{\{j\}} = A^{\{j\}} \times \mathbb{K}_i \times \mathbb{K}_{ij}.$$

Proof. Identify an element $((a_i, t_i), (a_{ij}, t_{ij}), t_j)$ with $((a_i, a_{ij}, t_j), t_i, t_{ij})$. □

Next we wish to compute explicit formulae for the maps appearing in this diagram. Using computations from Example 1 and 3, we get (target first, source second):

$$\boxed{\begin{array}{ccc} U^{\{1,2\}} & \rightrightarrows & U^{\{2\}} \times \mathbb{K}_1 \times \mathbb{K}_{12} \\ & & \\ (v_0, v_1, v_2, v_{12}, t_1, t_2, t_{12}) & \overset{\pi_1}{\mapsto} & (v_0 + t_1 v_1, v_2 + t_{12} v_1 + t_1 v_{12} + t_2 t_{12} v_{12}, t_1, t_2, t_{12}) \\ & \overset{\pi_0}{\mapsto} & (v_0, v_2, t_1, t_2, t_{12}) \\ \downarrow_{\pi_1} \downarrow_{\pi_0} & & \downarrow_{\pi_1} \downarrow_{\pi_0} \\ (v_0 + t_2 v_2, v_1 + t_2 v_{12}, t_1 + t_{12} t_2, t_2) & & (v_0 + t_2 v_2, t_1 + t_{12} t_2, t_2) \\ (v_0, v_1, t_1, t_2) & \overset{\pi_0}{\mapsto} & (v_0, t_1, t_2) \\ & \overset{\pi_1}{\mapsto} & (v_0 + t_1 v_1, t_1, t_2) \\ & & \\ U^{\{1\}} \times \mathbb{K}_2 & \rightrightarrows & U \times \mathbb{K}_1 \times \mathbb{K}_2 \end{array}}$$

Similarly there is a diagram of unit maps δ

$$\boxed{\begin{array}{ccc} U^{\{1,2\}} & \leftarrow & U^{\{2\}} \times \mathbb{K}_1 \times \mathbb{K}_{12} \\ \uparrow & & \uparrow \\ U^{\{1\}} \times \mathbb{K}_2 & \leftarrow & U \times \mathbb{K}_1 \times \mathbb{K}_2 \end{array}}$$

explicitly given by

$$\boxed{\begin{array}{ccc} (v_0, 0, v_2, t_1, t_2, t_{12}) & \hookleftarrow & (v_0, v_2, t_1, t_2, t_{12}) \\ & & \\ (v_0, v_1, 0, t_1, t_2, 0) & & (v_0, 0, t_1, t_2, 0) \\ \uparrow & & \uparrow \\ (v_0, v_1, t_1, t_2) & & (v_0, t_1, t_2) \\ & & \\ (v_0, 0, t_1, t_2) & \hookleftarrow & (v_0, t_1, t_2) \end{array}}$$

Finally we also can "derive" with respect to the index $\{2\}$ the product map

$$* = *_1 : U^{\{1\}} \times_{U \times \mathbb{K}_1} U^{\{1\}} \to U^{\{1\}}, \quad (v_0, v_1; v_0', v_1'; t_1) \mapsto (v_0', v_1 + v_1'; t_1).$$

This formula is *linear*, hence the derived map is easy to compute (since (t_1, t_2, t_{12}) is kept constant, we suppress it from the notation): whenever the matching conditions hold,

$$(v_0, v_1, v_2, v_{12}) *_1^{\{2\}} (v'_0, v'_1, v'_2, v_{12}) = (v'_0, v_1 + v'_1, v'_2, v_{12} + v'_{12}).$$

On the other hand, the groupoid $(U^{\{1,2\}}, U^{\{1\}} \times \mathbb{K}_2)$ comes with another product $*_2$ which we shall denote by $\bullet$ and given by, whenever defined,

$$(v_0, v_1, v_2, v_{12}) \bullet (v'_0, v'_1, v'_2, v'_{12}) = (v'_0, v'_1, v_2 + v'_2, v_{12} + v'_{12}).$$

Theorem 6.7. *The four sets and the maps appearing in the "square" diagrams*

$$\begin{array}{ccc} U^{\{1,2\}} & \rightrightarrows & U^{\{2\}} \times \mathbb{K}_1 \times \mathbb{K}_{12} \\ \downdownarrows & & \downdownarrows \\ U^{\{1\}} \times \mathbb{K}_2 & \rightrightarrows & U \times \mathbb{K}_1 \times \mathbb{K}_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} U^{\{1,2\}} & \leftarrow & U^{\{2\}} \times \mathbb{K}_1 \times \mathbb{K}_{12} \\ \uparrow & & \uparrow \\ U^{\{1\}} \times \mathbb{K}_2 & \leftarrow & U \times \mathbb{K}_1 \times \mathbb{K}_2 \end{array}$$

together with the two products $* := *_1^{\{2\}}$ and $\bullet = *_2$ on $U^{\{1,2\}}$, and with the product on $U^{\{1\}}$ and on $U^{\{2\}}$ from the preceding chapter, satisfy the following properties:

(1) each "edge data" of the square defines a groupoid,
(2) each pair of arrows ((π_1, π_1), etc.) defines a morphism of groupoids,
(3) the two products $*$ and $\bullet$ satisfy the following interchange law*: whenever all expressions are defined, then*

$$(a * b) \bullet (c * d) = (a \bullet c) * (b \bullet d).$$

Proof. (1) The vertical edges form groupoids: this is a direct application of Theorem 5.10. Idem for the lower horizontal edge (where just a variable t_2 is added in all formulae from Theorem 5.10). Concerning the upper horizontal edge, the data form a groupoid since we have applied the "functor of deriving with respect to $\{2\}$" to the data of a groupoid, and hence we get again the data of a groupoid. (See next chapter for more on this procedure.)

(2) For horizontal pairs of maps, this is a direct application of Theorem 5.12: indeed, the horizontal pairs of arrows are all of the form $(f^{\{2\}}, f \times \mathrm{id})$, where f is one of the structure maps π_1, etc., of the groupoid $U^{\{1\}}$. For the vertical pairs of maps, we use again that "deriving with respect to $\{2\}$" is a functor.

(3) A conceptual interpretation of the interchange law will be given in the next chapter. Let's give here a computational proof, based on the preceding formulae:

$$\begin{aligned} &((a_0, a_1, a_2, a_{12}) * (b_0, b_1, b_2, b_{12})) \bullet ((c_0, c_1, c_2, c_{12}) * (d_0, d_1, d_2, d_{12})) \\ =&(b_0, a_1 + b_1, b_2, a_{12} + b_{12}) \bullet (d_0, c_1 + d_1, d_2, c_{12} + d_{12}) \\ =&(d_0, c_1 + d_1, b_2 + d_2, a_{12} + b_{12} + c_{12} + d_{12}). \end{aligned}$$

On the other hand,

$$\begin{aligned} &((a_0, a_1, a_2, a_{12}) \bullet (c_0, c_1, c_2, c_{12})) * ((b_0, b_1, b_2, b_{12}) \bullet (d_0, d_1, d_2, d_{12})) \\ =&(c_0, c_1, a_2 + d_2, a_{12} + c_{12}) * (d_0, d_1, b_2 + d_2, b_{12} + d_{12}) \\ =&(d_0, c_1 + d_1, b_2 + d_2, a_{12} + c_{12} + b_{12} + d_{12}). \end{aligned}$$

Now, using *commutativity* of addition $+$ in V, we have

$$a_{12} + b_{12} + c_{12} + d_{12} = a_{12} + c_{12} + b_{12} + d_{12},$$

and the interchange law follows. □

Definition 6.8. *We say that the properties (1) – (3) define on U^2 the structure of a* double groupoid. *See next chapter for more on this!*

Theorem 6.9. *Fix $\mathbf{t} = (t_1, t_2, t_{12}) \in \mathbb{K}^3$ such that $t_{12} = 0$. Then the same formulae as above define a double groupoid*

$$\begin{array}{ccc} U_{\mathbf{t}}^{\{1,2\}} & \rightrightarrows & U_{t_2}^{\{2\}} \\ \downdownarrows & & \downdownarrows \\ U_{t_1}^{\{1\}} & \rightrightarrows & U \end{array} \qquad \text{and} \qquad \begin{array}{ccc} U_{\mathbf{t}}^{\{1,2\}} & \leftarrow & U_{t_2}^{\{2\}} \\ \uparrow & & \uparrow \\ U_{t_1}^{\{1\}} & \leftarrow & U \end{array}$$

having the additional property that it is edge-symmetric, *in the sense that all formulae remain unchanged under simultaneous exchange $t_1 \leftrightarrow t_2$ and $v_1 \leftrightarrow v_2$. In particular, when $t_1 = t_2 =: t$, the* flip *or* exchange *$v_1 \leftrightarrow v_2$ defines an automorphism of the double groupoid.*

Proof. If $t_{12} = 0$, then in the explicit formulae from the preceding pages, we have $t_1 + t_2 t_{12} = t_1$, and thus $\mathbf{t}$ becomes a "silent" variable (all maps are the identity in the $\mathbf{t}$-component). Hence all properties make sense for $\mathbf{t}$ fixed. Moreover, symmetry under $t_1 \leftrightarrow t_2$ and $v_1 \leftrightarrow v_2$ is seen by direct inspection of the formulae. □

Note. Edge symmetry is the "conceptual" version of Schwarz' theorem!

Example. If, in the preceding theorem, we let $t_1 = 0 = t_2$, then we get the *double tangent bundle* denoted by

$$\begin{array}{ccc} TTU = U_0^{\{1,2\}} & \rightrightarrows & TU = U_0^{\{2\}} \\ \downdownarrows & & \downdownarrows \\ TU = U_0^{\{1\}} & \rightrightarrows & U \end{array}$$

See next chapter for more on this!

Example. If in the preceding theorem we let $t_1 = 1 = t_2$, then we get the *double pair groupoid*: recall (Prop. 5.11) that $U_1 \cong U \times U$. Applying this twice, we get

$$\begin{array}{ccc} U \times U \times U \times U \cong U_{(1,1)}^{\{1,2\}} & \rightrightarrows & U \times U \cong U_1^{\{2\}} \\ \downdownarrows & & \downdownarrows \\ U \times U \cong U_1^{\{1\}} & \rightrightarrows & U \end{array}$$

See next chapter for more on this!

Summing up, for $(t_1, t_2) \to (0, 0)$, we get a continuous "contraction" of the double pair groupoid towards the double tangent groupoid. The gain of our apparently more complicated definition of differential, invoking 7 variables, is that the higher differential f^2 is highly structured and incorporates not only "infinitesimal" data, but also "finite" ones.

7. Seventh lecture: small n-fold cats and n-fold groupoids

The preceding chapter has furnished an example of a "higher order categorical structure". Historically, Charles Ehresmann was the first to define and use double groupoids, which then, together with other "higher gadgets", entered into other theories – see the n-lab or [Br2] for more information on the historical development. Let's look at some more exemples, before giving general definitions and returning to U^{n}.

7.1. **Example: the double pair groupoid.** Let M be a set. Recall that the *pair groupoid* $\mathsf{PG}^1 M$ of M is given by the two projections $M \times M \downdownarrows M$, unit section the diagonal map $M \to M \times M$ and product $(x,y) * (y,z) = (x,z)$ and inverse $(x,y)^{-1} = (y,x)$. Every map $f : M \to M'$ induces a groupoid morphism $\mathsf{PG}^1 f = (f \times f, f)$. Let $\mathsf{PG}^{\{k\}}$ be a copy of $\mathsf{PG}^{\{1\}}$, "of k-th generation". By definition, the *double pair groupoid* is

$$\mathsf{PG}^{\{1,2\}} M := \mathsf{PG}^{\{2\}}(\mathsf{PG}^{\{1\}} M).$$

This is a *pair of pairs* of sets, $((M^4, M^2), (M^2, M))$, and a projection $\pi_\sigma : M^2 \to M$ gives rise to a projection $\mathsf{PG}^{\{2\}} \pi_\sigma : M^4 \to M^2$. Together with projections $\pi_\sigma^{\{2\}}$, these fit into a diagram

$$\begin{array}{ccc} M^4 & \rightrightarrows & M^2 \\ \downdownarrows & & \downdownarrows \\ M^2 & \rightrightarrows & M \end{array}, \qquad \begin{array}{ccc} (x_0,x_1,x_2,x_{12}) & \mapsto & (x_0,x_2) \\ \downarrow & \pi_0 & \downarrow \\ (x_0,x_1) & \mapsto & x_0 \end{array} \qquad \begin{array}{ccc} (x_0,x_1,x_2,x_{12}) & \mapsto & (x_1,x_{12}) \\ \downarrow & \pi_1 & \downarrow \\ (x_2,x_{12}) & \mapsto & x_{12} \end{array}$$

and likewise we get a diagram of unit sections. Moreover, on M^4 there are two products given by (under the matching conditions)

$$(x_0,x_1,x_2,x_{12}) *_1^{\{2\}} (y_0,y_1,y_2,y_{12}) = (x_0,y_1,x_2,y_{12}),$$
$$(x_0,x_1,x_2,x_{12}) *_2 (y_0,y_1,y_2,y_{12}) = (x_0,x_1,y_2,y_{12}).$$

Applying $\mathsf{PG}^{\{3\}}$ to these things, we get $((M^8, M^4, M^4, M^4), (M^4, M^2, M^2, M))$, a "pair of pairs of pairs", which can be represented by a *cube* of sets and maps.

7.2. **Example: the double tangent bundle.** Let $M \subset V$ be a subset of a vector space (or a smooth manifold). Recall that the *tangent bundle* is the groupoid with $\pi_0 = \pi_1 =: \pi$ and projection

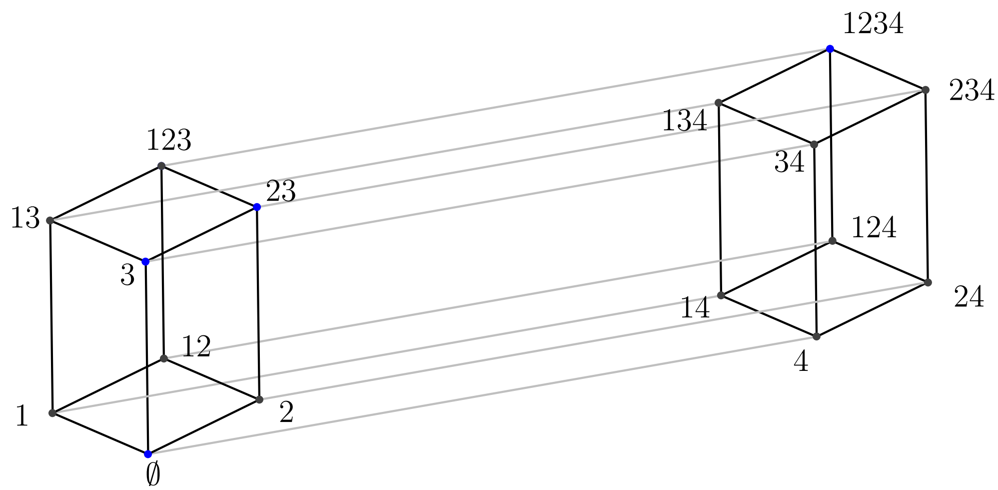
$$\pi : TM = M \times V \to M, \quad (x,v) \mapsto x$$

and groupoid law $(x,v) * (x,w) = (x, v+w)$ and unit section $\delta(x) = (x,0)$. If $f : M \to N$ is of class C^1, then $Tf : TM \to TN$ is a groupoid morphism. Hence we can "derive" π and get $T\pi : TTM \to TM$. As before, this gives rise to a diagram (since $\pi_0 = \pi_1$, one gets what is called a *double vector bundle*):

$$\begin{array}{ccc} TTM = M \times V \times V \times V & \to & TM = M \times V \\ \downarrow & & \downarrow \\ TM & \to & M \end{array}, \qquad \begin{array}{ccc} (x_0,v_1,v_2,v_{12}) & \mapsto & (x_0,v_2) \\ \downarrow & \pi & \downarrow \\ (x_0,v_1) & \mapsto & x_0 \end{array}$$

Iterating, we get a *hypercube of projections on the n-th order tangent bundle $T^n M$.*

7.3. **Notation: hypercubes.** The *n-dimensional hypercube* is given by the *vertex set* $\mathcal{P}(\{1,\ldots,n\})$. An *(oriented) edge* of the hypercube is a pair (β,α) of vertices such that α has exactly one element more than β. A *face* of the hypercube is given by two different edges $(\beta,\alpha),(\delta,\alpha)$ at a common vertex α. See here for more information on hypercubes, but mind that we are only interested here in its combinatorial structure, not by realisations as "solids" in $\mathbb{R}^n$. Exercise: show that the n-dimensional hypercube has 2^n vertices, $n\times 2^{n-1}$ edges and $n(n-1)\times 2^{n-3}$ faces. Here is the image of a 4-*cube*, also called *tesseract*. As the image suggests, an n-cube can be "generated" by taking an $n-1$-cube, translating it (so we produce a new copy of "next generation") and then joining corresponding vertices by "new" edges (in grey):



7.4. **Two definitions of higher gadgets.** We shall now give two definitions of *n-fold categories and groupoids*, and then prove that they are equivalent. The first is abstract and conceptual, and the second algebraic and computable. As to the first, it proceeds by induction – in fact, it is the short definition one finds on the n-lab; but we shall care more about "smallness" questions than done there:

Definition 7.1. *Let $\mathcal{T}$ be a concrete category. A* small cat of type $\mathcal{T}$ *is a set C carrying both a structure of type $\mathcal{T}$ and the structure of a small cat $(C_0,C_1,\pi_\sigma : C_1 \downdownarrows C_0,*)$, so $C=C_0\dot\cup C_1$, such that all structure maps of the small cat C are compatible with the structure $\mathcal{T}$. This includes the assumption that C_0,C_1 and $C_1\times_{C_0}C_1$ also carry structures of type $\mathcal{T}$, and hence it makes sense to require that π_σ,δ and $*$ are morphisms for $\mathcal{T}$. A* morphism between two small cats of type $\mathcal{T}$, *say C and C', is a map $f:C\to C'$ which is a functor (for the small cat-structures) and a structure-preserving map for $\mathcal{T}$. This defines a new concrete category $\underline{\mathcal{T}\text{-Cat}}$. In the same way the concrete category $\underline{\mathcal{T}\text{-Goid}}$ of* groupoids ot type $\mathcal{T}$ *is defined.*

For $n=1$, a small 1-fold cat *is a small cat, and this defines the concrete category $\underline{\mathrm{Cat}}_1$ of small cats. A* small n-fold cat *is a small cat C of type $\underline{\mathrm{Cat}}_{n-1}$, and a morphism of small n-fold cats is a functor $f:C\to C'$ which is also a morphism with respect to the $\underline{\mathrm{Cat}}_{n-1}$-structures on C and C'. This defines a concrete category $\underline{\mathrm{Cat}}_n$. In the same way, the concrete category $\underline{\mathrm{Goid}}_n$ of* n-fold groupoids *is defined.*

This definition is fairly abstract. To formulate it in more concrete terms, notice that a small cat is a *pair* of sets (plus structure); a small double cat hence a *pair of pairs of sets* (i.e., four sets), and so on: a small n-fold cat will be given by a *hypercube of sets*, plus structure:

Definition 7.2. *A small n-fold cat C is given by the following data:*

1. *a family of sets $(C_\alpha)_{\alpha\in\mathcal{P}(\mathsf{n})}$, indexed by vertices of the natural hypercube $\mathcal{P}(\mathsf{n})$,*
2. *a structure of small cat $(\pi^e : C_\alpha \downdownarrows C_\beta, *_e, z_e)$, called* edge-cat*, for every (oriented) edge $e = (\beta,\alpha;\mathsf{n})$ of the hypercube,*
3. *such that for every face $(\beta,\alpha),(\delta,\alpha)$ of the hypercube, the edge categories form a small double cat:*

$$\begin{array}{ccc} \alpha & \to & \beta \\ \downarrow & & \downarrow \\ \delta & \to & \gamma \end{array} \qquad \text{gives a small double cat} \qquad \begin{array}{ccc} C_\alpha & \rightrightarrows & C_\beta \\ \downdownarrows & & \downdownarrows \\ C_\delta & \rightrightarrows & C_\gamma \end{array}$$

which means that:

(a) *each pair of maps (such as (π_1,π_1), (δ,δ)) is a morphism of small cats,*
(b) *the two laws $* := *^{\beta,\alpha}$ and $\bullet := *^{\delta,\alpha}$ on the vertex set C_α satisfy the* interchange law*:*

$$(a * b)\bullet(c*d) = (a\bullet c)*(b\bullet d).$$

To shorten notation, we often write $(C_\alpha)_{\alpha\in\mathcal{P}(\mathsf{n})}$ to denote the whole structure. Morphisms of small n-fold cats *are families of maps $f_{\alpha;\mathsf{n}} : C_{\alpha;\mathsf{n}} \to C'_{\alpha;\mathsf{n}}$ such that, for each edge (β,α), the pair (f_β, f_α), is a morphism of small cats. If all edge categories are groupoids, then these data define an n-*fold groupoid*.*

Theorem 7.3. *Both preceding definitions are equivalent.*

Proof. For experts in category theory, this may appear "trivial": see the very short proof in [FP10], Prop. 2.5. However, since I do not claim to be an expert, I have written up a fairly detailed proof that one may find here (Theorem B.2). One proceeds by induction. For $n = 1$, the claim is obviously true. Now assume it holds for $n \in \mathbb{N}$. In order to prove that it holds also on the level of $n+1$, assume first that C is a small $n+1$-fold cat in the sense of the first definition, that is $C = (C_0, C_1)$ with C_0 and C_1 small n-fold cats. By the induction hypotheses, both C_0 and C_1 are given by families indexed by an n-cube. Putting them together, we get a family indexed by an $n+1$-cube (see the illustration). Conditions (2) and (3a) now follow fairly directly from the definition. It remains to show that the interchage laws (3b) must hold. This comes from the fact that the product map

$$A : C_1 \times_{C_0} C_1 \to C_1, \quad (a,b) \mapsto a*b$$

is also required to be a morphism (with respect to the product $\bullet$), which means that

$$A((a,b)\bullet(c,d)) = A(a,b)\bullet A(c,d).$$

But $(a,b)\bullet(c,d) = (a\bullet c, b\bullet d)$, whence $A(a,b)\bullet A(c,d) = A(a,b)\bullet A(c,d)$, which is precisely the interchange law. This proves one direction. But all arguments can be reversed, so that also the second definition implies the first. $\square$

Remark/exercise. Let $(G,\cdot)$ be a group. Show that the group law: $\cdot : G\times G \to G$ is a group morphism, if and only if, the group is commutative (e.g., $G=(V,+)$ a vector space!). Note that this is the precise analog of the argument given above, in case where both laws $*$ and $\bullet$ agree. Compare also with the proof of Part (3) p. 33.

7.5. **Examples.** The n-fold pair groupoid of a set M is defined inductively by

$$\mathsf{PG}^{\mathsf{n}}(M) := \mathsf{PG}^{\{n\}}(\mathsf{PG}^{\{1,\ldots,n-1\}}(M)).$$

Theorem 7.4. *The n-fold pair groupoid is an n-fold groupoid.*

Proof. The main point is that the symbol $\mathsf{Q} := \mathsf{PG}$ is a "groupoid-rule": a functor from the ccat <u>Set</u> to the ccat <u>Goid</u>. Thus applying $\mathsf{PG}^{\{2\}}$ to anything gives a groupoid; but if the anything is itself a groupoid, then since PG is a functor, the groupoid properties are transformed into properties of the same kind, so we get a groupoid. Taken together, these structures define a groupoiod of type <u>Goid</u>. Similarly for $n>2$ (see also Appendix C, loc. cit.). □

Exercise. For this example it is fairly easy to give explicit descriptions of the "vertex sets" corresponding to a vertex α, and of the "edge projections" corresponding to an edge (β,α). See Appendix C, loc. cit..

Theorem 7.5. *Let M be an (open) subset of a real vector space V, or a smooth manifold. The* n-fold tangent bundle *defined by $T^nM := T(T^{n-1}M)$ is an n-fold groupoid (where everywhere $\pi_0=\pi_1$).*

Proof. Same arguments as above. □

The double tangent bundle is frequently used in differential geometry. The higher tangent bundles are useful too, but much less exploited. Recall that, for $n=1$, the "tangent groupoid", defined as by Alain Connes, or as $U^{\{1\}}$ (lecture 5), gives a sort of interpolation between the two preceding examples. This remains true for higher structures: there is an n-fold groupoid interpolating between $\mathsf{PG}^{\mathsf{n}}U$ and $T^{\mathsf{n}}U$:

Theorem 7.6. *Let V be a $\mathbb{K}$-vector space and $U\subset V$ a set. Then $U^{\{n\}}$, as defined in the preceding chapter, is an n-fold groupoid.*

Proof. The proof goes as above: the symbol $\mathsf{Q}=\mathsf{Q}^{\{j\}}$ defined by

$$U \mapsto (U^{\{j\}}, U\times \mathbb{K}_j), \qquad f\mapsto (f^{\{j\}}, f\times \mathrm{id})$$

is a groupoid rule, hence iterating it we get an n-fold groupoid. There is just one slightly technical point: one has to pay attention that Q is compatible with domains of definitions of products $*$. To ensure this, one has to establish the property

$$\mathsf{Q}(E\times_F G) = \mathsf{Q}E \times_{\mathsf{Q}F} \mathsf{Q}G.$$

The proof is not really difficult; for details we refer to Appendix E, loc. cit. □

The n-fold groupoid U^{n} contains, in a sense, the whole information of (local) differential geometry at order n. It contains even more information then what is usually used, thus allowing for much greater generality. On the other hand, due to this rich information contents, studying the structure of U^{n} is a bit complicated. I will try, in the following lectures, to explain more about the motivation, and about topics related to this.

8. Eighth lecture: Concepts of infinitesimal geometry

What nowadays is called *differential geometry*, started with Gauss and Riemann, who sort of combined "pure", or synthetic, Euclide-style, geometry, with Descartes-style analytic geometry, and in particular with Newton's and Leibniz's differential and integral calculus. Until today, the interplay between between "analytic" and "synthetic" ways of thinking is a major feature of geometry. In these lectures, I would like to discuss some aspects of differential, or infinitesimal, geometry (without pretention of being exhaustive, or even of being serious):

(1) What is a "manifold", or a "space"? To be more precise, seen with a physicist's eye, the question can be put like this: *what is an "invariant" object of geometry, and what is merely dependent on our choice of coordinate system? what does it mean that something is "independent of coordinates"?* This question is fundamental for the very structure of physics, and even today it is not so easy to answer it. Elie Cartan, in "Leçons sur la géométrie des espaces de Riemann" (Gauthier-Villars 1928), writes: *"La notion générale de variété est assez difficile à définir avec précision. Une surface donne l'idée d'une variété à deux dimensions..."* Indeed, it is useful to start by looking at the surface of our terrestral sphere, and by looking at the ancient topic of representing it by maps. However, be aware that (in contrast to the sphere) we cannot "see from the outside" the four-dimensional universe where we are living, and thus it is much harder to say what kind of "space" it really is.

(2) What does the term "infinitesimal" mean? Leibniz, Euler, Fermat, and many other pioneers of "infinitesimal calculus", used this term as if "infinitely small objects" really existed: e.g., in a letter from to Tschirnhaus, Leibniz writes: *"Le calcul infinitésimal consiste à considérer en plus des caractères x, y, etc., les infiniment petits $d\overline{x}$, $d\overline{y}$, et des choses analogues."* Physics people still think in these terms, whereas mathematics people usually say that infinitesimals do not really exist: only *limits* really exist. But that's not the whole story... Let's start with this:

8.1. **The infinitely small.**

It's not easy to guess how mathematicians really thought, over 200 years ago. For instance, they certainly have used two different aspects of "infinitely small objects", whithout, however, clearly distinguishing them:

(a) *Euler* has used infinitesimal quantities like dx, in the sense that it is a non-zero number, whose inverse $\omega := \frac{1}{dx}$ is *infinitely big* (that is, *bigger than any natural number* $n \in \mathbb{N}$). We speak of *invertible infinitesimals*.

(b) *Fermat* used "very small" quantities ε, which are so small that ε^2 can be neglected – that is, in certain computations one may simply put $\varepsilon^2 = 0$. We speak of *nilpotent infinitesimals*. The differential of f is characterized by

$$f(x + \varepsilon v) = f(x) + \varepsilon df(x)v \tag{F}$$

since all higher order terms in the Taylor expansion "are zero". For instance, if $f(x) = x^3$, to show that $f'(x) = 3x^2$, i.e., $df(x)v = 3x^2v$, proceed like this:

$$f(x + \varepsilon v) = (x + \varepsilon v)^3 = x^3 + 3x^2\varepsilon v + 3x(\varepsilon v)^2 + (\varepsilon v)^3 = x^3 + \varepsilon(3x^2)v.$$

After a period of despise, infinitesimals have been rehabilateed in the 20-th century: there are rigourous theories using them. Since an invertible element can never be nilpotent ($ab = 1$ implies $a^k \neq 0$, for all $k \in \mathbb{N}$), this lead to two very different approaches:

(a) Euler's use is justified by non-standard analysis: one constructs a field $^*\mathbb{R}$ which contains $\mathbb{R}$, along with elements like ω (it is a *non-archimedian, ordered field extension of* $\mathbb{R}$). This is an interesting theory, but we won't study it in these lectures – see Chapter 12 of [Nu] for a good account.

(b) A quite influential paper [We53] by André Weil was the starting point for using nilpotent infinitesimals in mathematics: on the one hand, they entered into algebraic geometry (work of Alexander Grothendieck), and on the other hand, via the so-called *Weil functors*, they entered also into differential geometry. The most elaborated theory, combining all these things, is a categorical approach, using *model and topos theory*, called (by Anders Kock and others) *Synthetic Differential Geometry* (SDG). Among other things, Fermat's computations there are fully justified.

The following is inspired by SDG, though entirely independent of it.

8.2. **Dual numbers.** Everybody knows that the complex numbers $\mathbb{C}$ can be constructed from $\mathbb{R}$ by defining $\mathbb{C} = \mathbb{R}^2 = \mathbb{R} \oplus i\mathbb{R}$, with addition as in $\mathbb{R}^2$ and multiplication by respecting the rule $i^2 = -1$. It is less well-known that, replacing $i^2 = -1$ by other rules, one still obtains nice associative products on $\mathbb{R}^2$, which are in general no longer *fields*, but still *commutative rings*. So let's replace $i^2 = -1$ by $\varepsilon^2 = 0$:

Proposition 8.1. *Let* $\mathbb{K} = \mathbb{R}$*, or any other field you like. Then the set* $\mathbb{K}[\varepsilon] := \mathbb{K} \times \mathbb{K}$*, with addition and multiplication defined by*

$$(a, b) + (a', b') := (a + a'; b + b'),$$
$$(a, b) \cdot (a', b') := (aa', ab' + a'b)$$

is a commutative ring with unit $1 = (1, 0)$*. The element* $\varepsilon := (0, 1)$ *satisfies* $\varepsilon^2 = 0$.

Proof. We give four proofs:

– first proof: direct computation – check associativity, etc. (straightforward),

– second proof ("algèbre 1"): this is the quotient ring $\mathbb{K}[X]/(X^2)$ where $\varepsilon := [X]$,

– third proof: identifying (a, b) with $\left(\begin{smallmatrix} a & 0 \\ b & a \end{smallmatrix}\right)$, check that it is a subring of $M(2, 2; \mathbb{K})$,

– fourth proof: recall from Def. 5.8 the tangent map Tf of a map f. Now for f take the product map $m : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $(x, y) \mapsto xy$, and compute Tm:

$$Tm((x, v), (y, v)) = \Big(xy, \Big[\frac{(x + tv)(y + tw) - xy}{t}\Big]_{t=0}\Big) = \big(xy, xw + yv\big)$$

Similarly, for the addition map $a : \mathbb{K}\times\mathbb{K} \to \mathbb{K}$, $(x, y) \mapsto x+y$, we get $Ta((x, v), (y, w)) = (x + y, v + w)$. Thus $\mathbb{K}[\varepsilon] = T\mathbb{K}$, equipped with Tm and Ta. Properties such as commutativity and associativity can be expressed by diagrams of maps, such as

$$\begin{array}{ccc} \mathbb{K}^3 & \overset{\mathrm{id}\times m}{\to} & \mathbb{K}^2 \\ m \times \mathrm{id} \downarrow & & \downarrow m, \\ \mathbb{K}^2 & \overset{m}{\to} & \mathbb{K} \end{array} \qquad \begin{array}{ccc} (x, y, z) & \mapsto & (x, yz) \\ \downarrow & & \downarrow \\ (xy, z) & \to & xyz \end{array}$$

Now recall that T is *functor*: applying T to such a diagram yields a diagram of the same form, but everywhere $\mathbb{K}$ is replaced by $T\mathbb{K}$, m by Tm and $\mathrm{id}_\mathbb{K}$ by $\mathrm{id}_{T\mathbb{K}}$.

We conclude that Ta, Tm are again commutative and associative, etc., and hence $(T\mathbb{K}, Ta, Tm)$ is again a ring. (Rk.: the property of being a *field* is not expressed by such diagrams, and hence $T\mathbb{K}$ is not a field, even if $\mathbb{K}$ is one.) ☐

Definition 8.2. *For any commutative ring $\mathbb{K}$, the ring $T\mathbb{K} = \mathbb{K}[\varepsilon]$ will be called the* tangent ring*, or* ring of dual numbers over $\mathbb{K}$*. We identify (a, b) with $a + \varepsilon b$.*

The last proof exhibits a general method: if A is an algebraic structure such as a ring, then its tangent object TA is a structure of the same kind. Recall that in a ring A, an element a is called *invertible* if there exists $a' \in A$ with $aa' = 1 = a'a$. Let $A^\times \subset A$ be the set of invertible elements.

Proposition 8.3. *If $\mathbb{K}$ is a ring, then the map*

$$\pi : T\mathbb{K} \to \mathbb{K}, \quad a + \varepsilon b \mapsto a$$

is a ring homomorphism. Moreover,

$$(T\mathbb{K})^\times = \{a + \varepsilon b \mid a \in \mathbb{K}^\times\} = T(\mathbb{K}^\times),$$

and then $(a + \varepsilon b)^{-1} = a^{-1} - \varepsilon a^{-2} b$. In particular, $(1 + \varepsilon b)^{-1} = 1 - \varepsilon b$.

Proof. $\pi((a + \varepsilon b)(a' + \varepsilon b')) = aa' = \pi(a + \varepsilon b)\pi(a' + \varepsilon b')$, and for inversion, several proofs possible: direct computation, or identifying $a+\varepsilon b$ with $\left(\begin{smallmatrix} a & 0 \\ b & a \end{smallmatrix}\right)$, this is Cramer's rule. Most conceptually: give a proof in the spirit of the fourth proof above! ☐

Corollary 8.4. *If $\mathbb{K}$ is a field, then $(T\mathbb{K})^\times = \mathbb{K}^2 \setminus L$, where L is the line $a = 0$. In particular, for $\mathbb{K} = \mathbb{R}$, the set of invertible elements is* open and dense *in $\mathbb{R}[\varepsilon]$.*

Proposition 8.5. *If V is a vector space over a field $\mathbb{K}$, then $TV = V \times V$, with addition and multiplication by elements from $T\mathbb{K}$ given by*

$$(x, v) + (x', v') := (x + x', v + v'),$$
$$(a + \varepsilon b) \cdot (x, v) := (ax, av + bx)$$

*is a $T\mathbb{K}$-*module.

Proof. Recall that a module over a ring is defined in the same way as a vector space, just by replacing the “base field” by a “base ring”. To check commutativity, associativity, distributivity, etc., one uses the same arguments as in the preceding two proofs. Formally, the computations are the same as before, by writing $x + \varepsilon v$ instead of (x, v). ☐

Again, instead of (x, v) we may write $x + \varepsilon v$. All computations are “as usual”, and just mind the relation $\varepsilon^2 = 0$. For instance,

$$(r + \varepsilon s)(x + \varepsilon v) = rx + \varepsilon(rv + sx).$$

The main result justifying the use of such notation, and of nilpotent infinitesimals in general, is the following:

Theorem 8.6. *Assume $\mathbb{K}$ is a topological ring whose set of units $\mathbb{K}^\times$ is open and dense in $\mathbb{K}$, and V, W topological $\mathbb{K}$-modules and U open in V. If $f : U \to W$ is of class C^2 over the ring $\mathbb{K}$, then the tangent map $Tf : TU \to TW$ is of class C^1* over the ring *$T\mathbb{K}$.*

Proof. Saying that f is of class C^1 over $\mathbb{K}$ can be expressed by a certain diagram of maps. Applying the tangent functor T to this diagram yields a diagram of the same form, where $\mathbb{K}$ is replaced by $T\mathbb{K}$, U, V, W by TU, TV, TW and f by Tf. But this means that Tf is of class C^1 over $T\mathbb{K}$. (See paper [19] on my homepage, Theorem 6.2, or book *Calcul différentiel topologique élémentaire*, Exercises B, for full details and further comments.) □

The theorem implies that writing expressions like $x + \varepsilon v$ really has an "intrinsic meaning". Using this notation,

$$Tf(x + \varepsilon v) = f(x) + \varepsilon df(x)v \tag{F'}$$

which is very close to Fermat's Formula (F) $f(x + \varepsilon v) = f(x) + \varepsilon df(x)v$.

8.3. **Second and higher order tangent rings $T^n\mathbb{K}$.** As in the previous lectures, things start to get really interesting when *iterating* them. The *second order tangent ring* is

$$TT\mathbb{K} = (\mathbb{K}[\varepsilon_1])[\varepsilon_2] = \mathbb{K} \oplus \varepsilon_1\mathbb{K} \oplus \varepsilon_2\mathbb{K} \oplus \varepsilon_1\varepsilon_2\mathbb{K}, \tag{8.1}$$

where $\varepsilon_1, \varepsilon_2$ are elements with $\varepsilon_i^2 = 0$ and $\varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_1$. Computations are made using these rules. For instance, let $\delta := \varepsilon_1 + \varepsilon_2 \in TT\mathbb{K}$. Then

$$\delta^2 = (\varepsilon_1 + \varepsilon_2)^2 = 2\varepsilon_1\varepsilon_2, \quad \delta^3 = 2\varepsilon_1\varepsilon_2(\varepsilon_1 + \varepsilon_2) = 0. \tag{8.2}$$

Likewise, all elements in $\varepsilon_1\mathbb{K} \oplus \varepsilon_2\mathbb{K} \oplus \varepsilon_1\varepsilon_2\mathbb{K}$ are *nilpotent of order* 3.

Definition 8.7. *The* n-th order tangent ring $T^n\mathbb{K}$ *is inductively defined by* $T^n\mathbb{K} := T(T^{n-1}\mathbb{K})$. *It can be identified with*

$$\mathbb{K}[\varepsilon_1, \ldots, \varepsilon_n] := (\mathbb{K}[\varepsilon_1, \ldots, \varepsilon_{n-1}])[\varepsilon_n],$$

where all ε_i *satisfy* $(\varepsilon_i)^2 = 0$. *For* $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subset \mathsf{n}$, *we write*

$$\varepsilon_\alpha := \varepsilon_{\alpha_1} \cdot \varepsilon_{\alpha_2} \cdots \varepsilon_{\alpha_k}.$$

Theorem 8.8. *The n-th order tangent ring $T^n\mathbb{K}$ is a ring of dimension 2^n over $\mathbb{K}$, given by a $\mathbb{K}$-basis of elements $(\varepsilon_\alpha)_{\alpha \in \mathcal{P}(\mathsf{n})}$, which are multiplied by the rule*

$$\varepsilon_\alpha \cdot \varepsilon_\beta = \begin{cases} 0 & \text{if} \quad \alpha \cap \beta \neq \emptyset, \\ \varepsilon_{\alpha\cup\beta} & \text{if} \quad \alpha \cap \beta = \emptyset. \end{cases}$$

Using these rules, computations are quite easy. For instance, the reader may compute the powers δ^k of the element

$$\delta := \varepsilon_1 + \ldots + \varepsilon_n.$$

One gets $\delta^2 = 2\sum_{(i,j):i<j} \varepsilon_i\varepsilon_j$, and so on. Finally, $\delta^n = n!\varepsilon_1 \cdots \varepsilon_n$ and $\delta^{n+1} = 0$. In fact, all elements of the form $\sum_{\alpha \neq \emptyset} \varepsilon_\alpha v_\alpha$ are nilpotent of order at most $n + 1$. By induction:

Theorem 8.9. *Let V be a $\mathbb{K}$-module. Then T^nV is a $T^n\mathbb{K}$-module.*

The reader may note that we are back in the context of Lecture 7: everything is structured by the n-hypercube $\mathcal{P}(\mathsf{n})$. Indeed, we have here an n-fold groupoid with “source = target”. More details can be found in reference [19] of my homepage, where the whole theory of local differential geometry is based on the algebra of $T^n\mathbb{K}$. I just mention here the formula for higher order tangent maps, iterating (F'): let's write elements of T^nU in the form $\mathbf{v} = \sum_{\alpha\in\mathcal{P}(\mathsf{n})} \varepsilon_\alpha v_\alpha$ with $x := v_\emptyset \in U$ and all other $v_\alpha \in V$. Applying (F') twice, we get

$$TTf(x + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_{12} v_{12}) =$$
$$= f(x) + \varepsilon_1 df(x)v_1 + \varepsilon_2 df(x)v_2 + \varepsilon_{12}\big(df(x)v_{12} + d^2f(x)(v_1, v_2)\big).$$

In loc. cit. Theorem 7.5., this is generalized for arbitrary n: the sum carries over all partitions Λ of α

$$T^n f(\sum_{\alpha\in\mathcal{P}(\mathsf{n})} \varepsilon_\alpha v_\alpha) = \sum_{\alpha\in\mathcal{P}(\mathsf{n})} \varepsilon_\alpha \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\Lambda \\ \alpha=\Lambda_1\cup\ldots\cup\Lambda_\ell}} d^\ell f(v_\emptyset)(v_{\Lambda_1}, \ldots, v_{\Lambda_\ell}).$$

To close this lecture, I would like to explain that the algebra of $T^n\mathbb{K}$ can be used to recover, in an elementary way, one of the central concepts of SDG:

8.4. **The infinitesimal neighborhood relations.** As A. Kock writes in [Ko10], most of the “synthetic infinitesimal geometry” is encoded in the structure of *k-th order infinitesimal neighborhoods*. In usual analysis courses, we learn the “basic idea of differential calculus”: differentiating at x means to make a kind of “infintely strong zoom” near x, and under this zoom, everything becomes linear: the differerential $df(x)$ is the linear map describing f when zooming at x, and the tangent space T_xM is the linear space describing the “first order infinitesimal neighborhood of x”. However, it is rarely taught what happens when we iterate this procedure: how does the second differential enter into the picture? Zooming once more around x does not give anything new, since a linear space and a linear map, under a zoom, remain the same (the differential of a linear map is the same map)! To understand things, one needs the notion of *second order infinitesimal neighborhood*, which consists of all the union of all first order neighborhoods of all points from the first order neighborhood T_xM. To explain this, first, a very general definition:

Definition 8.10. *A* neighborhood relation *on a set M is a relation $N \subset (M \times M)$ which is* reflexive and symmetric*. We often write $x \sim y$ instead of $(x, y) \in N$ (although the relation is not supposed to be transitive!). The set*

$$U_x := \{y \in M \mid x \sim y\}$$

*is called the N-*neighborhood of x*. Reflexivity means that $x \in U_x$, and symmetry that: $y \in U_x \Leftrightarrow x \in U_y$.*

Example 1. On $M = \mathbb{R}^n$ (or any other metric space, with distance d), define $x \sim y$ iff $d(x, y) < 1$. Then $U_x = B_1(x)$ is the ball centered at x and with radius 1. This relation is not transitive (in general) – make a drawing!

Example 2. On $M = \mathbb{K}^n$, define $x \sim y$ iff $\exists j \in \mathsf{n} : x_j = y_j$. For $n = 2$, U_x is the union of the vertical and the horizontal line passing through x. For general n, U_x

is the union of all n coordinate hyperplanes passing through x. This relation is not transitive. Make a drawing!

Example 3. On $M = \mathbb{K}^n$, define $x \sim y$ iff $\exists j \in \mathsf{n} : \forall i \neq j : x_i = y_i$. For $n = 2$, this is the same as in Example 2, but not for $n > 2$, where U_x is the union of all coordinate axes passing through x. Make a drawing (for $n = 3$)...

Definition 8.11. *Given a "first order" neighborhood relation $\sim_1 = N_1$, define the "second and higher order neighborhood relations" $\sim_n = N_n$ by*

$$x \sim_2 y \iff \exists z \in M : x \sim_1 z, z \sim_1 y,$$
$$x \sim_n y \iff \exists z \in M : x \sim_{n-1} z, z \sim_1 y.$$

Note that these are again neighborhood relations, that $N_1 \subset N_2 \subset \ldots$, and that

$$x \sim_k y,\ y \sim_\ell z \ \Rightarrow\ x \sim_{k+\ell} z.$$

Example 0. If N happens to be an equivalence relation,then $\sim_n = \sim_1$, for all n.

Example 1. In Example 1 above, with $M = \mathbb{R}^n$, we have $x \sim_k y$ iff $d(x, y) < k$. Exercise: prove this! (Hint: one should not make the error to believe that this holds for any metric space. For instance, if the metric space is ultrametric, then show that $\sim_1$ is an equivalence relation, whence $\sim_n = \sim_1$!)

Example 2. In Example 2, $\sim_2$ is the "all-relation". Indeed, if $x, y \in \mathbb{K}^n$, let $z := (x_1, y_2, \ldots, y_n)$, then $x \sim_1 z$ and $z \sim_1 y$.

Example 3. For N as in Example 3 above, for $n = 3$, $\sim_2$ is not the "all-relation", but $\sim_3$ is. Prove this! Generalize for any $n \in \mathbb{N}$.

The neighborhood relation of infinitesimal geometry will be similar as in Example 3, but takes account of the "hypercube structure". For $n = 2$, the following definition says that $\mathbf{v} = v_0 + \varepsilon_1 v_1 + \varepsilon_2 v_2 + \varepsilon_{12} v_{12}$ and $\mathbf{w} = w_0 + \varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_{12} w_{12}$ are *second order infinitesimal neighbors in TTV* iff $v_0 = w_0$, and are *first order infinitesimal neighbors in TTV* iff $w_0 = v_v$ and $[w_1 = v_1$ or $w_2 = v_2]$. For fixed w_0, this can be represented by taking $\varepsilon_1 V, \varepsilon_2 V, \varepsilon_{12} V$ as three axes, and then the first order neighborhood is the union of two planes intersecting along a parallel of the third axis $\varepsilon_{12} V$.

Definition 8.12. *For $\mathbf{v} = \sum_{\alpha \in \mathcal{P}(\mathsf{n})} \varepsilon_\alpha v_\alpha$ and $\mathbf{w} = \sum_{\alpha \in \mathcal{P}(\mathsf{n})} \varepsilon_\alpha w_\alpha \in T^n U$ define*

$$\mathbf{v} \sim_1 \mathbf{w} \ \text{iff}\ \exists j \in \mathsf{n} : \forall \beta \in \mathcal{P}(\mathsf{n} \setminus \{j\}) : v_\beta = w_\beta.$$

In other words, there exists an index j such $\mathbf{v}$ and $\mathbf{w}$ have same image under the j-th of the n projections $T^n U \to T^{n-1} U$.

This is a neighborhood relation, called the *first order infinitesimal neighborhood relation*: reflexivity and symmetry are obvious. Note that the relation is not transitive (because if $x \sim_1 y$ and $y \sim_1 z$, the j need not be the same for both).

Exercise: describe the k-th order neighborhood relation $\sim_k$, and show that $\sim_n$ is an equivalence relation, but $\sim_k$ for $k < n$ is not.

One of the main goals of the synthetic theory is to give a geometric description of notions such as affine connections and parallel transport – for more on this, see Kock's book.

9. Ninth lecture: What is a manifold?

In this lecture I would like to give some kind of reply to Elie Cartan's dictum "La notion générale de variété est difficile à définir avec précision" – of course, there is a by now standard definition of manifold, but it took very long to arrive at this point. The "difficulty" appears constantly in the dialog between mathematicians and physicists: let us take the example of the sphere $M = S^2$, which may be identified with the earth's surface. This surface can be represented by an *atlas*, that is, a *collection of charts* covering the whole surface. A *chart* is a bijection of some part of the earth's surface with some part of $V = \mathbb{R}^2$ (our drawing plane). On chart overlaps, coordinate data tell us how to translate the image from one chart into another. This, roughly, is the modern definition of manifold.

Mathematicians usually forget that it took a long time for humans to realize that we are living on the surface of a globe, and not on a flat plane stretching infinitely far into all directions. But as long as humans were not able to travel into space and to *see the globe from outside*, this fact could only be explained indirectly, by *measurements*, by *comparing different observations*, and finally proposing a *better model* which leads to better predictions and explanations. In the same way, Einstein proposed a better model for the whole universe – but so far nobody has been able to "see it from outside"... That's the way physics proceeds, but not mathematics!

In the same way, the notion of manifold has two sides: for a physicist, the "real world" is what you can read in the charts, the measurements, the coordinate descriptions, the observations by various observers – let's call this the "gluing data". From this point of view, the "space M" definitely is a kind of idealization – the "ideal world" that mathematicians prefer but that you cannot see directly. Mathematically, the "ideal world" and the "real world" are equivalent – this is a theorem, but as far as I know, its proof has never been written up in a completely formalized way. I have tried to formalize it as far as possible – see here. Comments and corrections are welcome!

As one may see, *groupoids* and *double categories* naturally enter the picture when one wants to describe also *morphisms between manifolds* (= smooth maps) in a formal way. This may help to better understand, formally, the distinction between active and passive transformation, important in physics and engeneering.

10. Tenth lecture: Excursion into the non-associative world

The most natural property of product maps is *associativity*, as we have seen in preceding lectures. Non-associative products often have an "exotic", or "exceptional" flavor. Recall:

Definition 10.1. *A* product *on a set M is a map $M \times M \to M$, $(a,b) \mapsto a \star b$ (or any other notation: $a \cdot b$, $[a,b]$ or ab instead of $a \star b$). A product may have, or not, some of the following properties:*

1. *it may be* commutative, associative*, or satisfy other "identies"; for instance:*
2. *a product is called* alternative *if it satisfies, for all $a, b \in M$,*
$$a(ab) = (aa)b, \qquad (ab)b = a(bb)\,,$$
3. *it may have a* neutral element *e, that is, $\forall a \in M$: $a \star e = a = e \star a$;*
4. *it may be a* latin square*, or* quasigroup*: $\forall a, b \in M$, the equation $x \star a = b$ has a unique solution $x \in M$, and so has the equation $a \star y = b$,*
5. *it may be a* loop *[fr:* boucle*], that is, a quasigroup with a neutral element e.*

Often, people are interested in *bilinear* products:

Definition 10.2. *An* algebra *(over a field $\mathbb{K}$) is a $\mathbb{K}$-vector space $\mathbb{A}$ together with a* bilinear *product map $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$. It is called* commutative (associative, alternative,...) *if the product has the corresponding property. It is called a* division algebra *if the set $\mathbb{A} \setminus \{0\}$ is a quasigroup. (Note that always $0 \cdot a = 0 = a \cdot 0$, by bilinearity.)*

Two questions:

1. Obviously, any associative algebra is alternative. Is the converse true? In other words, give an example of an alternative, non-associative algebra!
2. Obviously, fields like $\mathbb{R}$ and $\mathbb{C}$ are division algebras. Are there other division algebras, that are not fields? Give an example!

It is not quite trivial to answer these questions, and, surprisingly, they are closely related to each other. The most famous example, both for an alternative and for a division algebra that is not a field, is the algebra of octonions. In this lecture, we explain how to construct them. The octonions have many interesting relations with geometry, see [Ba02, CS03, Fau14, Nu].

10.1. The Cayley-Dickson construction.

The Cayley-Dickson construction is the "doubling process" to construct

1. the complex numbers $\mathbb{C}$ from $\mathbb{R}$ by $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, $i^2 = -1$,
2. the quaternions $\mathbb{H}$ from $\mathbb{C}$ by $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$, $j^2 = -1$,
3. the octonions $\mathbb{O}$ from $\mathbb{H}$ by $\mathbb{O} = \mathbb{H} \oplus f\mathbb{H}$, $f^2 = -1$.

At each stage, some properties change, and some remain:

1. the complex numbers are *associative and commutative*,
2. the quaternions are *assocative*, but no longer commutative,
3. the octonions are no-longer associative, but they are *alternative*,
4. each of the three algebras is a *division algebra*,
5. at each step we have what is called an *involution*:

Definition 10.3. *An* involution *of an algebra $\mathbb{A}$ is a map $\mathbb{A} \to \mathbb{A}$, $a \mapsto a^*$ that*

(1) *is of order two:* $\forall a \in \mathbb{A}$: $(a^*)^* = a$,
(2) *is an automorphism of addition:* $\forall a, b \in \mathbb{A}$, $(a+b)^* = a^* + b^*$,
(3) *is an anti-automorphism of the product:* $\forall a, b \in bA$: $(ab)^* = b^* a^*$, $1^* = 1$.

(Examples: $a^* = a$ *when* $\mathbb{A}$ *is commutative;* $z^* = \overline{z}$ *when* $\mathbb{A} = \mathbb{C}$*;* $X^* = X^t$ *when* $\mathbb{A} = M(n, n; \mathbb{K})$*.)*

The Cayley-Dickson construction works also if we replace the conditions $i^2 = -1$, etc., by $i^2 = \lambda$, with $\lambda \in \mathbb{R}$, etc.: then properties (1), (2), (3), (5) remain true, but in general we no longer have division algebras.

Definition 10.4. *Assume* $\mathbb{A}$ *is an algebra over* $\mathbb{K}$ *with product* ab *and an involution denoted by* $a \mapsto \overline{a}$*. Fix also* $\lambda \in \mathbb{K}$*. Then the* Cayley-Dickson extension of $\mathbb{A}$ *is the set* $\mathbb{B} := \mathrm{KD}(\mathbb{A}, \lambda) := \mathbb{A} \times \mathbb{A}$ *with sum and product defined by*

$$(a, b) + (a', b') := (a + a'; b + b'),$$
$$(a, b) \cdot (a', b') := \big(aa' + \lambda \overline{b'} b, b'a + b\overline{a'}\big)$$

We define also $(a, b)^* := (\overline{a}, -b)$*.*

Example 1. Let $\mathbb{A} = \mathbb{R}$ with $\overline{a} = a$. Then:
(a) if $\lambda = -1$, then $\mathrm{KD}(\mathbb{R}, -1) = \mathbb{C}$, and $(a, b)^* = (a, -b)$ its complex conjugate,
(b) if $\lambda = 0$, then $\mathrm{KD}(\mathbb{R}, 0) = \mathbb{R}[\varepsilon] = T\mathbb{R}$ are the dual numbers over $\mathbb{R}$,
(c) if $\lambda = 1$, then $\mathrm{KD}(\mathbb{R}, 1) \cong \mathbb{R}[X]/(X^2 - 1) \cong \mathbb{R} \times \mathbb{R}$, sometimes called the paracomplex numbers.

Theorem 10.5. *Assume* $\mathbb{A}$ *is an associative algebra over* $\mathbb{K}$ *with an involution* $a \mapsto \overline{a}$ *such that* $\overline{a} = a$ *iff* $a \in \mathbb{K}1$*, and let* $\lambda \in \mathbb{K}$*. Then the Cayley-Dickson extension* $\mathbb{B} = \mathrm{KD}(\mathbb{A}, \lambda)$ *satisfies:*
(1) $\mathbb{B}$ *is alternative if and only if* $\mathbb{A}$ *is associative,*
(2) $\mathbb{B}$ *is associative if and only if* $\mathbb{A}$ *is associative and commutative,*
(3) $\mathbb{B}$ *is commutative if and only if* $\mathbb{A}$ *has trivial involution.*

Proof. We will not give here a full proof of this central result – see e.g., [Fau14] for a clear and elegant presentation. Let's illustrate the proof of (2) by the following example: essentially, one can reduce things to a matrix computation. The proof of (1) cannot be reduced to a matrix computation and must be done “by hand”; that is the main reason why (1) is more difficult and longer. □

Example 2. Let $\mathbb{A} = \mathbb{C}$ with $\overline{a}$ complex conjugation and λ real. Define the following matrix $M(a, b) = \left(\begin{smallmatrix} a & \lambda b \\ \overline{b} & \overline{a} \end{smallmatrix}\right)$ with $a, b \in \mathbb{C}$, and compute, using commutativity of $\mathbb{C}$ (!),

$$M(a, b) \cdot M(a', b') = \begin{pmatrix} a & \lambda b \\ \overline{b} & \overline{a} \end{pmatrix} \cdot \begin{pmatrix} a' & \lambda b' \\ \overline{b'} & \overline{a'} \end{pmatrix} = \begin{pmatrix} c & \lambda d \\ \overline{c} & \overline{d} \end{pmatrix} = M(c, d)$$

with $(c, d) = (aa' + \lambda b\overline{b'}, ab' + b\overline{a'}) = (a, b) \cdot (a', b')$. The same computation always works if $\mathbb{A}$ is commutative and associative, and this proves:

Theorem 10.6. *Assume* $\mathbb{A}$ *is commutative and associative and* $\overline{\lambda} = \lambda$*. Then* $\mathrm{KD}(\mathbb{A}, \lambda)$ *is an associative algebra which can be identified with the subalgebra* $\mathbb{B}$ *of* $M(2, 2; \mathbb{A})$ *of matrices of the form* $M(a, b) = \left(\begin{smallmatrix} a & \lambda b \\ \overline{b} & \overline{a} \end{smallmatrix}\right)$ *with* $a, b \in \mathbb{A}$*.*

Now define Hamilton's *quaternions*, $\mathbb{H} := \mathbb{H} = \mathbb{C} \oplus j\mathbb{C} = \mathrm{KD}(\mathbb{C}, -1)$, the Cayley-Dickson extension with $\lambda = -1$. According to the preceding example it can be realized as a matrix algebra. An $\mathbb{R}$-basis is given by the four matrices $1 = M(1,0)$, $i = M(i,0)$, $j = M(0,1)$, $k = M(0,i)$. The reader should compute the multiplication table and check that it coincides with Hamilton's famous relations, which according to the folklore he carved into Broom Bridge.

Theorem 10.7. *The quaternions form a division algebra.*

Proof. More generally, in the situation of Theorem 10.6, for any matrix $M = M(a,b)$, the matrix M^* coincides with the adjugate matrix (fr: comatrice) of M. Hence an element $M = M(a,b)$ is invertible if, and only if, its determinant $a\overline{a} - \lambda\overline{b}b$ is invertible in $\mathbb{A}$, and then its inverse is $M^{-1} = \frac{1}{\det M} M((a,b)^*)$. In particular, if $\mathbb{A} = \mathbb{C}$ and $\overline{a}$ complex conjugation and $\lambda = -1$, then *every non-zero element* of $\mathbb{B}$ is invertible, hence $\mathbb{B}$ is a division algebra. □

Definition 10.8. *The algebra of* octonions $\mathbb{O}$ *is defined as* $\mathbb{O} := \mathrm{KD}(\mathbb{H}, -1)$.

Theorem 10.9. *The octonions are an alternative, non-associative division algebra.*

Proof. See, e.g., [Fau14]. □

To see that the octonions are not associative, one may also consider the $\mathbb{R}$-basis $e_1, \ldots, e_8$ as listed here and read off from the multiplication table: $(e_1e_2)e_4 = e_3e_4 = e_7$, which is different from $e_1(e_2e_4) = e_1e_6 = -e_7$.

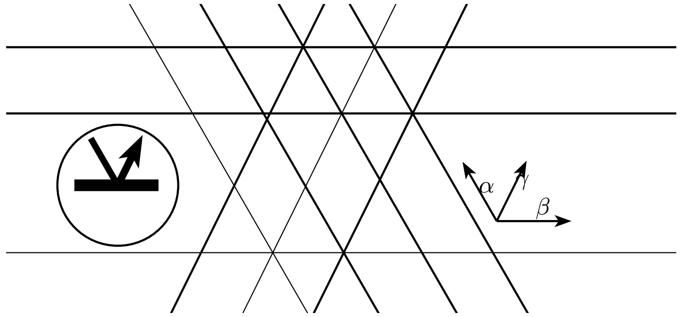
10.2. **Link with geometry: three-webs.** One of the many links of the octonions with geometry can be described as follows (cf. [AG06, CS03, Pi55]). Recall from Chapter 5 the notion of *transversality of two equivalence relations*: $\alpha \top \beta$ iff each equivalence class of α intersects each class of β in exactly one element (def. 5.2). Recall that then M is equivalent to a direct product $A \times B$ with $A = M/\alpha$ and $B = M/\beta$ (prop. 5.3).

Definition 10.10. *A* (generalized) d-web *on a set M is given by a family $\alpha_1, \ldots, \alpha_d$ of equivalence relations on M such $\alpha_i \top \alpha_j$ for all $i \neq j$.*

The case $d = 3$ is most interesting. We shall write $(\alpha, \beta, \gamma) = (\alpha_1, \alpha_2, \alpha_3)$, and

$$A = M/\alpha, \qquad B = M/\beta, \qquad C = M/\gamma\,.$$

When drawing pictures, we shall represent equivalence classes by "lines", and call them *α-lines, β-lines, γ-lines.* In principle, nothing distinguishes the three classes from each other, but to introduce some artifiicial distinction, we shall give them more fancy names: call α *collector*, β *base*, and γ *emitter*, and symbolize the 3-web by the transistor symbol. In the following, we have represented several lines of each type. The lines are "straight", since this is easier to draw than "families of curved lines". To produce the geogebra-files (from which the following figures are imported), just choose three directions (the three vectors labelled α, β, γ), and then, in all following constructions, you are only allowed to use lines or segments that are parallel to one of the three directions. Points lying on a line parallel to α then represent points that are equivalent under α, etc.

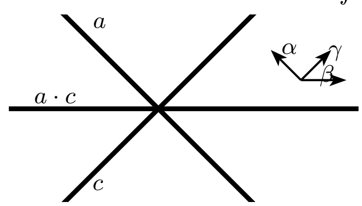


Remark 10.1. To motivate the fancy terminology: by definition of a 3-web, each class of the base β is a bisection of (α, γ), that is, it can be seen as the unit section of the pair-pregroupoid $A \times C$ (cf. Section 4.2, Theorem 4.9). In other words, a 3-web can be seen as a "pair groupoid + some additional strucure" (where (α, γ) represents the pair groupoid, and β the "additional structure"). Since a groupoid is an algebraic structure, one may expect that the additional structure is an even stronger algebraic structure. This is indeed true:

Definition 10.11. *With notation as above, we define a product map*

$$A \times C \to B, \quad (a, c) \mapsto a \cdot c := [a \cap c]_\beta$$

(the β-line through the unique intersection element of a and c):



In the same way, we can define 5 other products $C \times A \to B$, $A \times B \to C$ (all denoted by a point $\cdot$), etc., called parastrophic *with the first product.*

Theorem 10.12. *For each $a \in A, b \in B, c \in C$: the equation $a \cdot x = b$ has a unique solution $x \in B$, and the equation $y \cdot c = b$ has a unique solution $y \in A$.*

Proof. Saying that $a \cdot x = b$ amounts to saying that the unique intersection element of a and x belongs to b, or, equivalently, that

$$a \cap x \cap b \neq \emptyset.$$

This condition is symmetric in a, x, b, and so is equivalent to $x = a \cdot b$ (for the parastrophic product $A \times B \to C$). Therefore the solution x exists, and it is unique. Same for $y \cdot c = b$. □

Definition 10.13. *A* dissociated quasigroup *is given by three sets A, B, C, together with a product $A \times B \to C$ such that, for each $a \in A, b \in B, c \in C$, the equation $a \cdot x = b$ has a unique solution $x \in B$, and the equation $y \cdot c = b$ has a unique solution $y \in A$. (Note that then $(a, b) \mapsto x$ and $(c, b) \mapsto y$ defines other products, called again* parastrophic *with the first one.)*

Remark 10.2. Let us identify $A \times C$ with M via $(a, c) \mapsto a \cap c$. Then α corresponds to the projection onto A and γ to the projection onto C, and β is given by: $(a, c) \sim_\beta (a', c')$ iff $a \cap c \sim_\beta a' \cap c'$, iff $a \cdot c = a' \cdot c'$.

Theorem 10.14. *Let A, B, C a dissociated quasigroup with product $A \times C \to B$. Then the set $M = A \times C$ is a 3-web when equipped with the three equivalence relations*

$$(a,c) \sim_\alpha (a',c') \text{ iff } a = a', \qquad (a,c) \sim_\gamma (a',c') \text{ iff } c = c',$$

$$(a,c) \sim_\beta (a',c') \text{ iff } a \cdot c = a' \cdot c'.$$

Summing up, dissociated quasigroups and 3-webs are equivalent data.

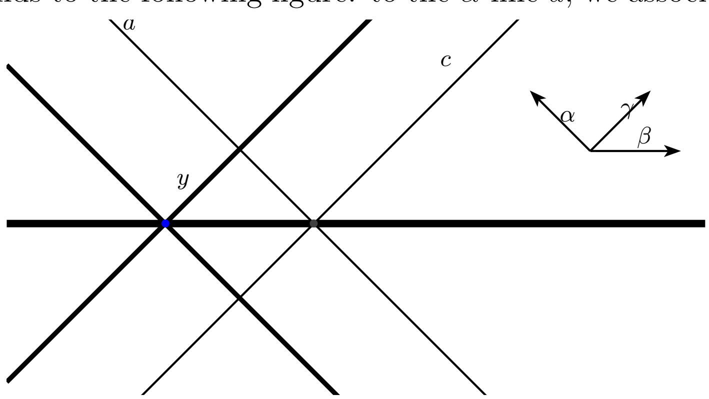
Proof. We have already seen that α and γ are transversal equivalence relations. Let's show that β is an equivalence relation transversal to α: β is an equivalence relation since it is the fiber of a map $M \to B$. It is transversal to α: indeed, the conditions $[(a,c) \sim_\alpha (x,y), (x,y) \sim_\beta (a',c')]$ are equivalent to $[x = a$ and $ay = a'c']$, which has *exactly one* solution (x,y), by definition of a dissociated quasigroup. In the same way, $\beta \top \gamma$. $\square$

Remark 10.3. We invite the reader to define *morphisms* of dissociated quasigroups, resp. of 3-webs, and to compare both notions.

10.3. **Base points, loops and quasigroups.** The game to be played next is: under the equivalence described above, translate algebraic conditions into geometric configurations, and vice versa! To play this game, we now need to *identify the sets A, B, C with each other*, in order to define "usual" (not dissociated) products, of the kind $A \times A \to A$, etc. The problem is that *there are many identifications of A with C*, and none of them deserves to be preferred! In fact, each choice of base point $y \in M$ gives us a bijection

$$A \to C, \quad a \mapsto c := a \cdot [y]_\beta = [(a \cap [y]_\beta)]_\gamma.$$

which corresponds to the following figure: to the α-line a, we associate the γ-line c,



Let us fix the base point $y \in M$. Using the various identifications given by y, there are essentially 6 ways to define a product $A \times A \to A$, $(u,v) \mapsto u \bullet v$ coming from the product $A \times B \to C$. Namely, to define $u \bullet v$, we have 3 choices to identify u with an element in A, B or C, and once this choice fixed, there remain 2 choices to identify v with an element of the remaining two spaces; then take their product as defined in the preceding section, and return to A. Again, we will call these 6 products *parastrophic*. Among these six products, two have $[y]_\alpha$ as neutral element, and the other four do not.

Theorem 10.15. *Let M be a 3-web and fix a base point $y \in M$. Then the following defines a loop on A with neutral element $[y]_\alpha$:*

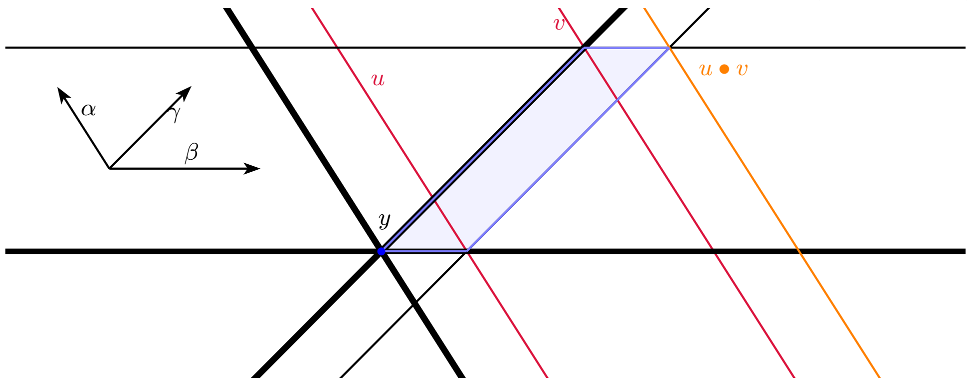
$$A \times A \to A, \quad (u,v) \mapsto u \bullet v := (u \cdot [y]_\beta) \cdot (v \cdot [y]_\gamma).$$

Proof. The law defines a quasigroup since it comes from the dissociated quasigroup $A \times B \to C$ by identifying $A = B$ and $A = C$ as mentioned above. Let us show that $[y]_\alpha$ is a neutral element:

$$[y]_\alpha \bullet v = ([y]_\alpha \cdot [y]_\beta) \cdot (v \cdot [y]_\gamma) = [y]_\gamma \cdot (v \cdot [y]_\gamma) = v,$$

and, in the same way, it is seen that $u \bullet [y]_\alpha = u$. □

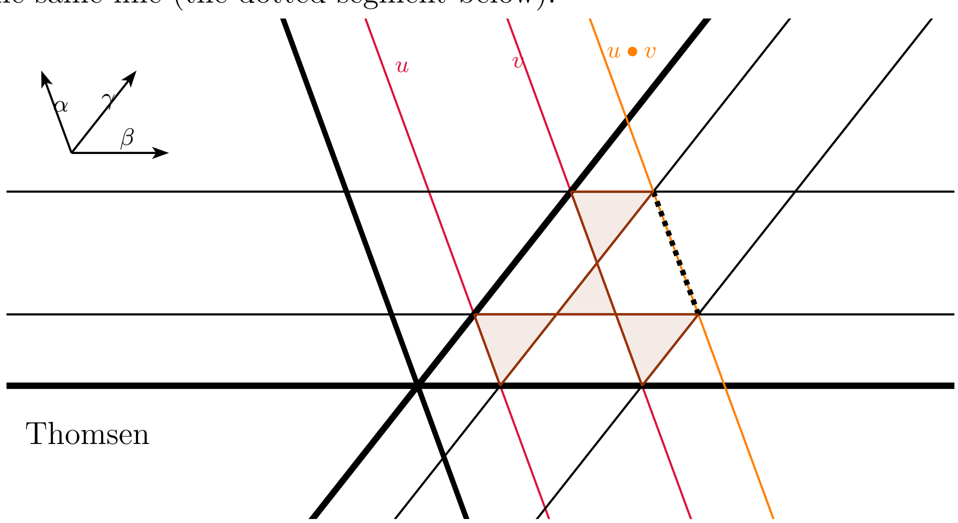
Here is a figure illustrating the construction of $u \bullet v = ((u \cap [y]_\beta)_\gamma \cap (v \cap [y]_\gamma)_\beta)_\alpha$.



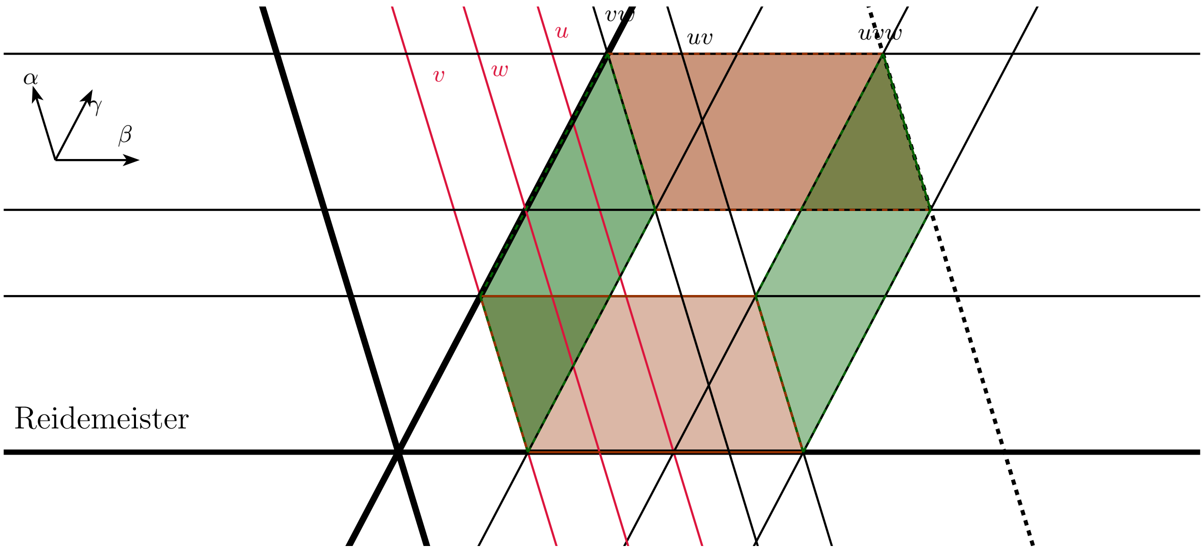
The theory of loops and quasigroups now starts by taking a closer look at this product and how geometric properties of the 3-web M are translated into algebraic properties of the product $A \times A \to A$. Each of the following algebraic properties (which may hold, or not!) translates into a certain geometric "configuration":

(1) commutativity into the so-called *Thomsen figure*,
(2) associativity into the *Reidemeister figure*,
(3) (left, middle or right) alternativity into the *(left, middle or right) Bol figure*,
(4) power associativity into the *hexagon figure*.

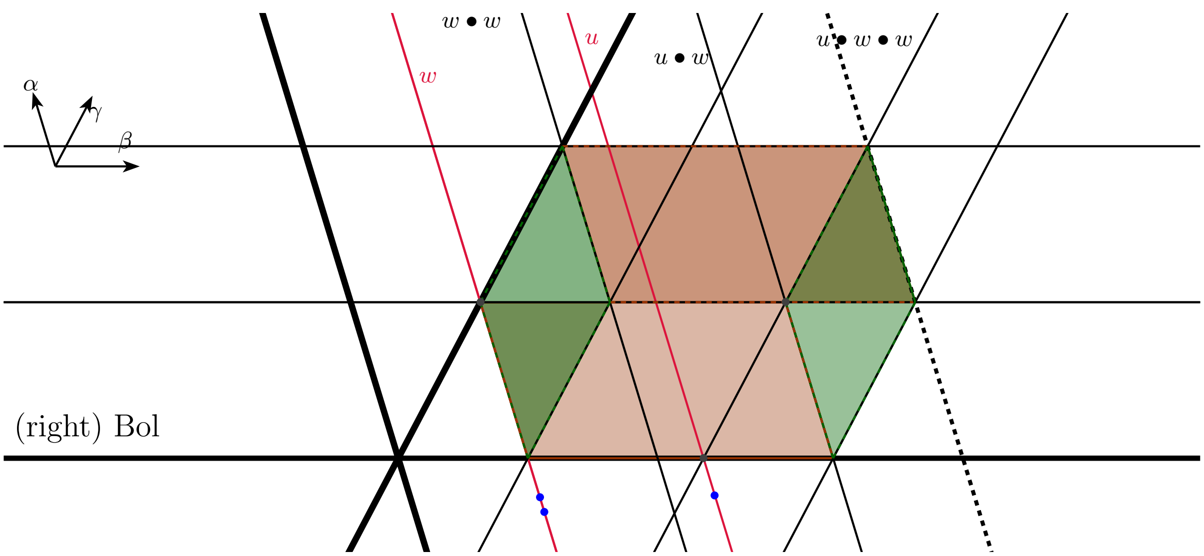
(1) Saying that $u \bullet v = v \bullet u$, amounts to saying that two different constructions give the same line (the dotted segment below):



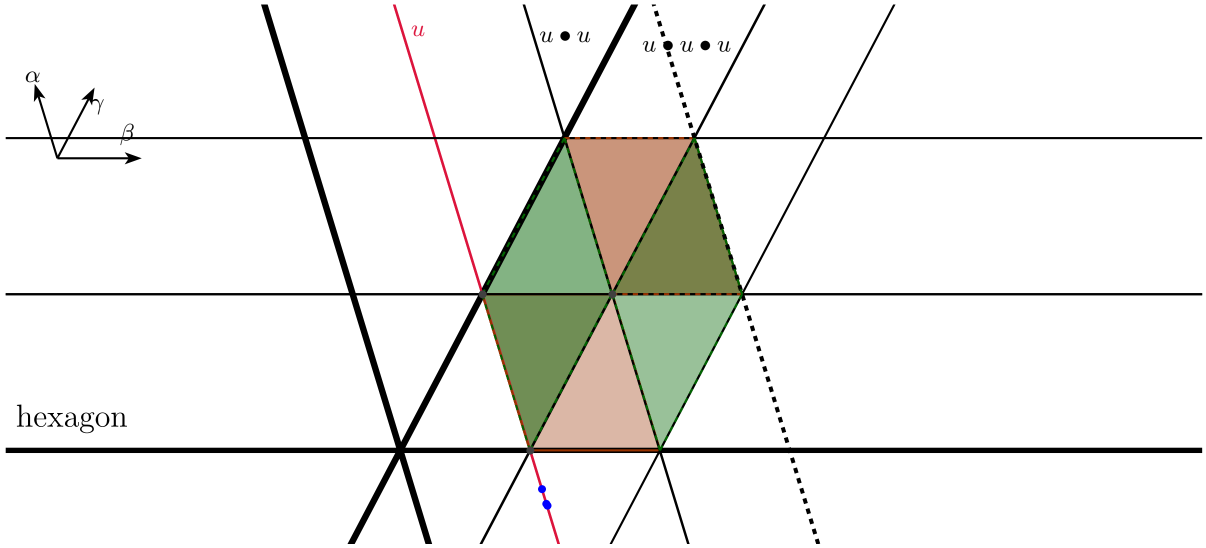
(2) Saying that $u \bullet (v \bullet w) = (u \bullet v) \bullet w$, amounts again to saying that two different constructions give the same (dotted) line. Note that the image looks like a view onto a three-dimensional cube. This is indeed a hint towards profound links with the "foundations of geometry" (cf. below):



(3) (Right) alternativity $u \bullet (w \bullet w) = (u \bullet w) \bullet w$ is a special case of associativity. To illustrate this, in the preceding geogebra-file, just move the lines v to agree with the line w. This gives:



(4) Finally, to illustrate $u^2 \bullet u = u \bullet u^2$, move the lines so that $u = v = w$, as follows:



To close these lectures, some final remarks.

Remark 10.4. The product $A \times A \to A$ described here should be seen as a multiplication of a "one-dimensional object". Indeed, one may identify A with the γ-line $[y]_\gamma$, by taking intersection with this line. This is indeed how most authors interprete such constructions – the reader should compare the preceding images with those given in [AG06], pages 216 – 220 !

It is possible, and interesting, to extend the definition of such products to define "two-dimensional products" (on M). For the more special context of projective planes, see here. Projective planes occupy a central place in the story of "foundations of geometry" (Grundlagen der Geometry, cf. Section 1.4). The close relation between 3-webs and projective planes has already been noticed and used in the classical textbook [Pi55] (Chapter 2); however, the presentation given there is very old-fashioned and definitely needs to be updated.

As explained in Hilbert's "Grundlagen", in dimension bigger than 2, things become more regular, and products tend to be always associative: the reason for this is that certain configurations (Desargues theorem, or cubes like the one from the Reidemeister configuration above) are *always* satisfied in 3-dimensional space, but not always in 2-space. Thus 3-webs in 2-space are particularly important since they allow to construct and describe "exceptional" geometries.

But the importance of 3-webs certainly goes beyond the construction of exceptional geometries, since they are naturally related to groupoids, and appear to be a very natural language to describe *connections* and *parallel transport*.[1]

[1] Cf. end of Chapter 8. The book *Smooth quasigroups and loops*, Kluwer Academic Publishers, Dordrecht, 1999, by Lev Sabinin, contains very interesting material. The book is hard to read, however, and it would certainly be worth to rewrite it in a more "synthetic" language.

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