SOME REMARKS ON TEACHING MATHS: THE UNIVERSAL SPACE

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For every affine space A over a field $\mathbb{K}$, there is a vector space $\hat{A}$, together with a canonical imbedding $A \hookrightarrow \hat{A}$, having the universal property that affine maps $f : A \to B$ canonically extend to linear maps $\hat{f} : \hat{A} \to \hat{B}$:

$$\begin{array}{ccc} A & \stackrel{f}{\to} & B \\ \downarrow & & \downarrow \\ \hat{A} & \stackrel{\hat{f}}{\to} & \hat{B} \end{array} \tag{0.1}$$

The brute-force construction $\hat{A} := A \oplus \mathbb{K}$ is unnatural since it requires the choice of a base point in A. A natural construction of this *universal space* is described in Chapter 3 of Marcel Berger's book "Geometry I". Berger qualifies this "a rather technical chapter". We would like to convince the reader that this need not be so, and that, by the way, we get a little bit more than what is usually mentioned: namely, the universal space is not only a vector space, but it is also, in a canonical way, a *semigroup*, and the maps $\hat{f}$ are, automatically, *semigroup morphisms*. Although this property may be "folklore", it seems that it has never really attracted attention – my interest in this property arose from my joint work with Michael Kinyon on *associative geometries* (see description of my research topics). The presentation given here is motivated by Section 1 of my paper [13], but is different and more general.

1. Construction of the universal space

As in the preceding note on affine spaces, we take the point of view that an affine space A over $\mathbb{K}$ is the same thing is a linear space over $\mathbb{K}$ with two structure maps, $S(a, b, c) = a - b + c$ and $P_r(b, c) = (1 - r)b + rc$. Since this causes no extra cost, we allow $\mathbb{K}$ to be an arbitrary base ring with unit 1. By "linear space over $\mathbb{K}$" we mean left $\mathbb{K}$-modules. *Translations* and *translation group* are defined by

$$\begin{aligned} T_{a,b} : A \to A, \quad x \mapsto T_{a,b}(x) := S(a, b, x), \\ \mathrm{Tran}(A) := \{f : A \to A \mid \exists u, v \in A : f = T_{u,v}\}. \end{aligned} \tag{1.1}$$

The homothety with center b and ratio r is

$$r_a : A \to A, \quad x \mapsto r_a(x) = P_r(a, x), \tag{1.2}$$

and we denote by H_a, resp. H, the sets of homotheties

$$\begin{aligned} H_a := \{r_a \mid r \in \mathbb{K}\}, \\ H(A) := \bigcup_{a \in A} H_a. \end{aligned} \tag{1.3}$$

Note that H_a is a semigroup for usual composition, but $H(A)$ is not. We denote by $\mathrm{Dil}(A)$ the *dilation semigroup* which is the subsemigroup of $\mathrm{Map}(A,A)$ generated by $H(A)$ and $\mathrm{Tran}(A)$:

$$\mathrm{Dil}(A) := \langle H(A), \mathrm{Tran}(A) \rangle \tag{1.4}$$

Recall finally that the set of self-mappings $\mathrm{Map}(A,A)$ is an affine space, with respect to pointwise defined structure maps. Since it has a canonical base point (the identity map id_A), it is in fact a $\mathbb{K}$-module.

Theorem 1.1. *The dilation semigroup* $\hat{A} := \mathrm{Dil}(A)$ *is an affine subspace of* $\mathrm{Map}(A,A)$ *containing* id_A*, hence is a linear subspace. With respect to the imbedding*

$$A \to \mathrm{Dil}(A), \quad a \mapsto 0_a,$$

it has the universal property (0.1). Moreover, there is a unique affine map

$$\kappa : \mathrm{Dil}(A) \to \mathbb{K} \qquad \text{such that} \qquad \kappa(r_a) = r,\ \kappa(T_{b,c}) = 1,$$

which admits the following splitting: for any choice of base point $o \in A$*,* $\mathrm{Tran}(A)$ *and* H_o *are submodules of* $\mathrm{Dil}(A)$ *whose sum is direct,*

$$\mathrm{Dil}(A) = \mathrm{Tran}(A) \oplus H_o.$$

Proof. Let $g \in \mathrm{Dil}(A)$. Because of the relation

$$r_a \circ T_{b,c} = T_{r_a(b), r_a(c)} \circ r_a,$$

we can write $g = T \circ h$ where T is a translation and h a composition of elements of $H(A)$. With respect to some base point $o \in A$, we have

$$r_a \circ s_b(x) = (1-r)a + r(1-s)b + rsx = T' \circ (rs)_o(x)$$

with some translation T', which shows that g can be written as $g = T'' \circ r_o$, so

$$\mathrm{Dil}(A) = \{ T_{b,o} \circ r_o \mid \ b \in A, r \in \mathbb{K} \}.$$

The set of such mappings is obviously stable under the pointwise ternary operations on mappings $S(f,g,h) = f - g + h$, $P_r(g,h) = rg + (1-r)h$ (since this independent of base point in $\mathrm{Map}(A,A)$, one may do the computation with respect to the "unnatural" but more usual base point 0_o, instead of the natural one). Thus $\mathrm{Dil}(A)$ is a linear space, and with respect to 1_o, $\mathrm{Tran}(A)$ and H_o are obviously direct summands. The map κ then can be defined as projection onto the second factor, and it is necessarily of this form. Moreover, the map $A \to \mathrm{Dil}(A)$, $a \to 0_a$ is affine:

$$S(0_a, 0_b, 0_c)(x) = 0_a(x) - 0_b(x) + 0_c(x) = a - b + c = 0_{S(a,b,c)}(x),$$
$$P_r(0_b, 0_c)(x) = (1-r)0_b(x) + r0_c(x) = (1-r)b + rc = 0_{P_r(b,c)}(x).$$

To finish the proof, let us now prove the universal property. Assume $f : A \to B$ is an affine map. We claim that there is a unique linear map $\hat{f} : \mathrm{Dil}(A) \to \mathrm{Dil}(B)$ such that

$$\hat{f}(r_a) = r_{f(a)}, \qquad \hat{f}(T_{b,c}) = T_{f(b),f(c)}.$$

Indeed, if $h(x) = rx + b$, after temporary choice of a base pont $o \in A$ and $o' := f(o) \in B$, the map $h \mapsto \hat{f}(h)$ defined by $(\hat{f}(h))y := ry + f(b)$, is linear since, for the direct sum decompositions with respect to o and o' it is given by the block matrix $\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and it sends r_a to $r_{f(a)}$ and $T_{b,o}$ to $T_{f(b),o'} = T_{f(b),f(o)}$, as required. Taking

$r = 0$, we see that $\hat{f}$ extends f in the sense of Diagram (0.1). Finally, $\hat{f}$ preserves the canonical base points: 1_o is sent to $1_{o'}$, hence $\hat{f}$ linear with respect to them. □

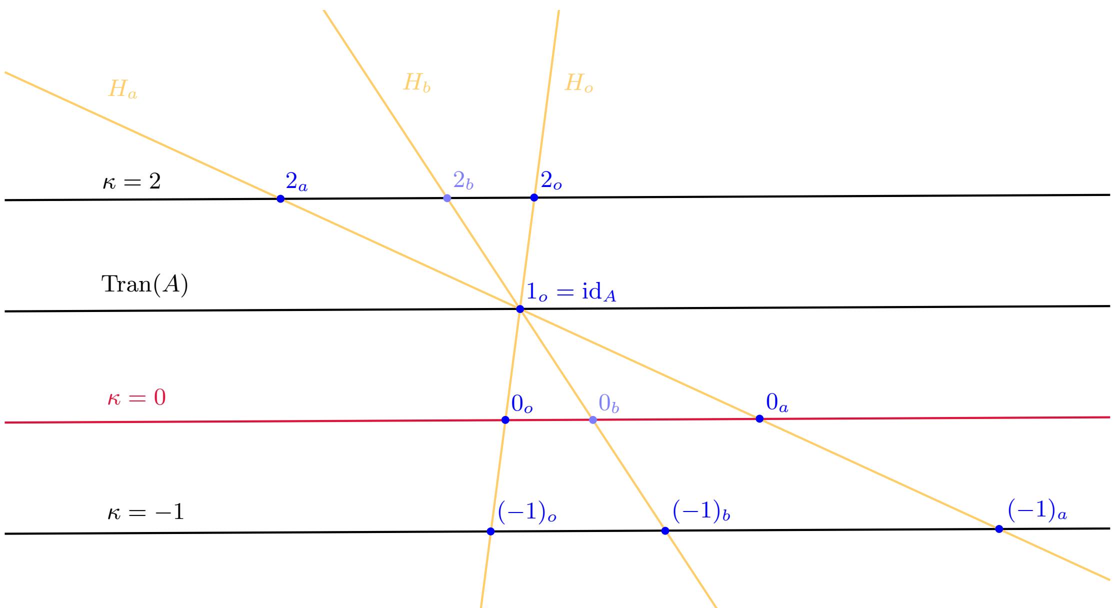
Remark. If $\mathbb{K}$ is *commutative*, then homotheties are *affine maps* $A \to A$, and hence $\mathrm{Dil}(V)$ is contained in the semigroup $\mathrm{Aff}(A, A)$ of affine self-maps of A.

2. Case of a (skew) field

If $\mathbb{K}$ is a *(skew) field*, then for every $r \neq 1$, the map $f(x) = rx + b$ has a fixed point p, namely $p = (1-r)^{-1}b$, and hence $f = r_p \in H(A)$. Thus

$$\mathrm{Dil}(V) = H(A) \cup \mathrm{Tran}(A), \qquad H(A) \cap \mathrm{Tran}(A) = \{\mathrm{id}_A\}.$$

Here is a figure: level sets of κ are horizontal lines, indexed by their level $\kappa(h)$; the translation group corresponds to level 1, whereas the canonical copy of A has level 0 (red). If 2 is invertible in $\mathbb{K}$, the set with level -1 also has nice properties: it realizes A as a set of point reflections (a point a is identified with its point reflection $(-1)_a$) which form a *symmetric space*.



Next, we show that the complement of the red line is the *group of invertible dilations*: its group structure is canonical with respect to the given data:

3. On the semigroup structure of the universal space

Definition 3.1. *We let*

$$\mathrm{Dil}^\times(A) := \kappa^{-1}(\mathbb{K}^\times) = \{T_{b,o} \circ r_o \mid b \in A, r \in \mathbb{K}^\times\}.$$

Theorem 3.2. *If A, B are affine spaces over $\mathbb{K}$ and $f : A \to B$ is an affine map, then its linear prolongation $\hat{f} : \mathrm{Dil}(A) \to \mathrm{Dil}(B)$ is a semigroup morphism, and so is the canonical map $\kappa : \mathrm{Dil}(A) \to \mathbb{K}$. It follows that $\mathrm{Dil}^\times(A)$ is a group, and that the restriction of $\hat{f}$ to $\mathrm{Dil}^\times(A)$, is a group morphism.*

Proof. Let $g, h \in \mathrm{Dil}(A)$. Fix a base point $o \in A$ and let $o' := f(o) \in B$ and write $g(x) = sx + w$, $h(x) = rx + v$, so $g \circ h(x) = srx + (sv + w)$, whence

$$(\hat{f}(g \circ h))(y) = sry + f(sv + w).$$

Since $f : (A, o) \to (B, o')$ is linear, we have $f(sv+w) = sf(v)+f(w)$. On the other hand, $(\hat{f}(g))(y) = sy + f(w)$ and $(\hat{f}(h))(y) = ry + f(v)$, so

$$(\hat{f}(g) \circ \hat{f}(h))(y) = s(ry + f(v)) + f(w) = sry + f(sv + w) = (\hat{f}(g \circ h))(y),$$

and $\hat{f}$ is a semigroup morphism. Taking for f the "final map" $A \to \{0\}$, we see that $\kappa = \hat{f}$ is a semigroup morphism. The rest follows from this. □

Proposition 3.3. *Left and right translations of the semigroup* $\mathrm{Dil}(A)$ *are affine maps, and the ternary product* $(xyz) := xy^{-1}z$ *on the group* $\mathrm{Dil}^\times(A)$ *is given by a rational formula. As a group,* $\mathrm{Dil}^\times(A)$ *is a semidirect product of* $\mathrm{Tran}(A)$ *with* $\mathbb{K}^\times$.

Proof. Identifying $\hat{A} = A \oplus \mathbb{K}$ with respect to the origin 0_o, the formula $g \circ h(x) = srx + (sv + w)$ translates to

$$(w, s) \cdot (v, r) = (sv + w, sr) = s(v, r) + (w, 0) = s((v, r) - (0, 1)) + (w, s)$$

or: $xz = z_2(x - y) + z$. All these are ways to write the product on the semidirect product $\mathrm{Tran}(A) \ltimes \mathbb{K}^\times$. For the inverse of x in $\mathrm{Dil}^\times(A)$ we get $x^{-1} = \frac{1}{x_2}((0,1)-x)+x$, which taken together leads to

$$(xyz) = \frac{z_2}{y_2}(x - y) + z.$$

This proves the claims. □

4. One step further

Now let us go the other way round: instead of $\hat{A}$, consider some linear space W together with a non-trivial linear form $\phi : W \to \mathbb{K}$. The reader may prove by direct computation that, for e with $\phi(e) = 1$ fixed,

$$x \cdot_e z := \phi(z)(x - e) + z$$

is a group law on $W^\times := \phi^{-1}(\mathbb{K}^\times)$, and more generally, the ternary product

$$(xyz) = \frac{\phi(z)}{\phi(y)}(x - y) + z,$$

defines a torsor structure on $W^\times$. But what is much more remarkable, the ternary product may be defined by the simple geometric (lattice-theoretic) formula given here (Section 2.5), or in [33] or there. Seen projectively, this translates as follows: *The set obtained by taking off two hyperplanes from a projective space carries a canonical torsor structure; this structure can be defined in purely geometric terms.*

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