0. Introduction

In this work we study the interplay of Lie- and Jordan theory on an algebraic and on a geometric level. We intend to continue it to a systematic study of the role Jordan theory plays in harmonic analysis. In fact, the applications of the theory of Jordan algebras to the harmonic analysis on symmetric cones (cf. the monograph [FK94]) were at the origin of the author's work in this area. Then Jordan algebras turned up in the study of many causal symmetric spaces (see Section XI.3), and soon it became clear that "generically" all symmetric spaces have a significant relation to Jordan theory. Since Jordan theory does not (yet) belong to the standard tools in harmonic analysis, the present text is designed to provide a self-contained introduction to Jordan theory for readers having some basic knowledge on Lie groups and symmetric spaces. Our point of view is geometric: throughout we introduce first the relevant geometric structures and deduce from their properties algebraic identities for the associated algebraic structures. Thus our presentation differs from related ones (cf. e.g. [FK94], [Lo77], [Sa80]) by the fact that it is nearly computation-free. Let us give now an overview of the contents. See also the introductions given at the beginning of each chapter.

0.1. Lie algebras and Jordan algebras. If we decompose the associative product of the matrix algebra $M(n,\mathbb{R})$ in its symmetric and skew-symmetric parts,

$$XY = \frac{XY+YX}{2} + \frac{XY-YX}{2}, \tag{0.1}$$

then the second term leads to the *Lie algebra* $\mathfrak{gl}(n,\mathbb{R})$ with product $[X,Y]=XY-YX$, and the first term leads to the *Jordan algebra* $M(n,\mathbb{R})$ with product $X\cdot Y=\frac{1}{2}(XY+YX)$.

Of fundamental importance in modern mathematics is the correspodence between Lie algebras and *Lie groups*: there is a functor which we call the *Lie functor for Lie groups* assigning to a Lie group its Lie algebra, and every (real, finite dimensional) Lie algebra belongs to some Lie group. One may ask whether there is also a *Jordan functor*: can we find a "global" or "geometric" object to which a given Jordan algebra is associated in a similar way as a Lie algebra to a Lie group?

0.2. Lie triple systems and Jordan triple systems. Lie groups are generalized by *symmetric spaces*. Correspondingly, Lie algebras are generalized by *Lie triple systems* (abbreviated LTS; cf. Def. I.1.1) which are nothing but the curvature tensor of a symmetric space, evaluated at the base point. For example, when we consider a Lie group as a symmetric space, we consider its Lie algebra as a LTS with the triple Lie bracket

$$[X,Y,Z] := \frac{1}{4}[[X,Y],Z]. \tag{0.2}$$

The category of connected simply connected symmetric spaces is equivalent to the category of (real, finite dimensional) Lie triple systems. This point of view has been systematically developed in the monograph [Lo69a]. (We give a summary of the basic facts in Chapter I.)

Just as Lie algebras are generalized by Lie triple systems, Jordan algebras are generalized by *Jordan triple systems* (abbreviated JTS; cf. Def. III.2.5). For instance, the space $M(n, \mathbb{R})$ with the triple product

$$T(X, Y, Z) = XYZ + ZYX \tag{0.3}$$

is a JTS. More generally, if $x \cdot y$ denotes the product of a Jordan algebra V, then

$$T(x, y, z) := 2(x \cdot (y \cdot z) - y \cdot (x \cdot z) + (x \cdot y) \cdot z) \tag{0.4}$$

defines a Jordan triple product on V. Again one may ask whether there is a *Jordan functor for Jordan triple systems*: is a JTS associated to a geometric object, just as a LTS is associated to a symmetric space?

0.3. Symmetric cones, bounded symmetric domains, symmetric R-spaces and "Makarevič spaces". The problem of defining a "Jordan functor" is of considerable interest because it is related to many topics in geometry and harmonic analysis on symmetric spaces. Let us illustrate this by recalling some important special cases in which such a functor is indeed known:

(a) By a result due to M. Koecher and E.B. Vinberg, Jordan algebras satisfying a certain positivity condition (called *formally real* in [BK66] and *Euclidean* in [FK94]) correpond bijectively to *symmetric cones* which are an important generalization of the cone of symmetric positive definite matrices (cf. [FK94]). More general Jordan algebras (the semisimple ones) correspond to other classes of non-convex cones which have been characterized in various ways; one of the earliest versions of these concepts is the notion of ω*-domain* in [BK66]. However, it seems that the first attempt to find a geometric object associated to a general Jordan algebra is the one proposed by the author in [Be94] (cf. Chapter II).

(b) From the point of view of harmonic analysis, symmetric cones are very interesting spaces (see [FK94]). Another particularly interesting class of symmetric spaces are the *Hermitian symmetric spaces*. This is a class of symmetric spaces admitting an *invariant complex structure*, generalizing the classical hyperbolic plane. M. Koecher discovered some thirty years ago an "elementary" approach to these spaces by Jordan theory (cf. [Koe69a,b]); the theory was developed more systematically by O. Loos ([Lo77]) who called the algebraic objects equivalent to bounded symmetric domains *positive Hermitian Jordan triple systems*. Some parts of this approach became known to a wider range of people working in harmonic analysis by the book of I. Satake [Sa80].

(c) *Symmetric R-spaces* have been defined in [Tak65] as symmetric spaces which can also be written as a quotient of a semisimple Lie group by a parabolic subgroup; in particular, they are compact. It turns out that they are precisely the real forms of the *compact Hermitian symmetric spaces*, and they correspond bijectively to *positive* or *compact Jordan triple systems* (cf. [Lo85]). Yet another characterization of these spaces is due to D. Ferus ([Fe80]): they are precisely the "immersed symmetric submanifolds" in Euclidean space; prime examples are the n-sphere and the orthogonal group in their usual realizations. This leads to what is called by Ferus the "extrinsic theory of symmetric spaces", namely the study of their realizations as submanifolds. Compared with the intrinsic theory, the *second fundamental form* comes in as an additional structure. This extrinsic approach was generalized by H. Naitoh ([Nai83]) to the pseudo-Riemannian case. His "pseudo-Riemannian symmetric R-spaces" correspond bijectively to Jordan triple systems with some invariant non-degenerate form. Although we will mainly be interested in an "intrinsic theory", it is worth remarking here that Jordan structures (unlike Lie structures) usually have several "different" geometric interpretations. The interplay between "intrinsic" and "extrinsic" theory leads to interesting and mainly unsolved geometric problems (see Chapter XI).

(d) A crucial point for the author's investigations was a paper by B. O. Makarevič ([Ma73]) where "open symmetric orbits in symmetric R-spaces" were classified. The most important class of these orbits is defined by a construction using Jordan-algebras; in [Be96b] we called such spaces *Makarevič spaces* and subsequently used them in the study of geometry and harmonic analysis on some symmetric spaces, in particular on some *causal symmetric spaces* (cf. [Be98a], [BeHi98]). However, the main lack in the concept of Makarevič spaces was that they were defined by a construction and not by an intrinsic characterization. This was overcome in [Be97b] where we replaced the notion of Makarevič spaces by the more general and more conceptual notion of *symmetric spaces with twist.* They are precisely the geometric objects corresponding to general Jordan triple systems, and the purpose of the present work is to develop their intrinsic theory in this generality.

0.4. Symmetric spaces with twist and the Jordan-Lie functor: formal properties. A *twist* is an additional structure on a symmetric space. This additional structure can be described in several ways. Before passing to a geometric description, let us first give its formal properties: if a symmetric space with twist is a symmetric space with an additional structure, then also a JTS must be a LTS with an additional structure. This is indeed true: every JTS T defines a LTS R_T via the formula

$$R_T(X,Y,Z) := T(Y,X,Z) - T(X,Y,Z) \tag{0.5}$$

(the notation R_T for a LTS is used because of the close relation with the curvature tensor, see above). The JTS T is called a *Jordan extension of* R_T; it is an additional structure on the LTS R_T. We call the correspondence

$$JTS \to LTS, \quad T \mapsto R_T \tag{0.6}$$

the *algebraic Jordan-Lie functor.* This is the algebraic version of the forgetful functor from the category of symmetric spaces with twist to the category of symmetric spaces which we call the *geometric Jordan-Lie functor.* We visualize the situation by the following commuting diagram of functors

$$\begin{array}{ccc} JTS & \leftarrow & \text{symmetric spaces with twist} \\ \downarrow & & \downarrow \\ LTS & \leftarrow & \text{symmetric spaces} \end{array} \tag{0.7}$$

where the left column is the algebraic and the right column the geometric Jordan-Lie functor. The top line is the Jordan functor for Jordan triple systems and the bottom line is the Lie functor for symmetric spaces. If we pass to the respective categories of *germs* of symmetric spaces, then the top and the bottom line become equivalences of categories. The diagram (0.7) summarizes the formal relations between Jordan- and Lie structures.

It is interesting to note the formal analogy of the additional structure given by a twist T on a symmetric space with the additional structure given by an affine connection ∇ on a manifold M: having in mind that $R := R_T$ is the curvature tensor of the associated symmetric space, formula (0.5) can also be written

$$R(X,Y) = T(Y,X) - T(X,Y). \tag{0.8}$$

This formula has a striking similarity with the formula

$$[X,Y] = \nabla_X Y - \nabla_Y X \tag{0.9}$$

expressing the Lie bracket of vector fields on a manifold having the *additional* structure of a torsionfree affine connection ∇.

0.5. Twisted complexifications and para-complexifications. Geometrically, the additional structure given by a twist on a symmetric space M can be interpreted as a *twisted complexification* or a *twisted para-complexification* of M. In order to illustrate what this means, consider the example of the real projective space $M = \mathbb{RP}^n$ which is a symmetric space $M = \mathrm{O}(n+1)/(\mathrm{O}(n) \times \mathrm{O}(1))$. It has a natural complexification given by the complex projective space $\mathbb{CP}^n = \mathrm{U}(n+1)/(\mathrm{U}(n) \times \mathrm{U}(1))$. We call this complexification "twisted" in order to distinguish it from the "straight" complexification $M_{\mathbb{C}} = \mathrm{O}(n+1, \mathbb{C})/(\mathrm{O}(n, \mathbb{C}) \times \mathrm{O}(1, \mathbb{C}))$. Every symmetric space $M = G/H$ has (locally) a unique straight complexification $M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$. In contrast, twisted complexifications are an additional structure of a symmetric space which in general need neither exist nor be unique. Another important example is given by the general linear group $M = \mathrm{Gl}(n, \mathbb{R})$ considered as a symmetric space. It has a twisted complexification given by the space $\mathrm{Gl}(2n, \mathbb{R})/\mathrm{Gl}(n, \mathbb{C})$ of complex structures on $\mathbb{R}^{2n}$.

If one replaces the condition $J^2 = -\mathbf{1}$ in the definition of a complex structure by $J^2 = \mathbf{1}$, one arrives at *paracomplex structures* or *polarizations.* A tensor field of this kind defines a product structure on the tangent spaces of a manifold. Again we distinguish "straight" and "twisted" polarizations. Straight polarizations lead to the trivial situation of a global direct product structure, whereas twisted polarizations lead to an interesting class of symmetric spaces (already investigated by S. Kaneyuki, cf. [KanKo85]) which as manifolds have locally, but not globally a product structure. A familiar example of such a space is the one-sheeted hyperboloid $M = \mathrm{SO}(2,1)/\mathrm{SO}(1,1)$ where the two transversal families of straight lines define the local product structure. One can speak of straight and twisted para-complexifications; the former are just direct products, and the latter are in a sense equivalent to twisted complexifications. For example, the twisted para-complexification of $\mathrm{Gl}(n, \mathbb{R})$ equivalent to the twisted complexification described above is the space $\mathrm{Gl}(2n, \mathbb{R})/(\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Gl}(n, \mathbb{R}))$ of para-complex structures on $\mathbb{R}^{2n}$.

The formal definition of "straight" and "twisted" is given in terms of the curvature tensor R and the invariant almost complex structure $\mathcal{J}$ of the complexified space (Chapter III; cf. Def. III.2.1).

0.6. Classification; existence and uniqueness problem. We now face two problems:
Existence problem: *when does a symmetric space admit a twisted complexification?*
Uniqueness problem: *if a symmetric space admits twisted complexifications, how many inequivalent ones are there?*

Equivalently, *which Lie triple systems lie in the image of the Jordan-Lie functor* (0.6)*, and what do the fibers of this functor look like?* General answers to these problems are not known, and we are invited to speculations. The best one could expect is that the Jordan-Lie functor can be imbedded into an exact sequence of functors which encodes in a sense the deeper algebraic relations between Jordan- and Lie structures.

For the time being, we can give partial answers by classification of simple real Lie- and Jordan triple systems. They are surprising enough: *every irreducible symmetric space of classical type admits a twisted complexification; the number of inequivalent twisted complexifications is either 1,2 or 3, and in "most" cases it is 1.* As for the exeptional irreducible symmetric spaces, the situation is less clear. For example, an exceptional Lie group never admits a twisted complexification. Thus we arrive at the surprising conclusion that *a Lie group is "classical" if and only if admits a twisted complexification.* The algebraic version of all of these statements goes back to the classification of simple JTS by E. Neher ([Ne85]). We present another type of classification which is influenced by the paper of B.O. Makarevič [Ma73] mentioned above (Ch. IV and XII). For every classical irreducible symmetric space, we specify all twists in all three aspects: Jordan extension, twisted complexification and para-complexification. In our opinion, this classification is much more appealing than the classification of irreducible symmetric spaces given by M. Berger ([B57]) because Jordan theory provides a means to understand the wealth of symmetric spaces, whereas Lie theory alone does not provide a satisfying organization principle.

0.7. Conformal group, Liouville- and Fundamental theorem. Symmetric spaces with twist have a rich geometric structure incorporating many elements of classical geometry which cannot be understood in the framework of general symmetric spaces. An important feature of symmetric spaces with twist is the existence of a *conformal group* which is much bigger than the "affine group" G appearing in the expression $M = G/H$. For instance, if $M = \mathrm{O}(n+1)/(\mathrm{O}(n) \times \mathrm{O}(1))$ is the real projective space, then the conformal group coincides with the usual projective group $\mathbb{P}\,\mathrm{Gl}(n+1,\mathbb{R})$ operating on M, and if $M = \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$ is the n-sphere, then the conformal group coincides with the classical "conformal group" $\mathrm{SO}(n+1,1)$ operating conformally on M in the usual sense. Classical results of geometry allow to characterize these groups: the projective group is precisely the group preserving "collinearity" (the fundamental theorem of projective geometry), and the classical conformal group is precisely the group preserving the Riemannian metric up to a scalar function (Liouville 1850). The conformal group in our general set-up is (in the irreducible case) characterized by a similar theorem which has aspects of both of the classical theorems mentioned above. It takes various interesting forms in other particular cases; for example, it contains the determination of the *causal group of special relativity* and of the *causal group of the de-Sitter and the anti de-Sitter model of general relativity.* In fact, the determination of the causal groups of some *causal symmetric spaces* was one of the starting points of the author's investigations ([Be96a], [Be96b]).

0.8. Algebraic structures of symmetric spaces with twist. In general a Lie group or a symmetric space is only *locally* determined by its Lie algebra resp. its LTS, and the canonical chart of a symmetric space, namely the *exponential map*, is only a locally defined diffeomorphism which is in general neither surjective nor has dense image.

For symmetric spaces with twist the situation looks much better: to any JTS belongs one *globally* symmetric space with twist (which is defined using the theory of the conformal group), and this space is entirely determined by its JTS (Chapter IX). Besides the exponential map, this space has another canonical chart which we call *Jordan coordinates.* It is a bijection between an open dense part of a vector space V (possibly non-connected) and an open dense part of the space. This chart is the true generalization of the celebrated *Harish-Chandra imbedding* of a Hermitian symmetric space whose most elementary example is the hyperbolic plane, imbedded as the unit disc in the complex plane.

Compared with the usual exponential map, Jordan coordinates have many surprising features. They are all related to the fact that Jordan coordinates are *algebraic*, whereas the exponential map is only analytic. Thus, if $M_0 = G/H$ is the identity component of the global space associated to a JTS, then elements of the group G are represented in Jordan coordinates V by *algebraic* (i.e. birational) maps. Even more, the whole algebraic structure of the global space can be described by a *rational analog of the Campbell-Hausdorff formula.* Recall that the usual Campbell-Hausdorff formula expresses the group multiplication on a neighborhood of the unit element of a Lie group in terms of the exponential map. It is a complicated analytic, but non-algebraic formula. Generalizing the theory of Lie groups, the theory of symmetric spaces M can be based on a *multiplicaton map* $\mu : M \times M \to M$ replacing the group multiplication of a Lie group ([Lo69a]; cf. Section I.4). Using the Campbell-Hausdorff formula, it is possible to find an analytic and local formula for the map μ expressed in exponential coordinates; we may call this a "Campbell-Hausdorff formula for symmetric spaces". In contrast to this situation, the expression for the multiplication map μ in Jordan coordinates is algebraic and global (it holds almost everywhere). Summing up, the global space comes together with a natural algebraic atlas such that the multiplication map is algebraic; we propose to call such spaces *real algebraic symmetric spaces* (Def. I.4.4). The space M itself is described in Jordan coordinates by algebraic conditions of the form

$$\{x \in V \mid \det B(x,x) \neq 0\}, \tag{0.10}$$

where

$$B : V \times V \to \mathrm{End}(V), \quad (x, y) \mapsto B(x, y) \tag{0.11}$$

is a polynomial map which plays an important role; we call it the *Bergman operator* because of its close relation with the Bergman kernel function in the case of a Hermitian symmetric space in Harish-Chandra realization.

Thus the integration of the infinitesimal theory to a local and global theory can be pushed much further for Jordan structures than for Lie structures. In this sense our approach follows the original line of Sophus Lie's investigations: understand the geometric and algebraic context in order to integrate differential equations.

0.9. The problem of the relations between algebras and triple systems. The intrinsic theory of symmetric spaces with twist leads automatically to the extrinsic theory: every such space has (by a construction following the one of D. Ferus in [Fe80]) a realization as a "symmetric submanifold" in a vector space. This realization is very similar to a realization of certain symmetric spaces studied first by K.H. Helwig ([Hw70]); let us call these spaces "Helwig spaces" (cf. Section II.4). Classification shows that "generically" both realizations coincide. It seems to be a quite hard problem to understand why this is the case. The algebraic heart of the problem is the relation between Jordan *algebras* and *triple systems* (Ch. XI): in the Lie category, it is almost trivial that triple systems have always a faithful imbedding into algebras as -1-eigenspaces of an involution. In the Jordan category, the corresponding statement is false (cf. [LoM77]), but "generically" it is true. Classification shows that a slightly modified construction is always possible, although no general proof for this fact is known. We call this the *extension problem for Jordan triple systems.* On a geometric level, the imbedding of a triple system into an algebra (with unit) corresponds to the realization of the corresponding symmetric space with twist as a Helwig-space. These spaces have very nice properties: they are imbedded in a bigger space similarly as orthogonal groups are imbedded in $\mathrm{Gl}(n, \mathbb{R})$; but as long as the extension problem remains open, one cannot develop a really satisfying theory of these spaces.

User's guide

We have divided the text into two parts: the first part (Chapters I – V) contains the theory of the Jordan-Lie functor; the main results are contained in Chapter III. The second part (Chapters VI – XI) contains the theory of the conformal group and of the global spaces; here the main chapter is Chapter X. In this part, the *integrability* of invariant almost complex structures and polarizations on symmetric spaces is used in an essential way, whereas in the first part only ordinary tensor calculus together with some basic theory of symmetric spaces (reviewed in Ch. I) is needed. This makes the theory presented in the first part very easy and general at the same time. Thus the reader looking for a rapid and easy-to-read introduction to Jordan triple systems and Jordan pairs should start by reading Chapter III and then take a glance at the other chapters of the first part. If he is not interested in Jordan *algebras*, he may skip Chapter II since the theory of Jordan algebras is treated independently of the other Jordan structures – in fact, in this chapter the integrability is already used in a somewhat hidden way.

The material presented in the second part is considerably more complicated and more advanced. In Chapters VI and VII we present two different ways of using efficiently the integrability: the approach in Chapter VI is direct; it leads rapidly to "integrated versions" of Jordan pairs and Jordan triple systems. Chapter VII is independent of Chapter VI; it uses the integrability only on the Lie algebra level and leads to a geometric version of the well-known *Kantor-Koecher-Tits construction.* In Chapters VIII and X we develop the analog of this construction on the group level and on the level of the symmetric spaces; thus the reader interested in the basic material of the second part should read Chapters VII, VIII and X. Chapters VI, IX and XI contain supplementary information on geometric and algebraic aspects of the theory and are independent of each other.

Finally, Chapter XII contains the classification and basic structural information on the classical simple objects (Jordan algebras, -triple systems and -pairs and the corresponding symmetric spaces). We hope that this chapter turns out to be useful in particular for non-specialists since most of the material can only be found in specialized literature and usually is not given in the concise form of tables.

In the following the reader interested in special topics finds a short guide how to get quickly to the main results:

- *Jordan algebras:* II, V.4, XI
- *symmetric submanifolds ("extrinsic theory")*: II.4, XI.4
- *(pseudo-) Hermitian symmetric spaces*: III.2, V.2, VI.1, X
- *(generalized) tube domains*: X.5, XI.2
- *causal symmetric spaces:* V.5, IX.2, XI.3
- *polarized and para-Hermitian symmetric spaces:* III.3, V.2, VI.2, VIII.3
- *Jordan pairs and 3-graded Lie algebras:* III.3, VII
- *examples:* I.6, IV.1, VIII.4, IX.2, X.2, XI.5; in particular:
    - *the general linear groups:* I.6.1, IV.1.1, X.2.7, XI.5.1
    - *Grassmannians, projective spaces:* I.6.3, IV.1.2, VIII.4.3, IX.2.1, IX.2.4, X.2.8, XI.5.1
    - *orthogonal groups and Lagrangians:* I.6.4, IV.1.3, VIII.4.2, XI.5.2
    - *spaces of complex structures ("Siegel-spaces"):* I.6.3, IV.1.1
    - *spheres, quadrics and hyperbolic spaces:* I.6.6, IV.1.5, VIII.4.4, IX.2.3, XI.5.3
- *classification and tables:* IV.2, XII

Let us finally give some comments for the more expert reader. A big portion of the results presented here is known and holds in most cases for quite general base fields or even base rings. However, the method we use seems to be new: the traditional approach starts with an axiomatic definition of some algebraic structure; next one derives by algebraic manipulations such as "polarization" new, often very complicated, identities ("Just as in case of Jordan algebras, a long list of identities is required..."; [Lo75, p. viii]); only then one can start to develop a "structure theory". We have tried to completely avoid this way of presentation by starting with geometric concepts (symmetric spaces, Lie groups) and adding additional geometric features (polarizations, almost complex structures). In our opinion, the clear distinction of the parts of the theory where integrability is used and where it is not used is an important gain of this method: it shows that certain identities belong to different parts of the theory – the reader may compare e.g. with the presentation from [Sa80], where the integrability enters only in a rather hidden way in loc.cit. p. 59, or from [Lo77] where it is assumed from the very beginning by considering spaces realized as bounded domains in some $\mathbb{C}^n$. Indeed, our method not only permits to derive many of the known identities in a geometric way, but it also permits to find new ones – which are so complicated that without a geometric interpretation they would be ununderstandable (cf. Cor. X.3.3).

Whereas the first part is at the same time completely general and more "functorial" than the usual approach (compare e.g. with [Sa80], where a in a sense "non-functorial" construction is used which by chance in the semisimple case turns out to be functorial), we introduce in the second part a slight restriction of generality: in Chapter XI we develop the global theory for spaces associated to "faithful" Jordan triple systems; this includes the semisimple case and the case of unital Jordan algebras. The assumption of faithfulness makes the exposition a bit less technical; it means that the "dual conformal group" is faithfully represented by its action on the original space – in the example of the projective spaces this means that there is a one-to-one correspondence between elements g^+ of the "original" projective group $\mathbb{P}\,\mathrm{Gl}(V)$ and elements g^- of the projective group $\mathbb{P}\,\mathrm{Gl}(V^*)$ of the "dual" projective space of hyperplanes in the given space, and fixing such a correspondence defines an automorphism Θ of the projective group. If we dropped the assumption of faithfulness, then g^+ no longer determines the pair (g^+, g^-), and the whole theory has to be formulated by using pairs. This is indeed the natural formulation in the context of *Jordan pairs*. However, in harmonic analysis we usually consider one space as "given" and the other, "dual" one to be a space of certain important objects on the given one; we did not want to lose this viewpoint in a more conceptual but too formal presentation.

For an algebraist, the main lack of our approach is certainly that it seems to be limited to the case of the base field of the real or complex numbers, and for a geometer, that the "geometry" mentioned in the title is really only "differential geometry". Our only excuse is that the present text is not meant to be the definite version of a theory but rather presents the author's personal and incomplete view on this rich field of mathematics – in the final Chapter XIII we give some comments on further topics and generalizations of our approach.

Acknowledgements. This work is a slightly extended version of the author's Habilitationsschrift. I am deeply indebted to Joachim Hilgert for his constant support and help during the time of its preparation. I also owe very much to Jacques Faraut for initiating my work on the area of Jordan theory, to Jean-Louis Clerc, Sigurdur Helgason and Wilfried Lex for reading the manuscript and especially to Ottmar Loos for his valuable comments and remarks.

This research was supported by the DFG-grant HI 412 5-1 which I gratefully acknowledge.

Clausthal, September 1999

Wolfgang Bertram

Chapter I: Symmetric spaces and the Lie-functor

There are four definitions of a symmetric space M:

(1) The global one: $M = G/H$ is a homogeneous space, where H is essentially the group of fixed points of an involution σ of a Lie group G.

(2) The infinitesimal one: M is a real manifold with a torsionfree affine connection ∇ whose curvature is covariantly constant: $\nabla R = 0$.

(3) The mixed one: M is a real manifold with an affine connection ∇ such that the geodesic symmetry s_p with respect to any point $p \in M$ is an automorphism of ∇.

(4) The algebraic one, which reflects axiomatically the properties of the map $\mu : M \times M \to M$ given by $\mu(x,y) := s_x(y)$ with s_x as in (3).

The definitions can be suitably modified in order to distinguish *globally* and *locally symmetric spaces* or to define *germs* of such. The definition (1) is used throughout in research texts on harmonic analysis on symmetric spaces, and therefore we will take it as starting point for our presentation, too. However, (1) is not really a definition of a symmetric space but rather of a *symmetric pair* (G, σ, H) which is not the same thing. Thus for a rigorous theory of symmetric spaces the best starting points are indeed (2) or (4). The purpose of this chapter is to explain (1) – (4), to discuss the relations between the various definitions and to draw some consequences. Since it is easy to work with (1), we start with this definition and describe in a self-contained way how to arrive at (2) – (4). The converse in either case requires considerably more work, and we refer to [Lo69a] and [KoNo69] for this.

Each definition leads to a certain notion of *homomorphisms* and thus to a category. The category corresponding to (1) is given by *symmetric pairs* and homomorphisms of Lie groups compatible with involutions; following Satake ([Sa80]) we call such homomorphisms *equivariant*. The second definition clearly leads to the notion of a *homomorphism of symmetric spaces* as a map which is *affine* (i.e. compatible) with respect to the respective connections. The third definition does not lead to a clear decision between these two possibilities. The fourth definition leads to an algebraic definition of homomorphisms; in [Lo69a] it is shown that for connected spaces it is equivalent to the notion of affine homomorphisms from (2). The categories (1) and (2) are not the same. Already the simple example of a vector space shows that to one space M in the sense of (2) one may associate several symmetric pairs (G, σ, H): a "small one" where G is just the translation group, and a "big one" where G is the full affine group of the vector space. But what is more serious, even if one chooses always the "smallest pair" $(G(M), \sigma)$, one gets a bijection of objects of categories, but not of the corresponding homomorphisms: in fact, the equivariant maps from (1) are always homomorphisms for (2), but the converse is false. Geometrically, this corresponds to the fact that smooth maps of manifolds do in general not induce homomorphisms of groups operating on them. From this point of view, it is remarkable that for *semisimple* symmetric spaces both notions of homomorphisms are indeed equivalent (see Ch.V, Th. V.1.9).

More important than the notion of homomorphisms is the *infinitesimal object* associated to a symmetric space with base point $o \in M$. In the case of definition (1), it is the Lie algebra $\mathfrak{g}$ of G together with the involution $T_e\sigma$; $(\mathfrak{g}, T_e\sigma)$ is called a *symmetric Lie algebra.* In the case of definition (2) the associated infinitesimal object is the tangent space T_oM, equipped with the trilinear map R_o obtained by evaluating the curvature tensor R at o; it is called the *associated Lie triple system* of M. In both cases we have a natural notion of homomorphisms, such that the infinitesimal object depends functorially on the category in question, and moreover, *germs* in the respective categories are entirely equivalent to the corresponding infinitesimal objects. We call the respective functors the *Lie functors for symmetric spaces.* Again the categories of symmetric Lie algebras and of Lie triple systems are not equivalent, but in the semi-simple case they are. For our purposes, the category of Lie triple systems is the "good" one.

Having defined the category of symmetric spaces and its homomorphisms, we are led – as in the category of groups – to the notion of a *representation* as a realization of a symmetric space as a space of invertible linear operators (Section 5). The experience shows that most symmetric spaces come along with a "natural representation". This is the guiding point of view for our presentation of examples (Section 6).

1. Lie functor: group theoretic version

1.1. The Lie functor for Lie groups. A *(real) Lie group* G is a topological group having the structure of a (real) differentiable manifold such that multiplication and inversion are smooth maps. The *Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ is the space of left-invariant vector fields on G; it is stable under the usual Lie-bracket of vector fields. Evaluation at the unit element e yields a vector space-isomorphism $\mathfrak{g} \to T_eG$. Then the differential at e of a homomorphism of Lie groups becomes a homomorphism of Lie algebras; thus we get a functor

$$\mathrm{Lie} : G \mapsto \mathfrak{g} = \mathrm{Lie}(G)$$

which we call the *Lie functor for Lie groups.* It is well-known that for any finite-dimensional real Lie algebra $\mathfrak{g}$ one can find a (unique) connected simply connected Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}$, and that in this sense the Lie functor is an equivalence of categories. If we don't want to use connected simply connected groups, we can replace them by the category of *germs of Lie groups.*

Complex Lie groups are defined as the real ones, but using the structure of a complex manifold instead of a real one. The corresponding Lie functor is a functor from complex Lie groups to complex Lie algebras. We may consider complex Lie groups as real Lie groups with an additional structure; thus we consider its Lie algebra as an algebra of real vector fields.

1.2. Symmetric pairs and equivariant maps. A *symmetric pair* (G, σ, H) is given by a Lie group G, a non-trivial involution σ of G and an open subgroup H of the fixed point group of σ. An *equivariant map of symmetric pairs* is a Lie group homomorphism respecting the given involutions and subgroups.

The infinitesimal version of a symmetric pair is a *symmetric Lie algebra* $(\mathfrak{g}, \dot\sigma)$, i.e. a Lie algebra $\mathfrak{g}$ together with a non-trivial involution $\dot\sigma$ of $\mathfrak{g}$. The corresponding homomorphisms are Lie algebra homomorphisms commuting with the respective involutions; we call them *equivariant.*

It is clear that the Lie functor yields a functor from symmetric pairs to symmetric Lie algebras, and on the level of *germs of symmetric pairs* it is an equivalence of categories. In this context we speak also of the homomorphisms as *germs of equivariant maps.*

1.3. Symmetric spaces and Lie triple systems. Following definition (1) mentioned in the introduction of this chapter, we say that, if (G, σ, H) is a symmetric pair, a *(homogeneous)* *symmetric space* is the associated homogeneous space $M = G/H$. The point $o := eH \in M$ is called the *base point*.

Every Lie group G can be considered as a symmetric space: let $G \times G$ act on G by $(g,h)\cdot x = gxh^{-1}$; then $G \cong G \times G/dia$, where $dia = \{(g,g)|g \in G\}$ is the diagonal subgroup, which is the fixed point group of the involution $(g,h) \mapsto (h,g)$.

The Lie functor for symmetric spaces is defined as follows: let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$ under the differential $\dot\sigma$ of the involution σ at the origin. Since $\dot\sigma$ is an involution of $\mathfrak{g}$, the space $\mathfrak{h} = \mathfrak{g}^{\dot\sigma}$ is a subalgebra of $\mathfrak{g}$ (it is the Lie algebra of H), and the -1-eigenspace $\mathfrak{q} = \mathfrak{g}^{-\dot\sigma}$ satisfies the rules $[\mathfrak{q},\mathfrak{q}] \subset \mathfrak{h}$, $[\mathfrak{h},\mathfrak{q}] \subset \mathfrak{q}$. This implies that

$$[[\mathfrak{q},\mathfrak{q}],\mathfrak{q}] \subset \mathfrak{q}.$$

For $X,Y,Z \in \mathfrak{q}$ we let

$$[X,Y,Z] := [[X,Y],Z], \quad R(X,Y)Z := -[X,Y,Z].$$

(In the next section we will interprete R as a curvature tensor.) Then for all $U,V,X,Y,Z \in \mathfrak{q}$ the following holds:

(LT1) $R(X,Y) = -R(Y,X)$,
(LT2) $[X,Y,Z] + [Y,Z,X] + [Z,X,Y] = 0$,
(LT3) $R(U,V)[X,Y,Z] = [R(U,V)X,Y,Z] + [X,R(U,V)Y,Z] + [X,Y,R(U,V)Z]$.

In fact, (LT1) is clear, (LT2) is just the Jacobi identity and (LT3) means that $R(U,V)$ is a *derivation* of the trilinear composition $[\cdot,\cdot,\cdot] : \mathfrak{q}\times\mathfrak{q}\times\mathfrak{q} \to \mathfrak{q}$. This follows easily from the fact that $\mathrm{ad}[U,V]$ is already a derivation of the bilinear composition $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, $(X,Y) \mapsto [X,Y]$.

Definition I.1.1. A vector space $\mathfrak{q}$ together with a trilinear map $[\cdot,\cdot,\cdot] : \mathfrak{q}\times\mathfrak{q}\times\mathfrak{q} \to \mathfrak{q}$ (which is equivalent to a bilinear map $R : \mathfrak{q}\times\mathfrak{q} \to \mathrm{End}(\mathfrak{q})$ defined by $R(X,Y)Z = -[X,Y,Z]$) having the properties (LT1), (LT2) and (LT3) is called a *Lie triple system* (LTS). A *homomorphism of Lie triple systems* is a linear map compatible with the respective triple products. ■

Summing up, we have associated to a symmetric space $M = G/H$ a LTS $\mathfrak{q}$. (We follow here a notation widely used, although the notation $\mathfrak{m}$ instead of $\mathfrak{q}$ would be more consequent.)

Example I.1.2. (The group case.) We realize a Lie group G as the symmetric space $G = G \times G/dia$ as explained above. Its LTS $\mathfrak{q}$ is

$$\mathfrak{q} = \{(X,-X)|\, X \in \mathfrak{g}\} \subset \mathfrak{g}\times\mathfrak{g}.$$

It is often convenient to identify this space with $\mathfrak{g}$ via $X \mapsto \frac{1}{2}(X,-X)$; in other words, the LTS $\mathfrak{q}$ is identified with the LTS defined on $\mathfrak{g}$ by $[X,Y,Z] = \frac{1}{4}[[X,Y],Z]$. (The factor $\frac{1}{2}$ is necessary in order to make this identification compatible with the evaluation of vector fields at the base point; cf. Section I.4). ■

Theorem I.1.3. *If $\mathfrak{q}$ is a finite-dimensional real LTS, then there exists a connected simply connected symmetric space M with associated LTS $\mathfrak{q}$.*

Proof. First we associate a Lie algebra $\mathfrak{g}$ to a LTS $\mathfrak{q}$: Let $\mathfrak{h} := \mathrm{Der}(\mathfrak{q}) \subset \mathrm{End}(\mathfrak{q})$ be the space of derivations of the Lie triple product on $\mathfrak{q}$; this is a Lie algebra. On the direct sum

$$\mathfrak{g} := \mathrm{Der}(\mathfrak{q}) \oplus \mathfrak{q}$$

we define a bilinear product $[\cdot,\cdot]$ by

$$[F,H] := F\circ H - H\circ F, \quad [H,X] := HX, \quad [X,H] := -[H,X], \quad [X,Y] := R(X,Y)$$

for $F, H \in \mathrm{Der}(\mathfrak{q})$, $X, Y \in \mathfrak{q}$. It clearly is antiymmetric and satisfies the Jacobi-identity because of (LT2); thus $\mathfrak{g}$ is a Lie algebra. Property (LT3) assures that $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$; since $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}$ are clear, the linear map

$$\sigma : \mathfrak{g} \to \mathfrak{g}, \quad (H, X) \mapsto (H, -X)$$

is an involution of the Lie algebra $\mathfrak{g}$ such that the abstract LTS $\mathfrak{q}$ is realized as the sub-LTS $\mathfrak{g}^{-\sigma}$ of $\mathfrak{g}$.

Now let G be the unique connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and lift σ to an involution σ of G. Then the fixed point group $H = G^\sigma$ is connected (cf. [Lo69a]), and $M := G/H$ is a connected simply connected symmetric space; clearly its associated LTS is $\mathfrak{q}$. ■

Definition I.1.4. If $\mathfrak{q}$ is a LTS, then the algebra

$$\mathfrak{g}(\mathfrak{q}) := [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q} \subset \mathrm{Der}(\mathfrak{q}) \oplus \mathfrak{q}$$

(cf. the preceding proof) is called the *standard-imbedding* of $\mathfrak{q}$. If $M = G/H$ is a connected symmetric space and $\mathfrak{q}$ the associated LTS, then we call $\mathfrak{g}(\mathfrak{q})$ the *displacement algebra of M*. The expression $M = G/H$ is called *prime* if G is a connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}(\mathfrak{q})$. ■

The space M constructed in the proof of Theorem I.1.3 becomes unique if we require $M = G/H$ to be prime. As an example of a non-prime expression one may take a vector space V written as a quotient $V = \mathrm{Aff}(V)/\,\mathrm{Gl}(V)$ of the affine group modulo the general linear group. The prime expression is simply $V = V/\{0\}$.

It is not advisable to restrict attention only to connected simply connected symmetric spaces. Therefore we will give a local version of the preceding theorem.

Definition I.1.5. An *equivariant map of symmetric spaces* is given by an equivariant map of the corresponding symmetric pairs (cf. Section I.1.2). Two symmetric spaces are called *locally isomorphic* if the associated germs of symmetric pairs are isomorphic. *Germs of symmetric spaces* are the isomorphism classes of symmetric spaces under the equivalence relation given by local isomorphism. *Equivariant maps of germs of symmetric spaces* are the equivariant maps of the corresponding symmetric Lie algebras. ■

Theorem I.1.6.

(1) *The map associating to a symmetric space its LTS induces a bijection from the class of germs of symmetric spaces in prime expression onto the isomorphism classes of finite-dimensional real Lie triple systems.*

(2) *The correspondence from* (1) *is a functor from the category of germs of symmetric spaces and their equivariant maps onto the category of equivalence classes of Lie triple systems.*

Proof. (1) The map is well-defined: if two symmetric spaces are locally isomorphic, then clearly the associated Lie triple systems are isomorphic. We construct an inverse map: given a LTS $\mathfrak{q}$, let M be the germ of the connected simply connected space defined in the previous theorem. It is easily checked that both constructions are inverses of each other.

(2) If φ is an equivariant map, then the differential $\dot{\varphi}$ is a homomorphism of symmetric pairs; in particular, its restriction to $\mathfrak{q}$ is a homomorphism of Lie triple systems. ■

The bijection from Part (1) of the preceding theorem is *not* an equivalence of categories. In other words, the converse of statement (2) does not always hold. We will see in the next chapter that LTS-homomorphisms are in fact equivalent to affine maps $M \to M'$ of germs of symmetric spaces. In general, maps between manifolds do *not* induce homomorphisms of groups operating on them, and nor do affine maps of symmetric spaces. It is all the more remarkable that they "nearly" do (cf. Prop. I.3.4 below).

1.4. Complex symmetric spaces and c-duality. *Complex symmetric spaces* are defined similarly as real ones, but using complex Lie groups and holomorphic involutions. *Complex* Lie triple systems are $\mathbb{C}$-trilinear maps satisfying (LT1) – (LT3). Then the complex analog of Theorem I.1.6 holds.

If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the decomposition of a symmetric Lie algebra, then one defines the *c-dual symmetric Lie algebra* as the subalgebra $\mathfrak{g}^c := \mathfrak{h} \oplus i\mathfrak{q}$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. If $\mathfrak{g}$ is the real form of $\mathfrak{g}_{\mathbb{C}}$ fixed under the conjugation τ, then $\mathfrak{g}^c$ is the real form fixed under $\tau\sigma_{\mathbb{C}} = \sigma_{\mathbb{C}}\tau$ where $\sigma_{\mathbb{C}}$ is the $\mathbb{C}$-linear extension of σ. Since $[iX, iY, iZ] = -i[X, Y, Z]$ in $\mathfrak{g}_{\mathbb{C}}$, the LTS $i\mathfrak{q}$ is isomorphic to the LTS $\mathfrak{q}$ equipped with the *negative* of its original Lie triple product. The germs of the symmetric spaces M and M^c corresponding to the pairs $\mathfrak{g}$ and $\mathfrak{g}^c$ can be considered as different real forms of the complex germ $M_{\mathbb{C}}$ corresponding to the complex pair $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{q}_{\mathbb{C}}$. Note that (although c-dual LTS can be realized on the *same* vector space) there exists no canonical realization (not even locally) of c-dual germs of symmetric spaces on the same underlying manifold. However, there is an important special case in which such a realization (the so-called *Borel-imbedding*) exists. This will be generalized in Section X.1.

2. Lie-functor: differential geometric version

2.1. The canonical vector field extension and the canonical connection of a symmetric space. A *vector field extension of a tangent vector* $v \in T_pM$ is a vector field $\mathbf{v}$ such that $\mathbf{v}_p = v$ (Def. I.A.1). If $M = G/H$ is a symmetric space (assumed to be connected) and $p \in M$, then every tangent vector $v \in T_pM$ has a *canonical vector field extension* denoted by $l_p v$ which is defined as follows: for $p = g.o \in M$ denote by $\mathfrak{g} = \mathfrak{h}^p \oplus \mathfrak{q}^p$ the decomposition of the Lie algebra $\mathfrak{g}$ of G w.r.t the involution $\sigma_p := g_* \circ \sigma \circ g_*^{-1}$ ($g_*(x) = gxg^{-1}$) of G. Then the evaluation map

$$\mathrm{ev}_p : \mathfrak{q}^p \to T_pM, \quad X \mapsto X_p$$

is bijective (in fact, since G operates transitively on M, the map $G \to M$, $g \mapsto g \cdot p$ is a submersion; i.e. $\mathfrak{g} \to T_pM$, $X \mapsto X_p$ is surjective, and $\mathfrak{h}^p$ is its kernel). We denote by

$$l_p : T_pM \to \mathfrak{q}^p \subset \mathfrak{X}(M)$$

its inverse; then $l_p v$ clearly is a vector field extension of v. Next we use the canonical vector field extension in order to define the *canonical connection of a symmetric space* (cf. Appendix B to this chapter for definition and elementary properties of affine connections on real manifolds).

Proposition I.2.1. *Let M be a symmetric space and l_p be defined as above. Then the formula*

$$(\nabla_X Y)_p := [l_p(X_p), Y]_p, \quad (X, Y \in \mathfrak{X}(M))$$

defines a torsionfree connection on M.

Proof. The defining relations of a connection are immediately verified. In order to prove that the torsion tensor $T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ vanishes at an arbitrary point $p \in M$, we use the tensor nature of T_∇: w.l.o.g. we may assume that $X = l_p X_p$, $Y = l_p Y_p$. Inserting this in the formulas for T_∇ and ∇, we get

$$(T_\nabla)_p(X_p, Y_p) = (\nabla_X Y)_p - (\nabla_Y X)_p - [X, Y]_p = [X, Y]_p - [Y, X]_p - [X, Y]_p = 0$$

since $[X, Y] \in \mathfrak{h}^p$ and thus $[X, Y]_p = 0$. ∎

(Remark: The principle of the proposition can be generalized – in fact *any* affine connection can be defined by a formula of the kind $(\nabla_X Y) = [l_p X_p, Y]_p$ with a suitable family $(l_p)_{p\in M}$ of vector field extensions defined by *adapted vector fields*, cf. [Hel78, p.36], [Lo69a, p.105].)

Recall from Appendix B.4 that a diffeomorphism g is called *affine* if $g_*\nabla = \nabla$, and a vector field X is called *affine* if $X \cdot \nabla = 0$.

Lemma I.2.2. *The group G is a group of affine diffeomorphisms, and the Lie algebra $\mathfrak{g}$ is a Lie algebra of affine vector fields w.r.t. the canonical connection defined in the previous proposition.*

Proof. The second statement is just the differentiated version of the first one, which is proved by a straightforward calculation, using that $l_{gp}(T_p g \cdot X_p) = g_*(l_p X_p)$:

$$\begin{aligned}
((g_*\nabla)_X Y)_p &= T_{g^{-1}p}g \cdot (\nabla_{T_{g^{-1}p}g \cdot X_p} g^* Y)_{g^{-1}p} \\
&= T_{g^{-1}p}g[l_{g^{-1}p}T_{g^{-1}p}g \cdot X_p, g^*Y]_{g^{-1}p} \\
&= T_{g^{-1}p}g[g^*(l_p X_p), g^*Y]_{g^{-1}p} \\
&= (g_*[g^*(l_p X_p), g^*Y])_p \\
&= [l_p X_p, Y]_p = (\nabla_X Y)_p.
\end{aligned}$$

■

2.2. The curvature tensor and the Lie functor. We continue to assume that M is a homogeneous symmetric space and ∇ its canonical connection defined by Prop. I.2.1.

Lemma I.2.3. *The covariant derivative of a tensor field S is given by*

$$(\nabla_X S)_p = ((l_p X_p) \cdot S)_p,$$

where $Z \cdot S$ is the ordinary Lie derivative.

Proof. Recall from [Hel78, p.42] that the covariant derivative ∇_X is uniquely determined by its values on $\mathfrak{X}(M)$ and on $\mathcal{C}^\infty(M)$ and by the property that it is a derivation of the mixed tensor algebra preserving types and commuting with contractions. It is clear that the right hand side of the stated formula has these properties; it coincides with the covariant derivative on $\mathfrak{X}(M)$ and on $\mathcal{C}^\infty(M)$ and thus coincides with the covariant derivative on the whole tensor algebra. ■

Lemma I.2.4. *For all smooth functions f on M, $p \in M$ and $X, Y \in \mathfrak{q}^p$,*

$$(\nabla \mathrm{d}f)(X,Y)_p = (\nabla \mathrm{d}f)(Y,X)_p = (XYf)(p),$$

$$(\nabla\nabla \mathrm{d}f)(X,Y,Z)_p = (XYZ\,f - [Y,[X,Z]]\,f)(p).$$

Proof. By the rules of the covariant derivative, $(\nabla \mathrm{d}f)(X,Y) = (XY - \nabla_X Y)f$ (cf. Appendix B.2), and $(\nabla \mathrm{d}f)(X\otimes Y - Y \otimes X) = T_\nabla(X,Y)f$. Thus the first equation follows from the fact that ∇ is torsionfree. For the second one, we use directly the definition of ∇ in Prop. I.2.1:

$$(XY - \nabla_X Y)_p = (XY)_p - [X,Y]_p = (XY)_p$$

since $[X,Y]_p \in \mathfrak{h}^p = 0$. Using the same argument and Lemma I.2.3 along with $l_p X_p = X$, we get further

$$\begin{aligned}
(\nabla\nabla \mathrm{d}f)(X,Y,Z)_p &= X_p(\nabla \mathrm{d}f)(Y,Z) - (\nabla \mathrm{d}f)_p([l_p X_p, Y]_p \otimes Z_p + Y_p \otimes [l_p X_p, Z]_p) \\
&= (X(YZ - \nabla_Y Z))_p f + 0 \\
&= (X_p YZ - [X, \nabla_Y Z]_p) f \\
&= (X_p YZ - [Y,[X,Z]]_p) f.
\end{aligned}$$

For the last equality we have used that $X \cdot \nabla = 0$ (Lemma I.2.2; note that $X \in \mathfrak{g}$), and therefore

$$[X, \nabla_Y Z]_p = (\nabla_{[X,Y]} Z + \nabla_Y [X, Z])_p = 0 + [l_p Y_p, [X, Z]]_p = [Y, [X, Z]]_p. \qquad \blacksquare$$

In a more global way, the preceding lemma may be written

$$(\nabla \mathrm{d} f)(X, Y)_p = (\nabla \mathrm{d} f)(Y, X)_p = X_p(l_p Y_p f),$$

$$(\nabla\nabla \mathrm{d} f)(X, Y, Z)_p = X_p(l_p Y_p \circ l_p Z_p) f - [l_p Y_p, [l_p X_p, l_p Z_p]]_p f$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and $p \in M$. These formulas permit to calculate the *curvature tensor* $R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z$ (cf. Appendix B.3).

Corollary I.2.5. *The curvature tensor at $p \in M$ is given by*

$$(R(X, Y)Z)_p = -[[l_p X_p, l_p Y_p], l_p Z_p]_p.$$

Proof. For any torsionfree connection we have the equation

$$(\nabla\nabla \mathrm{d} f)(X \otimes Y \otimes Z - Y \otimes X \otimes Z) = -R(X, Y)Z \cdot f,$$

cf. Appendix B.3. By the preceding lemma, the left hand side, evaluated at p for $X, Y, Z \in \mathfrak{q}^p$, equals

$$(XYZ - YXZ)_p - ([Y, [X, Z]] - [X, [Y, Z]])_p = [X, Y]_p \circ Z - [[Y, X], Z]_p = [[X, Y], Z]_p,$$

whence $(R(X, Y)Z)_p = -[[X, Y], Z]_p$. $\blacksquare$

Comparison with the definition of the associated LTS of a symmetric space shows now that the curvature tensor at a fixed point defines a LTS on the corresponding tangent space. If we fix a base point o, then the Lie functor from the preceding section can now be interpreted as assigning to M the tangent space $T_o M$ together with the Lie triple product $-R_o$.

Proposition I.2.6. *The curvature tensor is covariantly constant:*

$$\nabla R = 0.$$

Proof. From the equation $g_* \nabla = \nabla$ it follows immediately that $g_* R = R$. This holds for all $g \in G$, and differentiating we obtain $X \cdot R = 0$ for $X \in \mathfrak{g}$. But then, using Lemma I.2.3,

$$(\nabla_X R)_p = (l_p X_p \cdot R)_p = 0$$

since $l_p X_p \in \mathfrak{q}^p \subset \mathfrak{g}$. $\blacksquare$

Definition I.2.7. A manifold M with an affine connection ∇ is called *affine locally symmetric* if ∇ is torsionfree and the condition $\nabla R = 0$ holds. $\blacksquare$

Summing up, we have shown that every symmetric space with the canonical connection from Prop. I.2.1 is an affine locally symmetric space. Conversely, every affine locally symmetric space is locally isomorphic to a symmetric space (cf. [KoNo69, Ch.XI., Th.1.1 and Th.1.5], [Lo69a, Th.II.4.9]).

2.3. Homomorphisms of symmetric spaces.

Definition I.2.8. A *homomorphism of symmetric spaces* is a smooth map between symmetric spaces which is affine w.r.t. the corresponding canonical connections (cf. Def. I.B.7); if the spaces in question have base points, then homomorphisms are required to preserve them. *Homomorphisms of germs of symmetric spaces* are defined as germs of homomorphisms of symmetric spaces. ■

Since the composition of affine maps is again affine, symmetric spaces and germs of symmetric spaces with their homomorphisms form a category.

Theorem I.2.9.

(i) *The category of connected simply connected symmetric spaces with base point is equivalent to the category of finite dimensional real Lie triple systems. The equivalence is given by evaluating the curvature tensor at the base point.*

(ii) *The category of germs of symmetric spaces is equivalent to the category of finite dimensional real Lie triple systems.*

Proof. Evaluation of the curvature at the base point is a functor since affine maps are compatible with the respective curvature tensors. According to Theorem I.1.3, for any LTS $(\mathfrak{q}, R_o)$ we can find a connected simply connected symmetric space whose curvature tensor has value R_o at the base point. This space and the corresponding germ are uniquely determined by R_o. It only remains to be shown that any LTS-homomorphism $\dot{\varphi} : T_oM \to T_{o'}M'$ has an extension to an affine map $\varphi : M \to M'$ of the corresponding (germs of) symmetric spaces.

In [Lo69a, Th.II.4.6] (cf. also [Hel78, Lemma IV.1.2]) it is proved that, since $\dot{\varphi}$ is compatible with the respective curvature tensors, the formula

$$\varphi(\operatorname{Exp} v) := \operatorname{Exp}'(\dot{\varphi} v)$$

(where Exp and Exp$'$ are the exponential maps of ∇ resp. ∇' at o resp. o') defines a local extension of $\dot{\varphi}$ to an affine map. This proves part (ii). In order to prove part (i), one shows that, if M is connected simply connected, φ extends to a global affine map $M \to M'$ (cf. [Lo69a, Th.II.4.12]). ■

Corollary I.2.10. *Any automorphism of a LTS $\mathfrak{q}$ has a unique extension to an automorphism of the corresponding germ of a symmetric space.* ■

Corollary I.2.11. *Under the correspondence* (ii) *of the preceding theorem, equivariant maps of germs of symmetric spaces are precisely the LTS-homomorphisms $\mathfrak{q} \to \mathfrak{q}'$ having an extension to a Lie algebra homomorphism $\mathfrak{g}(\mathfrak{q}) \to \mathfrak{g}(\mathfrak{q}')$ of the corresponding standard-imbeddings.* ■

3. Symmetries and the group of displacements

We are going to explain definition (3) of a symmetric space mentioned in the introduction of this chapter.

Lemma I.3.1. *If $M = G/H$ is a symmetric space, then the symmetry*

$$s_o : M \to M, \quad g.o \mapsto \sigma(g).o$$

is an automorphism of M.

Proof. We note first that s_o is well-defined; in fact, $\sigma(gh).o = \sigma(g).o$ for all $h \in H$. Now the property $s_o(g.x) = \sigma(g).s_o(x)$ shows that s_o is equivariant. According to Cor. I.2.11, it is a homomorphism $M \to M$. ■

The symmetry s_p w.r.t. the point $p = g.o \in M$ is defined by

$$s_p = g \circ s_o \circ g^{-1};$$

it is again affine w.r.t. the canonical connection. It is an *extension* of the tangent map $-\mathbf{1}_p := -\operatorname{id}_{T_pM}$ in the sense of Definition I.A.1.

Theorem I.3.2. *For an affine connection ∇ on a manifold M the following are equivalent:*

(1) *∇ is affine locally symmetric (cf.* Def. I.2.7*).*

(2) *For all $p \in M$ the tangent map $-\mathbf{1}_p$ extends to a locally defined affine diffeomorphism s_p of M.*

Proof. Assume (2) holds. Since s_p is affine, the induced action $(s_p)_*$ of s_p on the mixed tensor algebra preserves the torsion, the curvature R and the tensor field ∇R. In particlar, the tangent map $T_p(s_p) = -\mathbf{1}_p$ is an automorphism of these tensor fields at the point p. The induced action of $-\mathbf{1}_p$ on a tensor $S_p : \otimes^k(T_pM) \to \otimes^l T_pM$ is given by

$$-\mathbf{1}_p \cdot S = \otimes^l(-\mathbf{1}_p) \circ S_p \circ \otimes^k(-\mathbf{1}_p) = (-1)^{k+l} S_p.$$

If S is the torsion tensor, then $l = 1, k = 2$, and thus $-\mathbf{1}_p$ preserves S_p iff $S_p = 0$, and similarly with $S = \nabla R$ ($l = 1, k = 4$). Since this holds for all $p \in M$, it follows that ∇ is torsionfree and that $\nabla R = 0$, whence (1).

Assume (1) holds. Since clearly $-\mathbf{1}_p$ is an automorphism of R_p, Cor. I.2.10 shows that $-\mathbf{1}_p$ has an extension to a local affine diffeomorphism. ∎

Definition I.3.3. If M is a symmetric space, the group $G(M)$ generated by all $s_p s_q$, $p, q \in M$, is called its *group of displacements.* ∎

Proposition I.3.4. *Assume $M = G/H$ is a connected globally symmetric space such that G acts effectively on M. Then $G(M)$ is a subgroup of G acting transitively on M. Its Lie algebra is the standard imbedding $[\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$ of the associated LTS $\mathfrak{q}$.*

Proof. If $p = g.o, q = h.o$, $g, h \in G$, then

$$s_p s_q = g s_o g^{-1} h s_o h^{-1} = g\sigma(g)^{-1}\sigma(h)h^{-1} \in G.$$

Taking $q = o$ and $g = \exp(X)$, $X \in \mathfrak{q}$, we get $s_p s_o = \exp(2X)$, proving that $\exp(\mathfrak{q}) \subset G(M)$. Moreover, since the subgroup generated by $\exp(\mathfrak{q})$ is transitive on M, it is easily seen that $G(M)$ is this subgroup (and is thus transitive on M). On the other hand, standard arguments (cf. [Lo69a, Lemma I.3.2]) show that the analytic subgroup of G whose Lie algebra is the standard imbedding of $\mathfrak{q}$ is the group generated by $\exp(\mathfrak{q})$ and is thus equal to $G(M)$. ∎

As pointed out already several times, homomorphisms of symmetric spaces do in general not induce homomorphisms of the corresponding groups of displacements, but a weaker statement holds:

Proposition I.3.5.

(i) *If $\varphi : M \to M'$ is a homomorphism of symmetric spaces, then for all $p \in M$,*

$$\varphi \circ s_p = s_{\varphi(p)} \circ \varphi.$$

(ii) *If $\varphi : M \to M'$ is a homomorphism of connected symmetric spaces and $g \in G(M)$, then there exists $g' \in G(M')$ such that*

$$\varphi \circ g = g' \circ \varphi.$$

Proof. (i) Both sides of the stated equation are affine maps sending p to $\varphi(p)$ and having tangent map $-T_p\varphi$ at p. By the uniqueness of affine extensions, they coincide.

(ii) Just write an element of $G(M)$ as a composition of symmetries and apply (i). ∎

Note that the decomposition of an element $g \in G(M)$ is not unique, and therefore the element g' from part (ii) is in general not uniquely determined. If it is unique for all g, then it follows immediately that $g \mapsto g'$ is indeed a well-defined homomorphism $G(M) \to G(M')$. The failure of uniqueness can in the general case be measured by introducing the group

$$G(\varphi) := \{(g,h) \in G(M) \times G(M') | \, \forall x \in M : \, \varphi(g.x) = h.\varphi(x)\} \subset G(M) \times G(M').$$

Part (ii) of the preceding proposition says that projection onto the first factor yields a surjection of $G(\varphi)$ onto $G(M)$.

Finally we mention that the map s_p can be interpreted as *geodesic symmetry* w.r.t. the point p. Recall from Appendix I.B.6 the definition of the *exponential map* $\mathrm{Exp} = \mathrm{Exp}_p : T_pM \to M$ of the connection ∇ w.r.t. the point p.

Corollary I.3.6. *For all $v \in T_pM$,*

$$s_p(\mathrm{Exp}(v)) = \mathrm{Exp}(-v).$$

Proof. Note that the symmetry s_0 of the symmetric space $\mathbb{R}$ is just change of sign and apply part (i) of the preceding proposition to the affine map $\mathbb{R} \to M$, $t \mapsto \mathrm{Exp}(tv)$. ∎

4. The multiplication map

We explain definition (4) of a symmetric space. To any symmetric space M we associate its *multiplication map*

$$\mu : M \times M \to M, \quad (x,y) \mapsto \mu(x,y) := s_x(y),$$

where the symmetries s_x as are as in the preceding section.

Example I.4.1. If $M = G$ is a Lie group and $o = e$, then we have $s_o(y) = y^{-1}$ and thus

$$\mu(x,y) = s_x(y) = l_x s_o l_x^{-1}(y) = x(x^{-1}y)^{-1} = xy^{-1}x.$$ ∎

Lemma I.4.2. *The following holds for all $x,y,z \in M$:*

(M1) $\mu(x,x) = x$

(M2) $\mu(x,\mu(x,y)) = y$

(M3) $\mu(x,\mu(y,z)) = \mu(\mu(x,y),\mu(x,z))$

(M4) *x is an isolated fixed point of $\mu(x,\cdot)$.*

Proof. (M1) rephrases that x is a fixed point of s_x; (M4) follows from the fact that $T_x(s_x) = -\mathbf{1}_x$; (M2) is just $(s_x)^2 = \mathrm{id}_M$, and (M3) is equivalent to

(M3') $\mu(s_x y, s_x z) = s_x\mu(y,z)$,

which means that s_x is an automorphism of μ. But it is clear that every affine diffeomorphism of M is an automorphism of μ, and since s_x is affine, (M3') holds. ∎

From Prop. I.3.5 (i) it follows that homomorphisms φ of connected symmetric spaces are compatible with multiplication maps, i.e.

$$\varphi \circ \mu = \mu' \circ (\varphi \times \varphi).$$

The following theorem is due to O. Loos (cf. [Lo69a, Chapter II]):

Theorem I.4.3. *A connected manifold M with a smooth multiplication map μ having properties* (M1) – (M4) *carries a natural structure of a symmetric space such that μ coincides with the multiplication map defined above. Homomorphisms $\varphi : M \to M'$ between connected symmetric spaces are precisely the smooth maps compatible with multiplication maps.* ■

The main step in the proof of this theorem is the construction of a canonical connection associated to a multiplication map and the verification that this connection is affine symmetric (cf. Def. I.2.7); then a version of Theorem I.2.9 becomes available in order to prove that the group generated by the $s_x s_y$ (where $s_x(y) := \mu(x, y)$) is a Lie group acting transitively on M.

The theory of symmetric spaces based on the product μ can be developed similarly to the theory of Lie groups based on the ordinary product, cf. [Lo69a]. In [Lo67] the more general situation obtained by omitting (M4) is considered. The spaces thus obtained, called *Spiegelungsräume* (reflection spaces), are fiber bundles over symmetric spaces.

Summarizing, one may give the following definitions:

Definition I.4.4.

(1) A *topological symmetric space* is a topological space M together with a continuous map $\mu : M \times M \to M$ satisfying (M1) – (M4).

(2) A *symmetric space* is a manifold M with a smooth map $\mu : M \times M \to M$ satisfying (M1) – (M4).

(3) A *real algebraic symmetric space* is a manifold M with a rational atlas (i.e. the transition functions are rational) and a map $\mu : M \times M \to M$ which is rational w.r.t. this atlas and satisfies (M1) – (M4).

(4) *Complex symmetric spaces* and *complex algebraic symmetric spaces* are defined similarly, using complex manifolds and holomorphic (resp. complex-rational) charts and multiplication maps. ■

Our definition of an algebraic symmetric space may seem a bit unusual from the point of view of contemporary algebraic geometry. However, we do not intend to develop a general theory of algebraic symmetric spaces. Rather, we are interested in a class of symmetric spaces defined by Jordan structures; they have a canonical algebraic atlas, and the algebraic formula for the multiplication map can be calculated explicitly (cf. Prop. II.2.9 and Th. X.3.2). The definition of algebraic symmetric spaces we have chosen is adapted to this situation.

If G is a group and we let $\mu(g, h) := gh^{-1}g$, then the properties (M1), (M2) and (M3) are easily verified. Thus any group with the discrete topology is a topological symmetric space, and any subset closed under μ is again a topological symmetric space. Subspaces are defined analogously to subgroups of groups. For instance, if $g \mapsto g^*$ is an anti-automorphism of a group G, then it is an automorphism of G as a symmetric space, and therefore the *space of $*$-symmetric elements*

$$S := \{g \in G | \, g = g^*\}$$

is a subspace of G. Spaces of this kind play an important role.

Lemma I.4.5. *Assume G is a Lie group and $*$ an involutive anti-automorphism of G. The group G acts on S by the formula $g.x = gxg^*$. Then every connected component of S is homogeneous under this action.*

Proof. The action is well-defined since $(gxg^*)^* = gxg^*$. Let S_o be a connected component of S. Since it is a connected symmetric space, Prop. I.3.4 shows that the group $G(S_o)$ acts transitively on S_o. Since

$$s_x s_y(z) = xy^{-1}zy^{-1}x = (xy^{-1})z(xy^{-1})^* = (xy^{-1}).z$$

and $G(S_o)$ is generated by the $s_x s_y$ with $x, y \in S_o$, S_o is contained in the orbit $G.z$ for any $z \in S_o$. ■

5. Representations of symmetric spaces

Definition I.5.1. A *representation of a symmetric space* M on a topological vector space V is a homomorphism $\pi : M \to \mathrm{Gl}(V)$ (i.e., we have $\pi(\mu(xy)) = \pi(x)\pi(y)^{-1}\pi(x)$ for all $x, y \in M$) such that the map

$$M \times V \to V, \quad (x, v) \mapsto \pi(x)v$$

is continuous. A *self-adjoint representation of a symmetric space* is a homomorphism of M into the symmetric space of self-adjoint (w.r.t. to a suitable bilinear form) elements of $\mathrm{Gl}(V)$. More generally, if $*$ is an involution of the symmetric space $\mathrm{Gl}(V)$, then a $*$*-representation* of M is a homomorphism into the space $S = \{g \in \mathrm{Gl}(V) | g = g^*\}$. *Intertwining operators* are defined in the obvious way; we call them also M*-homomorphisms.* ■

It is clear that group homomorphisms are also homomorphisms of symmetric spaces (in fact, one may verify that they are even equivariant maps of symmetric spaces), and thus we get

Lemma I.5.2. *A representation $\pi : G \to \mathrm{Gl}(V)$ of a Lie group is a representation of G considered as a symmetric space.* ■

The quadratic map. This map plays a similar role as the adjoint representation in the theory of Lie groups. Recall that, multiplied by a factor 2, the standard-imbedding $\mathfrak{q} \to \mathfrak{g} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$ is a homomorphism of Lie triple systems when we consider $\mathfrak{g}$ as in Ex. I.1.2 as a LTS. The global version of this homomorphism is as follows:

Proposition I.5.3.

(i) *For any symmetric space M, the map*

$$Q : M \to G(M), \quad p \mapsto s_p s_o$$

is a $\mathrm{Aut}(M)$-equivariant homomorphism of symmetric spaces. Its image is contained in the space

$$S := \{g \in G(M) |\, g^{-1} = \sigma(g)\},$$

and $Q : M \to S$ is a local isomorphism.

(ii) *If we identify $\mathfrak{g}$ with the antidiagonal in $\mathfrak{g} \times \mathfrak{g}$ via $X \mapsto \frac{1}{2}(X, -X)$, then the differential of Q at the origin is the standard-imbedding multiplied by a factor 2:*

$$\dot{Q} : \mathfrak{q} \to \mathfrak{g}, \quad X \mapsto 2X.$$

Proof. (i) We show that Q is $\mathrm{Aut}(M)$-equivariant and hence a homomorphism: The involution of $\mathrm{Aut}(M)$ is given by $\sigma(g) = s_o g s_o$, and the involution of $\mathrm{Aut}(M) \times \mathrm{Aut}(M)$ by $\sigma'((g, h)) = (h, g)$. Since for all $g \in \mathrm{Aut}(M)$,

$$Q(g.x) = s_{g.x} s_o = g s_x g^{-1} s_o = g s_x s_o \sigma(g)^{-1} = g Q(x) \sigma(g)^{-1},$$

the map Q belongs to the homomorphism

$$\widetilde{Q} : \mathrm{Aut}(M) \to \mathrm{Aut}(M) \times \mathrm{Aut}(M), \quad g \mapsto (g, \sigma(g))$$

which clearly satisfies $\widetilde{Q} \circ \sigma = \sigma' \circ \widetilde{Q}$, and therefore Q is equivariant.

The calculation

$$\sigma(Q(x)^{-1}) = s_o(s_p s_o)^{-1} s_o = s_p^{-1} s_o = s_p s_o = Q(x)$$

shows that $Q(M) \subset S$. Since the tangent space of S at the identity is precisely the subspace $\mathfrak{q}$ of $\mathfrak{g}$, the last statement will follow from part (ii).

(ii) We differentiate $\widetilde{Q}$ at the origin and restrict to $\mathfrak{q}$. We obtain

$$\dot{Q} : \mathfrak{q} \to \mathfrak{g} \times \mathfrak{g}, \quad X \mapsto (X, \dot{\sigma}(X)) = (X, -X).$$

Under the identification given in the lemma, this proves the claim. ∎

Part (i) may also be verified by proving that $M \to \mathrm{Aut}(M)$, $x \mapsto s_x$ is a homomorphism of symmetric spaces; $\mathrm{Aut}(M) \to \mathrm{Aut}(M)$, $g \mapsto g\, s_o$ is also a homomorphism, und Q is the composition of these two. Note further that the identification given in (ii) is more natural than the one given by $X \mapsto (X, -X)$. In fact, when we view all spaces as spaces of vector fields on $G(M)$, then the term $(X, -X)$ corresponds to $(X^r + X^l)$, where X^l is the left-invariant and X^r the right-invariant vector field corresponding to $X \in \mathfrak{g}$. We have $\frac{1}{2}(X^r + X^l)_e = X^l_e = X^r_e$, i.e. our identification is compatible with identifying $\mathfrak{g} \subset (\mathfrak{g} \times \mathfrak{g})$ with $T_e G$ via the evaluation map.

Corollary I.5.4. *If $\pi : G(M) \to \mathrm{Gl}(V)$ is a group representation, then*

$$\pi' := \pi \circ Q : \quad M \to \mathrm{Gl}(V)$$

is a representation of a symmetric space. If π is finite-dimensional and $\dot{\pi}$ is the differential of π at the origin, then

$$\dot{\pi}' : \mathfrak{q} \to \mathfrak{gl}(V), \quad X \mapsto 2\dot{\pi}(X)$$

is the differential of π' at the origin.

Proof. This follows immediately by combining Lemma I.5.2 and Prop. I.5.3. ∎

In particular, we may compose Q with the adjoint representation $\mathrm{Ad} : G(M) \to \mathrm{Gl}(\mathfrak{g})$ and then get a representation $\mathrm{Ad} \circ Q : M \to \mathrm{Gl}(\mathfrak{g})$. It is clear that an intertwining operator for π is also one for π'; thus $\pi \mapsto \pi'$ is a functor $\mathrm{Rep}(G(M)) \to \mathrm{Rep}(M)$ of categories of representations. There is no general functor in the other direction. However, if $\pi : M \to \mathrm{Gl}(V)$ is an equivariant representation, then the group homomorphism $\widetilde{\pi} : G(M) \to \mathrm{Gl}(V) \times \mathrm{Gl}(V)$ combined with the projection onto the first factor yields a group representation $\widehat{\pi} : G(M) \to \mathrm{Gl}(V)$.

Definition I.5.5. The map $Q : M \to G(M)$, $p \mapsto Q(p) = s_p s_o$ is called the *quadratic map of* M (w.r.t. the base point o). The *powers* of an element $x \in M$ (w.r.t. o) are defined for $n \in \mathbb{Z}$ by

$$x^{2n} := Q(x)^n o, \quad x^{2n+1} := Q(x)^n x.$$ ∎

(In [Lo69a, p.64] the map Q is called the "quadratic representation"; we have changed the terminology since Q is not a representation in the sense of Def. I.5.1. However, this is true in the case of a Jordan algebra, see the next chapter. Note further that x^2 and $\mu(x, x)$ have different meanings.)

Lemma I.5.6. *For all $x, y \in M$ and $m, n \in \mathbb{Z}$,*

(i) $Q(Q(x)y) = Q(x)Q(y)Q(x)$,

(ii) $\mu(x^n, x^m) = x^{2n-m}$,

(iii) $(x^m)^n = x^{mn}$.

Proof. (i) Clearly the quadratic map commutes with base-point preserving homomorphisms $\varphi : M \to M'$ in the sense that

$$\varphi(Q(x)y) = Q'(\varphi(x))\varphi(y).$$

Applying this to the homomorphism $Q : M \to G(M)$ itself and observing that $Q'(g)h = ghg$ in the group case, we get (i).

(ii) We note first that $s_o(x^k) = x^{-k}$ holds for all k. Now we have to distinguish two cases. Using that $Q(x)^k$ is for all k an automorphism of μ, we get

$$\begin{aligned}\mu(x^{2k}, x^m) &= \mu(Q(x)^k o, x^m) = Q(x)^k \mu(o, Q(x)^{-k} x^m) \\ &= Q(x)^k s_o Q(x)^{-k} s_o s_o(x^m) = Q(x)^{2k} x^{-m} = x^{4k-m}\end{aligned}$$

and, observing that $s_x Q(x)^n = s_x s_x s_o Q(x)^{n-1} = Q(x)^{1-n} s_o$,

$$\begin{aligned}\mu(x^{2k+1}, x^m) &= \mu(Q(x)^k x, x^m) = Q(x)^k \mu(x, Q(x)^{-k} x^m) \\ &= Q(x)^k s_x Q(x)^{-k} x^m = Q(x)^k Q(x)^{k+1} s_o(x^m) = x^{4k+2-m}.\end{aligned}$$

(iii) Using part (i), we get $(x^{2m})^{2n} = Q(Q(x)^m e)^n e = Q(x)^{2mn} e = x^{4mn}$, and similarly for other exponents. ■

Proposition I.5.7. *For all* $v \in T_o M$,

$$\mathrm{Exp}(v) = \exp(l_o v).o,$$

where $\exp : \mathfrak{g} \to G(M)$ *is the usual exponential map of the Lie group* $G(M)$.

Proof. The quadratic map is a local isomorphism onto the space S of symmetric elements (Prop. I.5.3). The latter is a sub-symmetric space of G. Its geodesics are given by the formula $\exp(tX)$ for $X \in \mathfrak{q}$ because the usual exponential map of a Lie group agrees with the exponential map corresponding to its symmetric space structure. By equivariance of the quadratic map,

$$Q(\exp(tX).o) = \exp(2tX).$$

It follows that $\gamma : \mathbb{R} \to M$, $t \mapsto \exp(tX).o$ is a geodesic. Deriving we obtain $\dot\gamma(0) = X_o$, whence $\mathrm{Exp}(X_o) = \exp(X).o$. ■

6. Examples

6.1. The classical groups. By "classical group" we mean the general linear groups $\mathrm{Gl}(n, \mathbb{F})$ over the base field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ or over the skew-field $\mathbb{F} = \mathbb{H}$ of quaternions and the orthogonal and unitary groups in the most general sense (which includes the symplectic groups); special linear groups will appear less frequently (actually, their role will be taken later by the projective groups). The classical groups are, in a natural way, subgroups of a group $\mathrm{Gl}(n, \mathbb{R})$ for some n (thus for us the term "classical group" will not mean an abstract group, but a group together with a distinguished representation), and as a special case of Ex. I.4.1, they are symmetric spaces with multiplication map

$$\mu(X, Y) = XY^{-1}X. \tag{6.1}$$

In order to fix some notation, let us recall the basic definitions.

Orthogonal and symplectic groups. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and assume a non-degenerate bilinear form on $V = \mathbb{F}^n$ be given by $(x|y) = x^t Ay$, $A \in \mathrm{Gl}(n, \mathbb{F})$; then $A^{-1}X^tA$ is the adjoint of X w.r.t. this form. The A*-orthogonal group* is the group

$$\mathrm{O}(A, \mathbb{F}) := \{g \in \mathrm{Gl}(n, \mathbb{F}) |\, A^{-1}g^tA = g^{-1}\}. \tag{6.2}$$

Standard forms are given by the following matrices

$$I_{p,q} := \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}, \quad J := J_n := \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}, \quad F := F_n := \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}. \tag{6.3}$$

We use also the classical notation

$$\mathrm{O}(p, q) := \mathrm{O}(I_{p,q}, \mathbb{R}), \quad \mathrm{O}(n) := \mathrm{O}(n, 0), \quad \mathrm{Sp}(n, \mathbb{F}) := \mathrm{O}(J, \mathbb{F}). \tag{6.4}$$

Note that the notation $\mathrm{O}(A, \mathbb{F})$ not only describes the classical group as an abstract group, but also its distinguished realization as a subgroup of $\mathrm{Gl}(n, \mathbb{F})$. A "base change" induces a conjugation of the corresponding group via the formula

$$\mathrm{O}(T^tAT) = T^{-1}\mathrm{O}(A)T. \tag{6.5}$$

For instance, we define the "real Cayley transform" by

$$R := \begin{pmatrix} \mathbf{1}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{1}_n \end{pmatrix}. \tag{6.6}$$

Then $R^t = 2R^{-1}$, and for all $A \in \mathrm{Gl}(n, \mathbb{F})$, the transformation formula

$$R \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} R^{-1} = \begin{pmatrix} -A & 0 \\ 0 & A \end{pmatrix} \tag{6.7}$$

and the corresponding group isomorphisms hold. In particular, $R^{-1}\mathrm{O}(F, \mathbb{F})R = \mathrm{O}(n, n)$. We define the *spaces of* A*-(skew-)symmetric matrices* by

$$\mathrm{Sym}(A, \mathbb{F}) := \{X \in M(n, \mathbb{F}) |\, A^{-1}X^tA = X\}, \tag{6.8}$$

$$\mathrm{Asym}(A, \mathbb{F}) := \{X \in M(n, \mathbb{F}) |\, A^{-1}X^tA = -X\}. \tag{6.9}$$

The space $\mathrm{Asym}(A, \mathbb{F})$ is the Lie algebra of $\mathrm{O}(A, \mathbb{F})$ (and may also be denoted by $\mathfrak{o}(A, \mathbb{F})$). It is a sub-LTS of the LTS $\mathfrak{gl}(n, \mathbb{F})$ with Lie triple product defined by

$$[X, Y, Z] = \frac{1}{4}[[X, Y], Z] = \frac{1}{4}(XYZ + ZYX - (YXZ + ZXY)). \tag{6.10}$$

Unitary groups. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ and ε an anti-automorphism of $\mathbb{F}$; it is identified with the corresponding map defined componentwise on $\mathbb{F}^n$ and on $M(n, \mathbb{F})$. We consider the sesquilinear form $(x|y) = x^tA\varepsilon(y)$ defined by a matrix $A \in \mathrm{Gl}(n, \mathbb{F})$. Then

$$X^* = \varepsilon^{-1}(A^{-1}X^tA) \tag{6.11}$$

is the adjoint of $X \in M(n, \mathbb{F})$ w.r.t. this form. The (A, ε)*-unitary group* is defined by

$$\mathrm{U}(A, \varepsilon, \mathbb{F}) := \{g \in \mathrm{Gl}(n, \mathbb{F}) |\, A^{-1}g^tA = \varepsilon(g)\}. \tag{6.12}$$

If ε is the standard-involution of the base field (i.e. complex conjugation of $\mathbb{C}$, resp. conjugation of $\mathbb{H}$ w.r.t. the center $Z(\mathbb{H})=\mathbb{R}$), then it may be omitted in the notation. We use also the classical notation

$$\begin{aligned} \mathrm{U}(p,q) &:= \mathrm{U}(I_{p,q},\mathbb{C}), \quad \mathrm{U}(n) := \mathrm{U}(n,0), \\ \mathrm{Sp}(p,q) &:= \mathrm{U}(I_{p,q},\mathbb{H}), \quad \mathrm{Sp}(n) := \mathrm{Sp}(n,0). \end{aligned} \tag{6.13}$$

The groups $\mathrm{U}(J,\mathbb{C})$ and $\mathrm{U}(n,n)$ are isomorphic; in fact,

$$C^{-1}\mathrm{U}(n,n)C = \mathrm{U}(J,\mathbb{C}), \tag{6.14}$$

where

$$C = \begin{pmatrix} -\mathbf{1} & -i\mathbf{1} \\ i\mathbf{1} & \mathbf{1} \end{pmatrix} \tag{6.15}$$

is the Cayley transform. The $\mathbb{H}$-unitary groups can be realized as subgroups of $\mathrm{Gl}(2n,\mathbb{C})$: the identification $\mathbb{H}^n = \mathbb{C}^n \oplus \mathbb{C}^n j = \mathbb{C}^{2n}$ (we let scalars act from the right on $\mathbb{H}^n$) defines an imbedding

$$\begin{aligned} M(n,\mathbb{H}) &= \{Z \in M(2n,\mathbb{C}) \mid \overline{Z} = JZJ^{-1}\} \\ &= \{\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \mid a,b \in M(n,\mathbb{C})\}. \end{aligned} \tag{6.16}$$

Then $\mathrm{Gl}(n,\mathbb{H})$ is identified with the corresponding matrices Z having non-zero determinant over $\mathbb{C}$. If ε is the canonical involution of $\mathbb{H}$, then the ε-sesquilinear form on $\mathbb{H}^n$ with matrix $I_{p,q}$ corresponds to the sesquilinear form on $\mathbb{C}^{2n}$ with matrix $\left(\begin{smallmatrix} I_{p,q} & 0 \\ 0 & I_{p,q} \end{smallmatrix}\right)$; from this we get

$$\mathrm{Sp}(p,q) \cong \mathrm{U}(2p,2q) \cap M(n,\mathbb{H}) \cong \mathrm{U}(2p,2q) \cap \mathrm{Sp}(2n,\mathbb{C}). \tag{6.17}$$

If φ is conjugation with the quaternion j, composed with the standard-involution, then the φ-sesquilinear form on $\mathbb{H}^n$ with matrix $\mathbf{1}_n$ corresponds to the $\mathbb{C}$-bilinear form on $\mathbb{C}^{2n}$ with matrix $\mathbf{1}_{2n}$, and we get

$$\mathrm{U}(\mathbf{1}_n,\varphi,\mathbb{H}) = \mathrm{O}(2n,\mathbb{C}) \cap M(n,\mathbb{H}) =: \mathrm{O}^*(2n). \tag{6.18}$$

This group is also isomorphic to $\mathrm{U}(I_{p,q},\varphi,\mathbb{H})$ $(p+q=n)$ and to $\mathrm{U}(J,\mathbb{H})$. We denote by

$$\mathrm{Herm}(A,\varepsilon,\mathbb{F}) = \{X \in M(n,\mathbb{F}) \mid A^{-1}X^tA = \varepsilon(X)\}, \tag{6.19}$$

$$\mathrm{Aherm}(A,\varepsilon,\mathbb{F}) = \{X \in M(n,\mathbb{F}) \mid A^{-1}X^tA = -\varepsilon(X)\}, \tag{6.20}$$

the *spaces of A-(skew-)Hermitian matrices.* The space $\mathrm{Aherm}(A,\varepsilon,\mathbb{F})$ is the Lie algebra of $\mathrm{U}(A,\varepsilon,\mathbb{F})$ and may therefore also be denoted by $\mathfrak{u}(A,\varepsilon,\mathbb{F})$. It is useful to keep in mind that the equation $i\,\mathrm{Herm}(n,\mathbb{C}) = \mathrm{Aherm}(n,\mathbb{C})$ has quaternionic analogues: the involutions ε and $\varphi = j_*\varepsilon = \varepsilon j_*$ of $\mathbb{H}$ being as above, we have

$$j\mathbf{1}_n \cdot \mathrm{Herm}(n,\varphi,\mathbb{H}) = \mathrm{Aherm}(n,\mathbb{H}), \quad j\mathbf{1}_n \cdot \mathrm{Aherm}(n,\varphi,\mathbb{H}) = \mathrm{Herm}(n,\mathbb{H}). \tag{6.21}$$

In fact, for any quaternionic matrix X we have $\varepsilon(j\mathbf{1}_nX)^t = \varepsilon(X^tj) = -\varepsilon(X^t)j = -\varphi(X^t)$.

6.2. Spaces of symmetric and of Hermitian matrices. The map "adjoint" $g \mapsto g^*$ is an automorphism of the symmetric space $\mathrm{Gl}(n,\mathbb{F})$. The corresponding space of symmetric elements is

$$M = \mathrm{Gl}(n,\mathbb{F}) \cap \mathrm{Sym}(A,\mathbb{F}), \text{ resp. } \quad M = \mathrm{Gl}(n,\mathbb{F}) \cap \mathrm{Herm}(A,\varepsilon,\mathbb{F}). \tag{6.22}$$

In general, M is not connected. The connected component containing the identity is homogeneous under the natural action of $\mathrm{Gl}(n,\mathbb{F})$ by $g.X = gXg^*$, and

$$M_{\mathbf{1}} \cong \mathrm{Gl}(n,\mathbb{F})/\mathrm{O}(A,\mathbb{F}), \text{ resp. } \quad M_{\mathbf{1}} \cong \mathrm{Gl}(n,\mathbb{F})/\mathrm{U}(A,\varepsilon,\mathbb{F}) \tag{6.23}$$

as a homogeneous symmetric space. Spaces of this kind will be considered in more detail in Chapter II.

6.3. Spaces of elements of order 2 and Grassmannians. For any Lie group G, the space $G^j := \{g \in G | \, g^2 = e\}$ of elements of order 2 is a subsymmetric space, namely it is the space fixed under the automorphism $j : g \mapsto g^{-1}$ of the symmetric space G. Let us exhibit the geometric significance of this space in case G is one of the classical groups defined above.

Case of the general linear groups. $G = \mathrm{Gl}(n,\mathbb{F})$,

$$M = \{X \in \mathrm{Gl}(n,\mathbb{F}) | \, X^2 = \mathbf{1}_n\}. \tag{6.24}$$

Note that the multiplication map of M takes the form $s_X Y = \mu(X,Y) = XY^{-1}X = XYX^{-1}$, and s_X is induced by a *linear* map. The space M is not connected; its connected components are the spaces

$$M_{p,q} := \{X \in \mathrm{Gl}(n,\mathbb{R}) | \, X^2 = \mathbf{1}_n, \, \mathrm{sgn}(X) = (p,q)\} \tag{6.25}$$

for p,q with $p+q=n$, where p is the dimension of the $+1$-eigenspace $E_1(X)$ and q the dimension of the -1-eigenspace $E_{-1}(X)$ of X. Associating to an element $X \in M_{p,q}$ the pair $(E_1(X), E_{-1}(X))$, we obtain a bijection of $M_{p,q}$ onto the open dense subset of elements (E,F) in the direct product $\mathrm{Gr}_{p,n}(\mathbb{F}) \times \mathrm{Gr}_{q,n}(\mathbb{F})$ of Grassmannians such that $E \cap F = 0$.

Since elements with $X^2 = \mathbf{1}$ can be diagonalized over $\mathbb{F}$, the action of $\mathrm{Gl}(n,\mathbb{F})$ by conjugation on $M_{p,q}$ is transitive. W.r.t. the base point $I_{p,q} \in M_{p,q}$, we write

$$M_{p,q} = \mathrm{Gl}(p+q,\mathbb{F})/(\mathrm{Gl}(p,\mathbb{F}) \times \mathrm{Gl}(q,\mathbb{F})) \tag{6.26}$$

as a homogeneous symmetric space. The corresponding involution σ of $G = \mathrm{Gl}(p+q,\mathbb{F})$ is given by conjugation with $I_{p,q}$. The tangent space of $M_{p,q}$ at the base point is given by

$$\mathfrak{q} = \{X \in M(n,\mathbb{F}) | \, I_{p,q}X = -XI_{p,q}\} = \{\begin{pmatrix} 0 & a \\ b^t & 0 \end{pmatrix} | \, a,b \in M(p,q;\mathbb{F})\}; \tag{6.27}$$

the Lie triple product is given by Eqn. (6.10) without the factor $\frac{1}{4}$ (it is not needed since we are not in a group case). As a vector space, $\mathfrak{q}$ is a direct product of $M(p,q;\mathbb{F})$ with itself.

Case of orthogonal groups. We consider the subspace fixed under the automorphism $g \mapsto g^{-1}$ of the symmetric space $\mathrm{O}(A,\mathbb{F})$. Since the conditions $X^2 = \mathbf{1}$ and $X^* = X^{-1}$ are equivalent to $X^2 = \mathbf{1}$ and $X = X^*$, this space can be described as

$$\begin{aligned} M = \{X \in \mathrm{O}(A,\mathbb{F}) | \, X^2 = \mathbf{1}_n\} &= \{X \in \mathrm{Gl}(n,\mathbb{F}) | X^2 = \mathbf{1}_n, X^* = X^{-1}\} \\ &= \mathrm{Gl}(n,\mathbb{F})^j \cap \mathrm{Sym}(A,\mathbb{F}); \end{aligned} \tag{6.28}$$

it is the subspace of (6.24) fixed under the automorphism $\tau(X) := X^*$. The condition $X = X^*$ implies that the eigenspaces $E_1(X)$ and $E_{-1}(X)$ are orthogonal; in particular the restriction of the form given by A is non-degenerate on the eigenspaces. In fact, the map $X \mapsto E_1(X)$ defines a bijection of M onto the (open dense) set of elements E in the Grassmannian $\mathrm{Gr}(\mathbb{F}^n) = \cup_p \mathrm{Gr}_{p,n}(\mathbb{F})$ such that the form given by A is non-degenerate on E.

If M is not empty, we fix a base point in M; replacing A if necessary by an equivalent matrix, we may assume that the base point is $I_{p,q}$. Then since $AI_{p,q} = I_{p,q}A$, we can write

$A = \left(\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right)$. According to Lemma I.4.5 the action of $\mathrm{O}(A, \mathbb{F})$ by conjugation on the corresponding connected component is transitive:

$$M_{I_{p,q}} = \mathrm{O}(\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \mathbb{F})_o / (\mathrm{O}(B, \mathbb{F}) \times \mathrm{O}(C, \mathbb{F}))_o. \tag{6.29}$$

Again the corresponding involution of the group is given by conjugation with $I_{p,q}$. The tangent space $\mathfrak{q}$ at the base point is given as the subspace of the space given by (6.27) fixed under $\tau(X) = X^*$:

$$\mathfrak{q} = \{\begin{pmatrix} 0 & X \\ BX^t C^{-1} & 0 \end{pmatrix} \mid X \in M(p, q; \mathbb{F})\}. \tag{6.30}$$

As a vector space this is isomorphic to $M(p, q; \mathbb{F})$. The most important special case is $\mathbb{F} = \mathbb{R}$ and $A = \mathbf{1}_{p+q}$; then

$$M_{I_{p,q}} = \mathrm{O}(p+q)/(\mathrm{O}(p) \times \mathrm{O}(q)) \tag{6.31}$$

is the whole Grassmannian $\mathrm{Gr}_{p,p+q}(\mathbb{R})$. For $\mathbb{F} = \mathbb{R}$, the choice $B = I_{k,l}$, $C = I_{i,j}$ leads to the space $\mathrm{O}(l+j, k+i)/(\mathrm{O}(k,l) \times \mathrm{O}(i,j))$ and the choice $B = J_r$, $C = J_s$ to $M_{I_{2r,2s}} = \mathrm{Sp}(r+s, \mathbb{R})/(\mathrm{Sp}(r, \mathbb{R}) \times \mathrm{Sp}(s, \mathbb{R}))$. One may call these spaces "of real Grassmannian type".

Case of unitary groups. The spaces of elements of order 2 in unitary groups can, similarly as above, be interpreted as open dense subsets of Grassmannians. The most important cases are here $A = \mathbf{1}_{p+q}$ in combination with the standard involutions of $\mathbb{C}$ resp. of $\mathbb{H}$ which yield the complex resp. quaternionic Grassmannians:

$$M_{I_{p,q}} = \mathrm{U}(p+q)/(\mathrm{U}(p) \times \mathrm{U}(q)), \text{ resp. } \quad M_{I_{p,q}} = \mathrm{Sp}(p+q)/(\mathrm{Sp}(p) \times \mathrm{Sp}(q)). \tag{6.32}$$

As above, other choices for B and C lead to the spaces $\mathrm{U}(l+j, k+i)/(\mathrm{U}(k,l) \times \mathrm{U}(i,j))$, $\mathrm{Sp}(l+j, k+i)/(\mathrm{Sp}(k,l) \times \mathrm{Sp}(i,j))$ and $\mathrm{SO}^*(2n)/(\mathrm{SO}^*(2p) \times \mathrm{SO}^*(2q))$ which are open dense in the complex resp. quaternionic Grassmannians and which may be called "of complex (quaternionic) Grassmannian type".

6.3. Spaces of complex structures. We say that an element g of a classical group $G \subset \mathrm{Gl}(n, \mathbb{R})$ is a *complex structure* if $g^2 = -\mathbf{1}_n$. All classical groups (leaving apart the special linear ones) are stable under the involution $g \mapsto -g^{-1}$. The fixed point space of this involution,

$$G^{-j} = \{g \in G \mid g^2 = -\mathbf{1}\}, \tag{6.33}$$

is a subsymmetric space, the *space of complex structures in* G.

Case of the general linear groups: $G = \mathrm{Gl}(2n, \mathbb{R})$;

$$M = \{X \in \mathrm{Gl}(2n, \mathbb{R}) \mid X^2 = -\mathbf{1}_{2n}\} \tag{6.34}$$

is the space of complex structures on $\mathbb{R}^{2n}$. The space M is connected: by linear algebra, every complex structure can be transformed to the normal form given by the matrix $J = J_n$ which we choose as base point in M. Thus

$$M = \mathrm{Gl}(2n, \mathbb{R}) / \mathrm{Gl}(n, \mathbb{C}) \tag{6.35}$$

as a homogeneous symmetric space. Note that conjugation by the matrix $I_{n,n}$ transforms J into the opposite complex structure $-J$. The tangent space $T_J M$ is naturally identified with the space

$$\mathfrak{q} := \{X \in M(2n, \mathbb{R}) \mid JX = -XJ\} = \{\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a, b \in M(n, \mathbb{R})\} \tag{6.36}$$

of anti J-linear maps. For $\mathbb{F} = \mathbb{C}$ we may define a space M by the same formula, but since $(iX)^2 = -X^2$ this gives us back spaces of elements of order 2. If $\mathbb{F} = \mathbb{H}$, then the matrix J is a complex structure in $\mathrm{Gl}(n, \mathbb{H})$. Using the realization (6.16) of $\mathrm{Gl}(n, \mathbb{H})$, we see that an element $Z \in \mathrm{Gl}(n, \mathbb{H})$ commuting with J must be real, i.e. $\overline{Z} = Z$. Identifying

$$\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{Gl}(n, \mathbb{H}) |\, a, b \in M(n, \mathbb{R})\} = \mathrm{Gl}(n, \mathbb{C}), \tag{6.37}$$

we get

$$\mathrm{Gl}(n, \mathbb{H})^{-j} \cong \mathrm{Gl}(n, \mathbb{H}) / \mathrm{Gl}(n, \mathbb{C}) \tag{6.38}$$

as a homogeneous symmetric space. Here the group involution is given as above by conjugation with J.

Case of the orthogonal groups ("Siegel-spaces"). The space

$$\begin{aligned} M = \{X \in \mathrm{O}(A, \mathbb{R}) |\, X^2 = -\mathbf{1}_{2n}\} &= \{X \in \mathrm{Gl}(2n, \mathbb{R}) |\, X^2 = -\mathbf{1}_{2n},\ X^* = X^{-1}\} \\ &= \mathrm{Gl}(2n, \mathbb{R})^{-j} \cap \mathrm{Asym}(A, \mathbb{R}) \end{aligned} \tag{6.39}$$

of A-orthogonal complex structures is the subspace of (6.33) fixed under the automorphism $\tau(X) = -X^*$. It may be empty, and in general it is not connected. If it is not empty, we assume w.l.o.g. that it contains the base point J, i.e. $J^t A J = A$, or $AJ = JA$ holds, and we may write $A = \left(\begin{smallmatrix} B & C \\ -C & B \end{smallmatrix}\right)$. As a homogeneous space, the component containing J is given by

$$M_J = \mathrm{O}(A, \mathbb{R})_o / (\mathrm{O}(A, \mathbb{R})_o \cap \mathrm{Gl}(n, \mathbb{C})). \tag{6.40}$$

The tangent space $T_J M$ is identified with the skew-symmetric elements in the space defined by (6.35):

$$\mathfrak{q} = \{X \in M(2n, \mathbb{R}) |\, XJ = -JX,\ X^t A = -AX\}. \tag{6.41}$$

Let us consider some special choices for A: If $A = \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right)$, then (using that $\mathrm{O}(\left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right), \mathbb{R}) \cap \mathrm{Gl}(n, \mathbb{C}) = \mathrm{U}(B)$),

$$M_J = \mathrm{O}(\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}, \mathbb{R})_o / \mathrm{U}(B). \tag{6.42}$$

Now let $B = I_{p,q}$; then

$$M_J \cong \mathrm{O}(2p, 2q)_o / \mathrm{U}(p, q). \tag{6.43}$$

Next, the choice $B = J$ yields

$$M_J \cong \mathrm{Sp}(2n, \mathbb{R}) / \mathrm{U}(n, n). \tag{6.44}$$

If $A = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{smallmatrix}\right)$, then

$$M_J = \mathrm{Sp}(n, \mathbb{R}) / (\mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{Gl}(n, \mathbb{C})) = \mathrm{Sp}(n, \mathbb{R}) / \mathrm{U}(n). \tag{6.45}$$

This space is known as the *Siegel-space.*

Case of $\mathbb{C}$-unitary groups. If X is a complex structure belonging to $\mathrm{U}(p, q)$, then iX is an element of order 2 in $\mathrm{U}(p, q)$. The corresponding spaces have already been discussed above.

Case of $\mathbb{H}$-unitary groups. This is case is similar to the case of orthogonal groups. It leads to the spaces $\mathrm{Sp}(p, q)/\mathrm{U}(p, q)$ (complex structures in $\mathrm{Sp}(p, q)$) and $\mathrm{O}^*(2n)/\mathrm{U}(p, 2n - p)$ (complex structures in $\mathrm{O}^*(2n)$).

6.4. Lagrangians and spaces of Lagrangian type. Next we consider subspaces which are simultaneously fixed under 2 involutions. Thus let us assume that G is a classical group which is stable under the involution $g \mapsto -g^* = -A^{-1}g^tA$ of $\mathrm{Gl}(m,\mathbb{R})$ and consider the space of A-skewsymmetric elements of order 2:

$$M = \{g \in G |\, g^2 = \mathbf{1}_m, g = -g^*\} = G^j \cap \mathrm{Asym}(A,\mathbb{F}). \tag{6.46}$$

Case of the general linear groups. Clearly $\mathrm{Gl}(n,\mathbb{F})$ is stable under $X \mapsto -X^*$. If $X^2 = \mathbf{1}$ and $X = -X^*$, then the restriction of the bilinear form given by A vanishes on the eigenspaces of X. It follows that the eigenspaces $E_1(X)$ and $E_{-1}(X)$ are maximal isotropic, i.e. Lagrangian subspaces for this form. Thus we obtain a bijection $X \mapsto (E_1(X), E_{-1}(X))$ of $M = \mathrm{Gl}(n,\mathbb{F})^j \cap \mathrm{Asym}(A,\mathbb{F})$ onto pairs (E,F) of Lagrangians (w.r.t. A) with $E \cap F = 0$.

Since $E_1(X)$ and $E_{-1}(X)$ are maximal isotropic and complementary, it follows that both have equal dimension denoted by $n = m/2$. We assume that the base point $I_{n,n}$ belongs to M. This forces $A = \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$ with $B = \pm C$ invertible. The tangent space of M at $I_{n,n}$ is given by taking skew-symmetric elements in (6.27):

$$\begin{aligned} \mathfrak{q} &= \{X \in M(2n,\mathbb{R}) |\, XI_{n,n} = -I_{n,n}X,\ X^* = -X\} \\ &= \{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} |b^t = -C^{-1}bB, a^t = -BaC^{-1}\} \end{aligned} \tag{6.47}$$

The group $\mathrm{O}(A,\mathbb{R})$ acts on M by conjugation, and the stabilizer of the base point $I_{n,n}$ is the group of matrices of the form $\left(\begin{smallmatrix} g & 0 \\ 0 & (CgC^{-1})^t \end{smallmatrix}\right)$ with $g \in \mathrm{Gl}(n,\mathbb{R})$; we identify it with $\mathrm{Gl}(n,\mathbb{R})$. For reasons of dimension, the action is transitive on the connected component of $I_{n,n}$. Thus $M_{I_{n,n}} = \mathrm{O}(A,\mathbb{R})_o / \mathrm{Gl}(n,\mathbb{R})$ as a homogeneous symmetric space (the group involution is given by conjugation with $I_{n,n}$). Now we distinguish the two cases: (1) $C = B$, (2) $C = -B$. In case (1) $\mathfrak{q}$, as a vector space, is a direct product $\mathrm{Asym}(C,\mathbb{R}) \times \mathrm{Asym}(C,\mathbb{R})$; in case (2) it is a direct product $\mathrm{Sym}(C,\mathbb{R}) \times \mathrm{Sym}(C,\mathbb{R})$. As a homogeneous symmetric space in case (1),

$$M_{I_{n,n}} = \mathrm{O}(\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}, \mathbb{R})_o / \mathrm{Gl}(n,\mathbb{R}), \tag{6.48}$$

and in case (2)

$$M_{I_{n,n}} = \mathrm{O}(\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}, \mathbb{R})_o / \mathrm{Gl}(n,\mathbb{R}). \tag{6.49}$$

Taking in case (1) $C = \mathbf{1}_n$, we get the space $\mathrm{O}(n,n)_o / \mathrm{Gl}(n,\mathbb{R})$; taking $C = F$ we get $\mathrm{Sp}(2k,\mathbb{R}) / \mathrm{Gl}(2k,\mathbb{R})$; taking in case (2) $C = \mathbf{1}_n$, we get the space $\mathrm{Sp}(n,\mathbb{R}) / \mathrm{Gl}(n,\mathbb{R})$, and taking $C = F$ we get $\mathrm{O}(2k,2k)_o / \mathrm{Gl}(2k,\mathbb{R})$. Similarly, for $\mathbb{F} = \mathbb{C}$ we obtain the spaces $\mathrm{Sp}(n,\mathbb{C}) / \mathrm{Gl}(n,\mathbb{C})$ and $\mathrm{SO}(2k,\mathbb{C}) / \mathrm{Gl}(k,\mathbb{C})$.

If $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, then we consider also the spaces

$$\mathrm{Gl}(n,\mathbb{F})^j \cap \mathrm{Aherm}(A,\varepsilon,\mathbb{F}). \tag{6.50}$$

Similar calculations as above lead, among others, to the spaces $\mathrm{U}(n,n) / \mathrm{Gl}(n,\mathbb{C})$, $\mathrm{Sp}(n,n) / \mathrm{Gl}(n,\mathbb{H})$ and $\mathrm{SO}^*(4n) / \mathrm{Gl}(n,\mathbb{H})$.

Case of orthogonal groups. Let A and D be two commuting matrices and consider the space

$$\begin{aligned} M &= \{X \in \mathrm{Gl}(2n,\mathbb{R}) |\, X^2 = \mathbf{1}_n, A^{-1}X^tA = -X, D^{-1}X^tD = X\} \\ &= \mathrm{O}(D)^j \cap \mathrm{Asym}(A,\mathbb{F}); \end{aligned} \tag{6.51}$$

this is the subspace of $\mathrm{Gl}(n,\mathbb{F})^j \cap \mathrm{Asym}(A,\mathbb{F})$ fixed under $\tau(X) = D^{-1}X^tD$. As above, it follows that $X \mapsto E_1(X)$ is a bijection of M onto the set of Lagrangian subspaces V w.r.t. A such that the restriction of the form given by D to V is non-degenerate. In particular, if $D = \mathbf{1}_{2n}$, N is isomorphic to the whole space of Lagrangian subspaces of A.

Assuming that the base point $I_{n,n}$ belongs to M, it follows that D is of the form $\left(\begin{smallmatrix} E & 0 \\ 0 & G \end{smallmatrix}\right)$. If $G = E$ and E and C commute, then A and D commute. The group $\mathrm{O}(D,\mathbb{R}) \cap \mathrm{O}(A,\mathbb{R})$ acts on M by conjugation, and the stabilizer of the base point $I_{n,n}$ is the intersection of $\mathrm{O}(D,\mathbb{R})$ with the group $\mathrm{Gl}(n,\mathbb{R})$ realized as in (6.48), (6.49); it is the group of matrices of the form $\left(\begin{smallmatrix} g & 0 \\ 0 & (CgC^{-1})^t \end{smallmatrix}\right)$ with $g \in \mathrm{O}(E,\mathbb{R})$. We identify it with $\mathrm{O}(E,\mathbb{R})$. Thus, as a homogeneous symmetric space,

$$M = (\mathrm{O}(\begin{pmatrix} 0 & C \\ \pm C & 0 \end{pmatrix}, \mathbb{R}) \cap \mathrm{O}(\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \mathbb{R}))/\mathrm{O}(E,\mathbb{R}) \tag{6.52}$$

(the group involution is again given by conjugation with $I_{n,n}$). The tangent space at the base point $I_{n,n}$ is given by intersecting the spaces defined by (6.27) and (6.47):

$$\mathfrak{q} = \{\begin{pmatrix} 0 & X \\ EX^tE^{-1} & 0 \end{pmatrix} \in M(2n,\mathbb{R}) |\, X^t = -CXB^{-1}\} \tag{6.53}$$

As a vector space, $\mathfrak{q}$ is isomorphic to $\mathrm{Asym}(C,\mathbb{R})$ (if $B = C$), resp. to $\mathrm{Sym}(C,\mathbb{R})$ (if $B = -C$). Now let us consider the two most important cases:

(1) If $A = J$ and $D = \mathbf{1}_{2n}$, then (6.52) reduces to

$$M = (\mathrm{O}(J,\mathbb{R}) \cap \mathrm{O}(2n))/\mathrm{O}(n) = \mathrm{U}(n)/\mathrm{O}(n), \tag{6.54}$$

the space of Lagrangians of the skew-symmetric form given by J. Here $\mathfrak{q}$ is isomorphic to $\mathrm{Sym}(n,\mathbb{R})$. The spaces $\mathrm{U}(p,q)/\mathrm{O}(p,q)$ are obtained similarly with $D = I_{p,q}$.

(2) If $A = \left(\begin{smallmatrix} C & 0 \\ 0 & C \end{smallmatrix}\right)$, it is convenient to diagonalize A via the matrix R given by Eqn. (6.6). Then the base point $I_{n,n}$ is replaced by the base point $RI_{n,n}R^{-1} = F$ (cf. Eqn. (6.7) and (6.3)). In other words, we consider the space

$$\begin{aligned} M' &:= R^{-1}MR = \{X \in \mathrm{Gl}(2n,\mathbb{R}) |\, X^2 = \mathbf{1}_{2n},\ X^tA' = -A'X,\ X^tD' = D'X\} \\ &= \mathrm{O}(D')^j \cap \mathrm{Asym}(A',\mathbb{F}), \end{aligned} \tag{6.55}$$

instead of M, where $A' = R^{-1}AR$, $D' = R^{-1}DR$. If $B = C = E$, then according to Eqn. (6.7), $A' = \left(\begin{smallmatrix} -C & 0 \\ 0 & C \end{smallmatrix}\right)$ and $D' = D$. The conditions from (6.51) now imply that $X \in M'$ anticommutes with $I_{n,n}$, and one easily deduces that

$$N' = \{\begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix} |\, g \in \mathrm{O}(C,\mathbb{R})\}. \tag{6.56}$$

As a symmetric space, this is isomorphic to $\mathrm{O}(C,\mathbb{R})$. In particular, the group $\mathrm{O}(n)$ is identified with the space of Lagrangians of $A' = I_{n,n}$, the groups $\mathrm{O}(p,q)$ can be realized as open dense subsets of this space, and $\mathrm{Sp}(n,\mathbb{R})$ can be realized as an open dense subset of the space of Lagrangians of the form given by J_{2n}.

Case of unitary groups. We consider the space

$$\mathrm{U}(D,\varepsilon,\mathbb{F})^j \cap \mathrm{Aherm}(A,\varepsilon,\mathbb{F}). \tag{6.57}$$

If the base point $I_{n,n}$ belongs to this space, we realize D and A as above and obtain the homogeneous symmetric spaces

$$M = (\mathrm{U}(\begin{pmatrix} 0 & C \\ \pm C & 0 \end{pmatrix}, \varepsilon, \mathbb{F}) \cap \mathrm{U}(\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \varepsilon, \mathbb{F}))/\mathrm{U}(E,\varepsilon,\mathbb{F}). \tag{6.58}$$

The most important cases are:

(1) If $A = J$ and $D = \mathbf{1}_{2n}$, then (6.58) reduces to

$$M = (\mathrm{U}(J, \varepsilon, \mathbb{F}) \cap \mathrm{U}(2n, \varepsilon, \mathbb{F}))/\mathrm{U}(n, \varepsilon, \mathbb{F}). \tag{6.59}$$

If ε is the standard conjugation of $\mathbb{F} = \mathbb{C}$, this space is isomorphic to $(\mathrm{U}(n) \times \mathrm{U}(n))/\mathrm{U}(n)$, i.e. to the group $\mathrm{U}(n)$. If ε is the standard conjugation of $\mathbb{F} = \mathbb{H}$, then this space is isomorphic to $\mathrm{U}(2n)/\mathrm{Sp}(n)$, the space of Lagrangians of the standard skew-sesquilinear form on $\mathbb{H}^n$. The spaces $\mathrm{U}(p,q)$ and $\mathrm{U}(2p,q2)/\mathrm{Sp}(p,q)$ are obtained similarly with $D = I_{p,q}$.

(2) If $A = \left(\begin{smallmatrix} 0 & C \\ C & 0 \end{smallmatrix}\right)$ and $E = C$, we transform again A via R and take F as base point. The same calculations as before show that the unitary group $\mathrm{U}(B, \varepsilon, \mathbb{F})$ is realized as an open dense subset of the variety of Lagrangians $\mathrm{Lag}(\begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix}, \varepsilon, \mathbb{F})$. In particular

$$\mathrm{U}(n) \cong \mathrm{Lag}(I_{n,n}, \tau, \mathbb{C}), \tag{6.60}$$

$$\mathrm{Sp}(n) \cong \mathrm{Lag}(I_{n,n}, \tau, \mathbb{H}). \tag{6.61}$$

6.5. c-duality. The *c-dual* spaces (cf. Section 1.4) of the symmetric spaces M considered so far are obtained as follows: the spaces we have looked at are all given by (at most 2) algebraic equations

$$M = \{g \in \mathrm{Gl}(n, \mathbb{R}) |\, F_1(g) = 0, F_2(g) = 0\}$$

with rational maps F_1, F_2. The space

$$M_{\mathbb{C}} = \{g \in \mathrm{Gl}(n, \mathbb{C}) |\, F_1(g) = 0, F_2(g) = 0\}$$

(where F_1, F_2 are the corresponding complex-rational maps) is a natural complexification of M, and complex conjugation τ in $M(n, \mathbb{C})$ defines a conjugation of $M_{\mathbb{C}}$ w.r.t. M (i.e. a $\mathbb{C}$-antilinear involution of $M_{\mathbb{C}}$ having fixed space M). Moreover, if $o \in M$ is a base point, then the symmety s_o was also defined by a rational map of $\mathbb{R}^n$; the corresponding complex-rational map will also be denoted by s_o. Then

$$M^c = (M_{\mathbb{C}})^{\tau s_o} = \{x \in M_{\mathbb{C}} |\, \tau(s_o(x)) = x\} \tag{6.62}$$

is the c-dual symmetric space of M.

Case of the general linear group. $M = \mathrm{Gl}(n, \mathbb{R})$, $o = \mathbf{1}_n$, symmetry $s_o(X) = X^{-1}$. Its c-dual real form is by definition the space

$$\begin{aligned} \mathrm{Gl}(n, \mathbb{C})^{\tau s_o} &= \{X \in \mathrm{Gl}(n, \mathbb{C}) | \overline{X} = X^{-1}\} \\ &= \{X \in \mathrm{Gl}(2n, \mathbb{R}) |\, XF = FX,\ I_{n,n} X I_{n,n} = X^{-1}\}. \end{aligned} \tag{6.63}$$

Applying to this space the automorphism $X \mapsto Y := I_{n,n}X$ of the multiplication map μ of $\mathrm{Gl}(2n, \mathbb{R})$, we end up with the two conditions $YF = -FY$, $Y^2 = \mathbf{1}_{2n}$. Thus the space (6.63) is isomorphic with the *space of real forms of* $\mathbb{C}^n$. As a homogeneous symmetric space,

$$\{Y \in \mathrm{Gl}(2n, \mathbb{R}) |\, YF = -FY,\ Y^2 = \mathbf{1}_{2n}\} = \mathrm{Gl}(n, \mathbb{C})/\mathrm{Gl}(n, \mathbb{R}). \tag{6.64}$$

In general, the c-dual of a space of group type G is given by $G_{\mathbb{C}}/G$.

Case of the space of complex structures. $M = \{g \in \mathrm{Gl}(2n,\mathbb{R}) |\, g^2 = -\mathbf{1}_{2n}\}$, $M_{\mathbb{C}} = \{X \in \mathrm{Gl}(2n,\mathbb{C}) |\, X^2 = -\mathbf{1}_{2n}\}$, base point J, and the conjugation w.r.t. the real form M is $\tau(X) = \overline{X}$. The symmetry s_J is given by $s_J(X) = JXJ^{-1}$. Thus

$$\begin{aligned} M^c &= \{X \in \mathrm{Gl}(2n,\mathbb{C}) |\, X^2 = -\mathbf{1}_{2n}, JXJ^{-1} = \overline{X}\} \\ &= \{X \in \mathrm{Gl}(n,\mathbb{H}) |\, X^2 = -\mathbf{1}_{2n}.\} \end{aligned} \tag{6.65}$$

This is the space of complex structures in $\mathrm{Gl}(n,\mathbb{H})$ which is isomorphic to $\mathrm{Gl}(n,\mathbb{H})/\mathrm{Gl}(n,\mathbb{C})$, cf. Eqn. (6.38). It is interesting to note that the c-dual construction thus automatically leads from $\mathbb{C}$-linear to $\mathbb{H}$-linear groups. (It would be nice if there were a similar construction leading automatically to exceptional groups...) More generally, the c-dual of a space of complex structures in a classical group is another space of complex structures in another real form of the corresponding complex group.

The general case. In most cases, the base point of M was either J or $I_{p,q}$, and the corresponding symmetry was given by conjugation with this matrix. In the former case the real form M^c is given by intersecting $M_{\mathbb{C}}$ with the algebra $M(n,\mathbb{H})$, in the latter case it is given by intersecting with the real form $\{X \in M(n,\mathbb{C}) |\, I_{p,q} X I_{p,q} = \overline{X}\}$ of $M(n,\mathbb{C})$. Thus one sees that the c-dual of a space of Grassmannian type is again of Grassmannian type, and the c-dual of a space of Lagrangian type is again of Lagrangian type.

6.6. Spheres and hyperbolic spaces. We consider the symmetric bilinear form $(x|y) = x^t A y$ on $V = \mathbb{F}^n$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) given by a non-singular symmetric matrix A.

Lemma I.6.1. *The quadratic hypersurface*

$$M = \{x \in \mathbb{F}^n |\, (x|x) = 1\} \tag{6.66}$$

is a symmetric space with multiplication map

$$\mu(x,y) = s_x(y) = 2(x|y)x - y. \tag{6.67}$$

The group $G(M)$ is equal to $\mathrm{SO}(A,\mathbb{F})$.

Proof. Note that the linear map s_x of V defined by $s_x(y) = 2(x|y)x - y$ is reflection w.r.t. the axis $\mathbb{F}x$. From this we get the properties (M1), (M2) and (M4). It is immediately verified that $\mathrm{O}(A,\mathbb{F})$ acts by automorphisms of μ, and since $s_x \in \mathrm{O}(A,\mathbb{F})$, property (M3) follows. The inclusion $G(M) \subset \mathrm{SO}(A,\mathbb{F})$ follows since $\mathrm{Det}(s_x s_y) = 1$. For the converse we may assume that $A = I_{p,q}$ and use the classical fact that the groups $\mathrm{SO}(p,q;\mathbb{F})$ are generated by reflections. ∎

The most important special case is $A = \mathbf{1}$ and $\mathbb{F} = \mathbb{R}$; then $M = S^n$ is the sphere. The case $A = I_{p,q}$, $\mathbb{F} = \mathbb{R}$ leads to $\mathrm{SO}(p,q)/\mathrm{SO}(p-1,q)$; for $\mathbb{F} = \mathbb{C}$ we obtain the complex sphere $S^n_{\mathbb{C}} = \mathrm{SO}(n+1,\mathbb{C})/\mathrm{SO}(n,\mathbb{C})$. In contrast to all preceding examples, the realization of M is not a representation of a symmetric space. If we wish to realize M in a faithful representation, we cannot use the quadratic map $Q : M \to \mathrm{SO}(A,\mathbb{F})$ since it is not injective (we have $s_x = s_{-x}$). In particular, $Q(S^n) \subset \mathrm{SO}(n+1)$ is isomorphic to the real projective space $\mathbb{RP}^n = \mathrm{Gr}_{1,n+1}(\mathbb{R})$.

A faithful representation can be constructed using Clifford algebras. Recall that the *Clifford algebra* $\mathrm{Cl}(A,\mathbb{F})$ associated to the quadratic form $q(x) = (x|x)$ is a 2^n-dimensional associative algebra with unit 1 containing V as a subspace such that the relation $x^2 = q(x)1$, or in other terms

$$\frac{xy + yx}{2} = (x|y)1 \tag{6.68}$$

holds for all $x, y \in V$.

Proposition I.6.2. *We consider M as a subset of $\mathrm{Cl}(A, \mathbb{F})$ via the imbedding $V \subset \mathrm{Cl}(A, \mathbb{F})$. Then M is a subsymmetric space of the space of elements of order 2 in the unit group of $\mathrm{Cl}(A, \mathbb{F})$.*

Proof. In the following all products are taken in $\mathrm{Cl}(A, \mathbb{F})$. If $x \in M$, then $x^2 = (x|x)1 = 1$, proving that M belongs to the space of elements of order 2. Furthermore, if $x, y \in M$, then

$$xy^{-1}x = xyx = (2(x|y) - yx)x = 2(x|y)x - y = \mu(x, y),$$

proving that M is a subsymmetric space of the unit group of $\mathrm{Cl}(A, \mathbb{F})$. ∎

We want to describe the image of the injective homomorphism $M \to \mathrm{Cl}(A, \mathbb{F})^\times$ in more detail. Recall from [LaMi89, p.13] the *Pin-group* which is the subgroup of the group of units of the Clifford algebra generated by elements $v \in V$ with $(v|v) = \pm 1$:

$$\mathrm{Pin}(A, \mathbb{F}) = \langle v \in V,\, (v|v) = \pm 1 \rangle \subset \mathrm{Cl}(A, \mathbb{F})^\times \tag{6.69}$$

Recall from [LaMi89, p.20] that the map

$$\widetilde{\mathrm{Ad}} : \mathrm{Pin}(A, \mathbb{F}) \mapsto \mathrm{O}(A, \mathbb{F}), \quad y \mapsto (v \mapsto \alpha(y) v y^{-1}) \tag{6.70}$$

is a covering of order 2 if $\mathbb{F} = \mathbb{R}$ and of order 4 if $\mathbb{F} = \mathbb{C}$. Here α is the involutive automorphism of $\mathrm{Cl}(A, \mathbb{F})$ determined by $\alpha(v) = -v$ for all $v \in V$. The inverse image of $\mathrm{SO}(A, \mathbb{F})$ under this covering is the *Spin group* $\mathrm{Spin}(A, \mathbb{F})$. It is equal to $\mathrm{Pin}(A, \mathbb{F})^\alpha$ ([LaMi89, p. 14]).

Proposition I.6.3.

$$M = \{x \in \mathrm{Pin}(A, \mathbb{F}) | \quad x^2 = 1,\, \mathrm{sgn}(\widetilde{\mathrm{Ad}}(x)) = (n-1, 1)\}$$

Proof. If $x \in M$, then $x \in \mathrm{Pin}(A, \mathbb{F})$, $x^2 = 1$ and $\widetilde{\mathrm{Ad}}(x)$ is reflection w.r.t. the orthocomplement of x, whence $\mathrm{sgn}(\widetilde{\mathrm{Ad}}(x)) = (n-1, 1)$. For the proof of the other inclusion, note that the right hand side is equal to

$$\widetilde{\mathrm{Ad}}^{-1}(N), \quad N = \{y \in \mathrm{O}(A, \mathbb{F}) |\, y^2 = 1, \mathrm{sgn}(y) = (n-1, 1)\}.$$

For every $y \in N$ we can find inverse images in M: y is self-adjoint and is therefore a reflection w.r.t. the orthocomplement $x^\perp$ of a vector x which we can normalize such that $(x|x) = 1$. Now x and $-x$ are the two preimages in $\mathrm{Pin}(A, \mathbb{F})$ of y in the real case, and $x, -x, ix, -ix$ are the preimages in the complex case. ∎

In particular, the sphere is realized as a subspace of $\mathrm{Pin}(n) := \mathrm{Pin}(\mathbf{1}_n, \mathbb{R})$.

6.7. Other spaces. *Oriented Grassmannians.* The group $\mathrm{SO}(p+q)$ acts transitively on the space $M_{p,q}$ of oriented p-planes in $\mathbb{R}^{p+q}$, and

$$M_{p,q} \cong \mathrm{SO}(p+q)/(\mathrm{SO}(p) \times \mathrm{SO}(q)) \tag{6.71}$$

as a homogeneous space. This space is a (simply connected) double cover of the Grassmannian $\mathrm{Gr}_{p,q}(\mathbb{R})$ (cf. [Hel78, p.453]). In particular, it is a homogeneous symmetric space. Similarly to the proof of Prop. I.6.3, one can prove that

$$M_{p,q} \cong \{g \in \mathrm{Pin}(n) |\, g^2 = 1, \mathrm{sgn}(\widetilde{\mathrm{Ad}}(g)) = (p, q)\}; \tag{6.72}$$

this defines a faithful representation of $M_{p,q}$. Of particular interest is the case $p = 2$: the spaces of oriented planes in $\mathbb{R}^{n+2}$ has yet another signification as we will see below.

Projective compactifications of quadratic hypersurfaces. Let $M_1 = \{x \in \mathbb{R}^n \,|\, x^t I_{p,q} x = 1\}$ ($p > 0$); as seen above, this is a symmetric space of the type $\mathrm{SO}(p,q)/\mathrm{SO}(p-1,q)$. The projective compactification of M_1 is given by

$$M = \{[z] \in \mathbb{P}(\mathbb{R}^{n+1}) \,|\, z^t I_{p,q+1} z = 0\}. \tag{6.73}$$

The actions of the group $\mathbb{P}\mathrm{O}(p,q+1)$ and of its maximal compact subgroup $\mathbb{P}(\mathrm{O}(p) \times \mathrm{O}(q+1))$ are transitive on M, and

$$\begin{aligned} M &\cong \mathbb{P}(\mathrm{O}(p) \times \mathrm{O}(q+1))/\mathbb{P}(\mathrm{O}(p-1) \times \mathrm{O}(1) \times \mathrm{O}(q) \times \mathrm{O}(1)) \\ &\cong (S^{p-1} \times S^q)/(\mathbb{Z}/(2)), \end{aligned} \tag{6.74}$$

where the quotient of the direct product $S^{p-1} \times S^q$ of spheres is taken w.r.t. the equivalence relation $(x,y) \cong (-x,-y)$. As a symmetric space, M is therefore locally isomorphic to a direct product of spheres.

Next we consider the complex case: $M_1 = \{x \in \mathbb{C}^n \,|\, z^t z = 1\} \cong \mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n-1,\mathbb{C})$,

$$M = \{[z] \in \mathbb{P}(\mathbb{C}^{n+1}) \,|\, z^t z = 0\}; \tag{6.75}$$

this space is compact and can therefore not be a complex symmetric space. In fact, the maximal compact subgroup $\mathrm{SU}(n+1) \cap \mathrm{SO}(n+1,\mathbb{C})) \cong \mathrm{SO}(n+1)$ of $\mathrm{SO}(n+1,\mathbb{C})$ acts transitively on M, and the stabilizer of the base point $[(0,\ldots,i,1)^t]$ is isomorphic to $\mathrm{SO}(n-1) \times \mathrm{SO}(2))$ (cf. Section IV.1.5), whence

$$M \cong \mathrm{SO}(n+1)/\mathrm{SO}(n-1) \times \mathrm{SO}(2)). \tag{6.76}$$

Thus M is a homogeneous symmetric space, isomorphic to the space of oriented 2-planes in $\mathbb{R}^{n+1}$. (For an explicit realization of this isomorphism cf. [Sa80, p.286].)

Appendix A: Tangent objects and their extensions

This appendix serves merely to fix our notation and to introduce some general terminology.

A.1. Basic notation and conventions.

Transformation Groups. Let M be a smooth manifold and G a group of diffeomorphisms of M. For a smooth function f on M we let

$$g^* f := f \circ g, \quad g_* f := f \circ g^{-1};$$

Then $g \mapsto g_*$ is homomorphism of G into the group of automorphisms of the function algebra $\mathcal{C}^\infty(M)$, and $g \mapsto g^*$ is a homomorphism of the opposite group G^{op} of G (which is by definition the group G with the usual product $m(x,y)$ replaced by $m^{op}(x,y) := m(y,x)$) into $\mathrm{Aut}(\mathcal{C}^\infty(M))$. Both actions can be canonically extended to actions of G resp. of G^{op} on the Lie algebra $\mathfrak{X}(M) = \mathrm{Der}(\mathcal{C}^\infty(M))$ of vector fields on M, on the algebra of tensor fields on M and on the algebra of differential operators on M. In all cases we write g_* for the canonical action of G and g^* for the canonical action of G^{op}. For instance, if X is a vector field (a derivation of $\mathcal{C}^\infty(M)$), then $(g_* X)f = g_*(X(g^* f)) = (X(f \circ g)) \circ g^{-1}$; the same formula holds for an arbitrary differential operator X.

Lie transformation groups and a sign-convention. If G is a Lie group of diffeomorphisms of M, then we identify its Lie algebra with a Lie algebra of vector fields on M by identifying $X \in \mathfrak{g}$ with the vector field X^* on M given by

$$(X^* f)(p) := \frac{d}{dt}|_{t=0} f(\exp_G(tX)p)$$

where $\exp_G$ is the exponential map of G. We denote by $\mathfrak{g}^{op}$ the opposite Lie algebra of $\mathfrak{g}$ (the Lie algebra $\mathfrak{g}$ with the usual Lie bracket replaced by its negative). Then $\mathfrak{g}^{op}$ is the Lie algebra of G^{op}, and the following diagram commutes (cf. [Hel78, p.122])

$$\begin{array}{ccc} G^{op} & \stackrel{g \mapsto g^*}{\longrightarrow} & \mathrm{Aut}(\mathcal{C}^\infty(M)) \\ \exp_G \uparrow & & \uparrow \exp \\ \mathfrak{g}^{op} & \stackrel{X \mapsto X^*}{\longrightarrow} & \mathfrak{X}(M)_c \end{array} \quad ;$$

here again $\exp_G$ denotes the exponential map of G which coincides with the exponential map of G^{op}, and $\exp$ is the map defined on the set of complete vector fields assigning to a complete vector field Z the automorphism $(\varphi_1)^*$, where $t \mapsto \varphi_t$ is the flow of Z. Of course, the reason for this situation is the fact that groups and Lie algebras are defined on levels which behave contravariantly with respect to each other (the space level and the function level). In principle, there are two ways to deal with this dilemma: either

(a) we agree that all Lie brackets are taken in $\mathfrak{X}(M)$; then we still denote the Lie algebra $\{X^* | X \in \mathfrak{g}\}$ by $\mathfrak{g}$ and keep in mind that it is the opposite of the Lie algebra of the transformation group G corresponding to $\mathfrak{g}$; or

(b) we apply the "opposite functor" to the above diagram, i.e. we agree to take all Lie brackets in $\mathfrak{X}(M)^{op}$; then $\mathfrak{g}$ is really the Lie algebra of the corresponding transformation group.

However, in case (b) we will have serious troubles when we want to use notions of differential geometry such as connections, and therefore *we use the sign convention* (a). It should be noted that many authors implicitly adopt the convention (b) by choosing the opposite sign in Equation (A.2) below (cf. e.g. [Lo77, p. 0.3]).

Vector fields on vector spaces, calculus. If M is an open domain in a vector space V, then we identify all tangent spaces canonically with V. Thus vector fields on M are identified with smooth maps $X : V \supset M \to V$ which operate on functions by

$$(Xf)(p) = (\partial_{X(p)} f)(p) = \mathrm{d}f(p) \cdot X(p). \tag{A.1}$$

Using the fact that the second differential of f is a symmetric bilinear form, one gets the formula

$$[X, Y](p) = \mathrm{D}Y(p) \cdot X(p) - \mathrm{D}X(p) \cdot Y(p). \tag{A.2}$$

Here

$$\mathrm{D}\varphi : V \supset M \to \mathrm{Hom}(V, W), \quad x \mapsto \mathrm{D}\varphi(x) \tag{A.3}$$

is the (first) total differential of a smooth map $\varphi : V \supset M \to W$ into a vector space W. (Note that, in accordance with the sign-convention (a) above, this equips $\mathfrak{gl}(V)$ with the opposite of its usual Lie bracket!) Iterating (A.3), we get the second total differential

$$\mathrm{D}^2\varphi = \mathrm{D}(\mathrm{D}\varphi) : V \supset M \to \mathrm{Hom}(V, \mathrm{Hom}(V, W)), \quad x \mapsto \mathrm{D}^2\varphi(x). \tag{A.4}$$

Under the canonical isomorphism $\mathrm{Hom}(V, \mathrm{Hom}(V, W)) = \mathrm{Hom}(V \otimes V, W)$, $\mathrm{D}^2\varphi(x)$ becomes a symmetric bilinear map; thus $\mathrm{D}^2\varphi$ takes values in the space $\mathrm{Hom}(S^2V, W)$, and so on for higher differentials. (In contrast to the total differential of maps between vector spaces, we denote the tangent map at x of a smooth map $\varphi : M \to N$ between abstract manifolds M and N by $T_x\varphi : T_xM \to T_{\varphi(x)}N$.)

The action of a diffeomorphism g of the vector space V on vector fields is described by

$$(g_*X)(p) = (\mathrm{D}g^{-1}(p))^{-1} \cdot X(g^{-1}(p)); \qquad (g^*X)(p) = (\mathrm{D}g(p))^{-1} \cdot X(g(p)). \tag{A.5}$$

Once more according to our sign-convention we have the formula

$$\frac{d}{dt}|_{t=0}(\exp(tY)_*X) = -[Y, X]. \tag{A.6}$$

We often write $\mathbf{v}$ for the constant vector field with value $v \in V$. If $t_v(x) = v + x$ denotes the translation by $v \in V$, then

$$\exp(\mathbf{v}) = t_v, \tag{A.7}$$

where $\exp$ is the exponential function of the Lie group V.

A.2. Extensions. In differential geometry the "extension" of a tangent object to a local or global object on the manifold plays an important role. We define:

Definition I.A.1. Let M and M' be real manifolds and $p \in M$, $q \in M'$.

(1) An *extension* of a linear map $a : T_pM \to T_qM'$ is a smooth map $\alpha : M \to M'$ such that

$$\alpha(p) = q, \quad T_p\alpha = a.$$

(2) A *vector field extension of a tangent vector* $v \in T_pM$ is a vector field $\mathbf{v} \in \mathfrak{X}(M)$ such that

$$\mathbf{v}_p = v.$$

(3) A *vector field extension of an endomorphism* $A \in \mathrm{End}(T_pM)$ is a vector field $X \in \mathfrak{X}(M)$ such that $X_p = 0$ and for all $v \in T_pM$ and vector field extensions $\mathbf{v}$ of v,

$$[\mathbf{v}, X]_p = A(v).$$

If the extended object is only defined on a neighborhood of p, then we speak of *local extensions*.■

Note that in (3) the value of $[\mathbf{v}, X]_p$ does not depend on the chosen vector field extension of v because the bracket of two vector fields vanishing at p vanishes again at p. We choose a chart V of M such that p corresponds to the origin $0 \in V$, consider X as a smooth function defined on a neighborhood of 0 and choose $\mathbf{v}$ to be constant on V. Then $[\mathbf{v}, X]_0 = \mathrm{D}X(0)v$ by formula (A.2), and therefore condition (3) means that

$$\mathrm{D}X(0) = A. \tag{A.8}$$

Definitions (2) and (3) are just the beginning of a chain of definitions relating tangent objects to vector fields vanishing of order k at p. For example, the next step would be to define a vector field extension of a symmetric bilinear map $B : T_pM \times T_pM \to T_pM$ as a vector field X vanishing of order 2 such that $B(v, w) = [\mathbf{v}, [\mathbf{w}, X]]_p$ for all vector field extensions $\mathbf{v}, \mathbf{w}$ of v, w.

Clearly extensions as defined above are highly non-unique. However, in the presence of additional requirements they may become unique. For instance, extensions of tangent maps to *affine maps* (i.e. maps compatible with affine connections) are always unique (cf. [Lo69a, p.24]).

Lemma I.A.2. *Let X be a vector field defined on a neighborhood of $p \in M$ and φ_t its local flow, assumed to be defined for all t belonging to some intervall I and x belonging to some neighborhood of p, and let $A \in \mathrm{End}(T_pM)$. Then the following are equivalent:*

(i) *X is a local vector field extension of A.*

(ii) *φ_t is a local extension of $e^{tA} \in \mathrm{Gl}(T_pM)$.*

Proof. It is classical that p is a fixed point of the flow φ_t iff X vanishes at p. Using a local chart V as in the remarks preceding Eqn. (A.7) and considering φ_t as a local diffeomorphism fixing the origin of V, we note that $\mathbb{R} \to \mathrm{Gl}(V)$, $t \mapsto \mathrm{D}(\varphi_t)(0)$ is (by the chain rule) a homomorphism. The implication (ii) $\Rightarrow$ (i) now follows by a straightforward differentiation. Conversely, if (i) holds, then

$$\frac{d}{dt}_{t=0} \mathrm{D}(\varphi_t)(0) = \mathrm{D}(\frac{d}{dt}_{t=0} \varphi_t)(0) = \mathrm{D}X(0) = A,$$

and integration yields (ii). ■

Example I.A.3. (The Euler operator.) If $M = V$ is a vector space and $A = \mathrm{id}_V$, then $e^t \,\mathrm{id}_V$ is a (global) extension of e^{tA}. The corresponding vector field extension of A is the vector field $E = E^0$ given by

$$E(p) = p, \tag{A.8}$$

which is called the *Euler operator*. From Eqn. (A.2) we get that $[E, Z](p) = \mathrm{D}Z(p) \cdot p - Z(p)$, and the *Euler differential equation* follows: Z is homogeneous polynomial of degree r on V if and only if

$$[E, Z] = (r - 1)Z. \tag{A.9}$$

■

Appendix B: Affine Connections

B.1. Affine connections. An *affine connection* on a real manifold M is a $\mathbb{R}$-bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto \nabla_X Y$$

of the space $\mathfrak{X}(M)$ of vector fields on M such that for all smooth functions f and vector fields X, Y on M,

$$\nabla_{fX} Y = f \nabla_X Y, \quad \nabla_X (fY) = (Xf)Y + f \nabla_X Y.$$

Example I.B.1. Let $M = V$ a real vector space and identify vector fields on V with smooth functions on V as explained in Appendix A.1, Eqn. (A.1). Immediate verificaton shows that the formula

$$(\nabla_X Y)(p) := (\mathrm{D}Y)(p) \cdot X(p) \tag{B.1}$$

defines an affine connection on V. We call it the *canonical connection of* V. ■

Let us describe the structure of the space

$$\mathrm{Conn}(M) \subset \mathrm{Hom}(\mathfrak{X}(M) \otimes_{\mathbb{R}} \mathfrak{X}(M), \mathfrak{X}(M))$$

of affine connections on M. If ∇ and ∇' are affine connections on M, then the defining properties show that the difference

$$(\nabla - \nabla')(X,Y) = \nabla_X Y - \nabla'_X Y$$

is function-linear both in X and in Y; in other words, $\nabla - \nabla'$ is a tensor field of type (2,1), i.e. a section of the bundle $\mathrm{Hom}(TM \otimes TM, TM)$, where TM is the tangent bundle. Thus $\mathrm{Conn}(M)$ is an affine space whose translation space is the space of $(2,1)$ tensor fields: if ∇^0 is *one* connection on M, then

$$\mathrm{Conn}(M) = \nabla^0 + \mathcal{C}^\infty(\mathrm{Hom}(TM \otimes TM, TM)),$$

and if f_i are functions such that $\sum_i f_i = 1$ and ∇_i are connections, then $\sum_i f_i \nabla_i$ is again a connection. In particular, given charts U_i and a subordinate partition of unity $1 = \sum f_i$, the formula $\nabla := \sum f_i \nabla^{(i)}$ defines a connection on M, where $\nabla^{(i)}$ are the canonical connections of the U_i's.

If ∇^0 is the canonical connection of a vector space V, then any other connection on V can be written $\nabla = \nabla^0 + C$ with a a smooth map $C : V \to \mathrm{Hom}(V \otimes V, V)$; the coefficients C_k^{ij} with $(\nabla'_{e_i} e_j)(x) = \sum_k C_k^{ij}(x) e_k$ are called *Christoffel symbols.*

B.2. Covariant derivative and the torsion tensor. The operator

$$\nabla_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

has a unique extension to a derivation

$$\nabla_X : T(\mathfrak{X}(M)) \to T(\mathfrak{X}(M))$$

of the *mixed tensor algebra* $T(\mathfrak{X}(M))$ of $\mathfrak{X}(M)$ which commutes with contractions and such that $\nabla_X f := Xf$ for a smooth function f (cf. [Hel78, p.42], [Lo69a, p.27]). Then $X \mapsto \nabla_X S$ for $S \in T(\mathfrak{X}(M))$ is function-linear, so that $(\nabla S)(X) := (\nabla_X S)$ defines a tensor ∇S whose covariance degree is one higher than the one of S.

As an example, consider the covariant derivative of the one form $\mathrm{d}f$, where f is a function:

$$\Gamma^2(X,Y)f := (\nabla \mathrm{d}f)(X,Y) = X\langle \mathrm{d}f, Y\rangle - \langle \mathrm{d}f, \nabla_X Y\rangle = (XY - \nabla_X Y)f.$$

Note that $\Gamma^2(X,Y)$ is a second order differential operator, depending function-linearly on X and Y since $\nabla \mathrm{d}f$ is a tensor. (It generalizes the classical *Hesse matrix* of second partial derivatives.) Thus

$$T_\nabla(X,Y) := (\Gamma^2)(X \otimes Y - Y \otimes X) = [X,Y] - \nabla_X Y + \nabla_Y X.$$

again depends function-linearly on X and Y; hence it is a tensor field, called the *torsion tensor of* ∇. A connection ∇ is torsionfree if and only if the corresponding Γ^2 is a *symmetric* function-bilinear map

$$\mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}^2(M),$$

where $\mathfrak{X}^2(M)$ is the space of second order differential operators. Note that the difference of two connections is the same as the difference of the corresponding Γ^2's. Therefore all torsionfree connections are obtained by taking *one* torsionfree connection and adding a *symmetric* bilinear map $\mathfrak{X}(M) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)$.

Example I.B.2. The canonical connection of a vector space is torsionfree: since the torsion T_∇ is a tensor, it suffices to verify that $(T_\nabla)(\mathbf{w},\mathbf{u})=0$, where for $v\in V$ we denote by $\mathbf{v}$ the constant vector field with value $v\in V$. But clearly $\nabla_{\mathbf{w}}\mathbf{u}=0$ and $[\mathbf{w},\mathbf{u}]=0$; therefore also $(T_\nabla)(\mathbf{w},\mathbf{u})=0$. Thus the commutator of two vector fields is given by

$$[X,Y](p)=(\nabla_X Y-\nabla_Y X)(p)=(\mathrm{D}Y)(p)\cdot X(p)-(\mathrm{D}X)(p)\cdot Y(p),$$

leading again to formula (A.2). ∎

B.3. Iterated covariant derivatives and the curvature tensor. Let us calculate

$$\begin{aligned}\Gamma^3(X,Y,Z)f &:= (\nabla\nabla \mathrm{d}f)(X,Y,Z)=(\nabla_X\nabla\mathrm{d}f)(Y,Z)\\ &= X\cdot((\nabla\mathrm{d}f)(Y,Z))-(\nabla\mathrm{d}f)(\nabla_X Y,Z)-(\nabla\mathrm{d}f)(Y,\nabla_X Z)\\ &= (XYZ-(X\nabla_Y Z+(\nabla_X Y)Z+Y(\nabla_X Z))+\nabla_{\nabla_X Y}Z+\nabla_Y(\nabla_X Z))f.\end{aligned}$$

Note that $\Gamma^3(X,Y,Z)$ is a third order differential operator, depending function-linearly on X,Y,Z since $\nabla\nabla\mathrm{d}f$ is a tensor. The symmetry relations of Γ^3 play a basic role in differential geometry. For simplicity we assume that ∇ is torsionfree, i.e. $[U,V]=\nabla_U V-\nabla_V U$. Then, as seen above, $\nabla\mathrm{d}f$ is symmetric in both arguments, and therefore $\nabla_X\nabla\mathrm{d}f$ is again symmetric in both arguments; i.e. Γ^3 is symmetric in Y and Z. It is in general not symmetric in X and Y:

$$\begin{aligned}\Gamma^3(X\otimes Y\otimes Z-Y\otimes X\otimes Z) &= [X,Y]Z-(\nabla_X Y-\nabla_Y X)Z+\\ &\qquad \nabla_{\nabla_X Y-\nabla_Y X}Z+[\nabla_Y,\nabla_X]Z\\ &= -T_\nabla(X,Y)Z+\nabla_{\nabla_X Y-\nabla_Y X}Z-[\nabla_X,\nabla_Y]T\\ &= (\nabla_{[X,Y]}-[\nabla_X,\nabla_Y])Z\\ &=: -R(X,Y)Z;\end{aligned}$$

where we used that $T_\nabla=0$. Note that $\Gamma^3(X\otimes Y\otimes Z-Y\otimes X\otimes Z)$ is a first order differential operator (if the torsion does not vanish it is a second order differential operator) depending function-linearly on X,Y,Z. The function-trilinear map R defined by the last equation is called the *curvature tensor* of ∇. A connection is called *flat* if it is torsionfree and $R=0$.

Example I.B.3. The canonical connection on a vector space is flat; this follows in the same way as we showed in Ex. I.B.2 that it is torsionfree. ∎

Remark I.B.4. By induction, one may define

$$\Gamma^k(X_1,\ldots,X_k)f:=(\nabla^k f)(X_1,\ldots,X_k)$$

and show that this is a k-th order differential operator depending function-linearly on the X_i's. It turns out that its symmetry relations under the action of the symmetric group Σ_k can all be described using the Γ^j's with $j<k$ together with the torsion tensor and the curvature tensor along with their covariant derivatives. These relations are not all independent: the linear relations between elements of Σ_k in this representation are given by the so-called *Bianchi-identities.* ∎

B.4. Automorphisms and derivations of a connection. The group of diffeomorphisms of M acts on the space of (torsionfree) connections as follows. Recall from Appendix A.1 the actions g_* and g^* on the the space of vector fields on M. Then for any connection ∇, the formula

$$(g_*\nabla)_X Y:=g_*(\nabla_{g^*X}g^*Y) \tag{B.2}$$

defines a connection $g_*\nabla$; if $g_*\nabla = \nabla$, then g is an automorphism of ∇, also called *an affine diffeomorphism* w.r.t. ∇. Clearly the torsion of $g_*\nabla$ is g_* applied to the torsion of ∇; therefore g_* preserves the space of torsionfree connections. We let $g^*\nabla = (g^{-1})_*\nabla$.

As an infinitesimal version of (B.2), the formula

$$(Z \cdot \nabla)_X Y := [Z, \nabla_X Y] - \nabla_{[Z,X]} Y - \nabla_X [Z, Y] \tag{B.3}$$

defines a tensor field of type (2,1) (symmetric if ∇ is torsionfree). If $Z \cdot \nabla = 0$, then Z is a derivation of ∇, also called an *affine vector field* w.r.t. ∇. Its local flow is a family of local automorphisms of ∇.

Example I.B.5. If ∇ is the canonical connection of a vector space V, we can describe the action of the diffeomorphism group more explicitely. From formula (A.5) one deduces that for all $u, v \in V$,

$$(g^*\nabla - \nabla)_p(u \otimes v) = (\mathrm{D}g(p))^{-1} \cdot ((\mathrm{D}^2 g)(p))(v \otimes u), \tag{B.4}$$

where $\mathrm{D}^2 g : V \to \mathrm{Hom}(S^2 V, V)$ is the ordinary second differential of $g : V \to V$. (In order to verify this formula, note that $g^*\nabla - \nabla$ is a tensor field of type (2,1); hence it suffices to determine its values on the constant vector fields.) ■

B.5 Affine maps. One would like to define the notion of an *affine map* between manifolds $M, \widetilde{M}$ with affine connections $\nabla, \widetilde{\nabla}$ as if $\nabla, \widetilde{\nabla}$ were tensor fields. However, this is not the case, and therefore we use the function-bilinear map Γ^2 defined in Section B.2.

Definition I.B.6. For a real manifold M we denote by

$$\mathfrak{X}^2(M) := Span_{\mathbb{R}}\{XY \mid X, Y \in \mathfrak{X}(M)\}$$

the space of second order differential operators without constant term and by

$$T_p^2 M := \{Z_p \mid Z \in \mathfrak{X}^2(M)\}$$

the *second order tangent space* at $p \in M$ (cf. [Lo69a, p.6]). If $\varphi : M \to M'$ is a smooth map, then its *second order tangent map* at $p \in M$ is defined by

$$T_p^2\varphi : T_p^2 M \to T_{\varphi(p)}^2 M', \quad (T_p^2\varphi \cdot Z_p) f := Z_p(f \circ \varphi);$$

cf. [Lo69a, p.9]. ■

If $m : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^2(M)$ is the natural composition $(X, Y) \mapsto XY$, then in Section B.2 we associated to any connection ∇ the map $\Gamma^2 := m - \nabla$.

Definition I.B.7. Let $M, \widetilde{M}$ be manifolds with connections $\nabla, \widetilde{\nabla}$ and corresponding maps $\Gamma^2, \widetilde{\Gamma}^2$. A smooth map $\varphi : M \to \widetilde{M}$ is called *affine* w.r.t ∇ and $\widetilde{\nabla}$ if for all $p \in M$ and $u, v \in T_p M$,

$$T_p^2\varphi \cdot \Gamma_p^2(u, v) = \widetilde{\Gamma}^2_{\varphi(p)}(T_p\varphi \cdot u, T_p\varphi \cdot v),$$

i.e. the diagram

$$\begin{array}{ccc} T_p^2 M & \to & T_{\varphi(p)}^2 \widetilde{M} \\ \uparrow & & \uparrow \\ T_p M \otimes T_p M & \to & T_{\varphi(p)}\widetilde{M} \otimes T_{\varphi(p)}\widetilde{M} \end{array}$$

commutes. ■

Example I.B.8. If $M = \widetilde{M}$, $\nabla = \widetilde{\nabla}$ and φ is bijective (or a local diffeomorphism), then φ is affine iff it is an automorphism of Γ^2. But since the natural map m is invariant under all diffeomorphisms, automorphisms of $\Gamma^2 = m - \nabla$ and automorphisms of ∇ are the same; thus the notion of affine maps given here extends the one from the previous section. In particular, if ∇ is the canonical connection of a vector space, formula (B.4) shows that φ is affine iff $\mathrm{D}^2\varphi = 0$, that is iff $\varphi(x) = Ax + b$ with $A \in \mathrm{Gl}(V)$ and $b \in V$. ■

Remark I.B.9. It is possible to introduce a general second differential

$$\mathrm{D}_p^{(2)}\varphi : T_pM \otimes T_pM \to T_{\varphi(p)}\widetilde{M}$$

of a smooth map $\varphi : M \to \widetilde{M}$ such that φ is affine iff $\mathrm{D}^{(2)}\varphi = 0$. This is done as follows: every connection defines a direct sum decomposition

$$T_p^2 M = T_pM \oplus \Gamma_p^2(T_pM \otimes T_pM),$$

cf. [Lo69a, p.7 and p.20], and we define $\mathrm{D}_p^{(2)}\varphi$ as the diagonal map in the following commutative diagram:

$$\begin{array}{ccccc}
T_p^2 M & = & T_pM & \oplus & \Gamma_p^2(T_pM \otimes T_pM) \\
T_p^2\varphi \downarrow & & \downarrow T_p\varphi & \swarrow & \downarrow T_p\varphi \otimes T_p\varphi \\
T_{\varphi(p)}^2 \widetilde{M} & = & T_{\varphi(p)}\widetilde{M} & \oplus & \Gamma_{\varphi(p)}^2(T_{\varphi(p)}\widetilde{M} \otimes T_{\varphi(p)}\widetilde{M}).
\end{array}$$

Using the maps Γ^k from Remark I.B.4, one can also define higher differentials at a point p as multilinear maps $\otimes^k T_pM \to \otimes^j T_pM$ for $j \leq k$. They have similar (but more complicated) properties as the ordinary higher differentials of smooth maps between vector spaces. ■

B.6. Geodesics and exponential map. If ∇ is an affine connection on M, then an affine map

$$\gamma : \mathbb{R} \supset I \to M$$

is called a *geodesic*. Here I is an open interval in $\mathbb{R}$, equipped with the canonical connection induced from $\mathbb{R}$. For $v \in T_pM$, there exists a unique geodesic γ_v with $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. The map

$$\mathrm{Exp} = \mathrm{Exp}_p : T_pM \supset U \to M, \quad v \mapsto \gamma_v(1),$$

where U is a suitable neighborhood of 0 in T_pM, is called the *exponential map of* ∇ *(at the point* p*)*. It is a local diffeomorphism (cf. [Lo69a, p.22]).

Notes for Chapter I.

I.1. – I.3. We have closely followed the presentation from [Lo69a] and [KoNo69]. The simple observation stated in Prop. I.2.1 and its consequences simplify some arguments given there. The terminology “prime” (Def. I.1.4) is used by O. Kowalski ([Kow80, p.41]). For a discussion of the relation between homomorphisms of LTS and equivariant maps we refer to [Jac51] (cf. also Notes to Ch.V).

I.4. The definition of symmetric spaces via the multiplication map (Def. I.4.4) is due to O.Loos ([Lo67], [Lo69a]).

I.5. Representations of Lie triple systems already appear in [Jac49], but representations of symmetric spaces seem not yet to have been considered as a proper subject in the literature. The quadratic map has been introduced by O. Loos ([Lo67], [Lo69a]).

I.6. Our presentation of examples is an enlarged version of [Lo69a, Section II.1.2]; see also [MnT86, Appendice 6.A]. Spaces of complex structures in classical groups are classified and investigated in [Bir98].

Appendix B. The treatment of the curvature in Section B.3 is inspired by the introduction of affine connections in [Lo69a, p.19]; see also [Po62]. As mentioned in Remarks I.B.4 and I.B.9, it is possible to extend this idea to a general theory of higher order differentials on affine manifolds (unpublished notes of the author).

Chapter II: Prehomogeneous symmetric spaces and Jordan algebras

Prehomogeneous symmetric spaces are symmetric spaces which are open orbits under the action of a linear group in a vector space. The interaction of the symmetric space structure with the flat structure of the ambient vector space gives rise to new features: a new, commutative algebra, the so-called *Lie triple algebra* emerges (Section 1). The most important examples of these algebras are *Jordan algebras*. They are characterized by the fact that the quadratic map of the corresponding symmetric space extends to a quadratic polynomial defined on the ambient vector space. We call the corresponding prehomogeneous symmetric spaces *quadratic* (Section 2). Important examples of such spaces are the general linear groups and the cones of positive definite symmetric matrices (Section 3). In the final section we introduce briefly two other classes of symmetric orbits in vector spaces; a typical example is given by the orthogonal groups which are certain closed symmetric orbits in the vector space of real or complex matrices.

1. Prehomogeneous symmetric spaces and Lie triple algebras

Definition II.1.1. Let G be a closed subgroup of the general linear group $\mathrm{Gl}(V)$ of a complex or real vector space V.

(1) An orbit $G.e \subset V$ $(e \in V)$ is called a *symmetric orbit* if G/H, where H is the stabilizer of e in G, is a symmetric space.

(2) If there exists a point $e \in V$ such that the orbit $\Omega := G.e$ is open in V, then we say that (G, V, e) is a *prehomogeneous vector space (with base point).*

(3) If $\Omega = G.e$ is an open symmetric orbit in V, we say that (G, σ, V, e) is a *prehomogeneous symmetric space.* Here σ is the involution of G belonging to the symmetric space $\Omega = G/H$. *Homomorphisms* of prehomogeneous symmetric spaces are linear base point-preserving maps which are, when restricted to the open symmetric orbit, homomorphisms of symmetric spaces. ■

The most important example of a prehomogeneous symmetric space is the group $\mathrm{Gl}(n, \mathbb{F})$ which is an open symmetric orbit in the matrix space $M(n, \mathbb{F})$. More examples are given in Section 3.

If (G, V, e) is a prehomogeneous vector space, the projection $G \to V, g \mapsto g.e$ is open, and therefore its differential at the base point

$$\mathrm{ev}_e : \mathfrak{g} \to V, \quad X \mapsto Xe = \frac{d}{dt}|_{t=0} \exp(tX).e$$

is surjective. Here $\mathfrak{g} \subset \mathfrak{gl}(V)$ denotes the Lie algebra of $G \subset \mathrm{Gl}(V)$; it is identified with the tangent space T_eG. The kernel of ev_e is $\mathfrak{h}$, the Lie algebra of the stabilizer H of the base point

e. If in addition the open orbit is symmetric in the sense of Def. II.1.1, we denote as usual by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the decomposition w.r.t. the differential of σ. Then $\kappa := \mathrm{ev}_e|_{\mathfrak{q}}$ is a bijection whose inverse we denote by $L : V \to \mathfrak{q}$. Now the action of $\mathfrak{g}$ on V restricted to $\mathfrak{q}$ yields a bilinear map (in other words, an *algebra structure*)

$$A_e : V \times V \to V, \quad (x, y) \mapsto xy := L(x)y,$$

and $L(x)$ is the *operator of left multiplication by* x in this algebra. The base point e becomes a right unit for this algebra: $L(x)e = \kappa(L(x)) = x$ for all x, whence $xy = L(x)L(y)e$. Moreover, since $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}$, it follows that $xy - yx = [L(x), L(y)] \cdot e \in \mathfrak{h} \cdot e = 0$, and the product is *commutative.* This implies that e is also a left unit, i.e. $L(e) = \mathrm{id}_V$. Hence $\exp(\lambda L(e)) = e^\lambda \mathrm{id}_V \in G$, and we thus see that all non-zero scalars belong to G in the complex case, and all positive scalars belong to G in the real case. Hence the open orbit Ω is a (in general non-convex) *cone.*

Lemma II.1.2. *The map $L : V \to \mathfrak{q}$ is H- and $\mathfrak{h}$-equivariant. In other words, H is a subgroup of the group $\mathrm{Aut}(V, A_e)$ of automorphisms of the algebra structure A_e, and $\mathfrak{h}$ is a subalgebra of the Lie algebra $\mathrm{Der}(V, A_e)$ of derivations of A_e.*

Proof. For $h \in H$, $h \cdot e = e$, and hence $\mathrm{ev}_e(h \circ X \circ h^{-1}) = h(Xe) = h(\mathrm{ev}_e(X))$, thus κ is H-equivariant. By derivation, it follows that κ is also $\mathfrak{h}$-equivariant, and the same is true for $L = \kappa^{-1}$.

Let us recall some basic notions related to algebras in order to explain the second statement. We identify $A_e : V \otimes V \to V$ and $L : V \to \mathfrak{q} \subset \mathrm{Hom}(V, V)$ under the canonical isomorphism $\mathrm{Hom}(V \otimes V, V) \cong \mathrm{Hom}(V, \mathrm{Hom}(V, V))$. Recall from elementary representation theory that $\mathrm{Gl}(V)$ acts in a natural way on these spaces such that they are both isomorphic to $V^* \otimes V^* \otimes V$ as $\mathrm{Gl}(V)$-modules. The *automorphism group* $\mathrm{Aut}(V, A_e)$ of A_e is the stabilizer of A_e in $\mathrm{Gl}(V)$; it is the group of invertible linear maps commuting with A_e, or (equivalently) commuting with L. Its Lie algebra is the Lie algebra $\mathrm{Der}(V, A_e) := \{X \in \mathfrak{gl}(V) | \, X \cdot A_e = 0\}$ of *derivations* of A_e. Here $X \cdot A = X \circ A - A \circ (X \otimes \mathrm{id} + \mathrm{id} \otimes X)$ is the natural action of $\mathfrak{gl}(V)$ on $\mathrm{Hom}(V \otimes V, V)$. The condition $X \cdot A = 0$ is equivalent to the rule $X(uv) = (Xu)v + u(Xv)$, where $uv = A(u \otimes v)$, and to $[X, L(v)] = L(Xv)$ for all $u, v \in V$. ■

(Remark. One can easily show that the association of the algebra A_e to a prehomogeneous symmetric space is functorial; the lemma can be seen as a consequence of this fact.) The lemma implies that

$$[L(V), L(V)] = [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h} \subset \mathrm{Der}(V, A_e). \tag{1.1}$$

Algebras with this property have been investigated by H. Petersson ([Pe67]) who introduced the following terminology.

Definition II.1.3. A commutative algebra (V, A) satisfying $[L(V), L(V)] \subset \mathrm{Der}(V, A)$ (where $L(v)x := A(v, x)$ is left multiplication by $v \in V$) is called a *Lie triple algebra.* ■

Putting $X = [L(x), L(y)] \in \mathrm{Der}(V, A_e)$, we have $[X, L(a)] = L(Xa)$, and (1.1) can be written explicitly in the form of the following identities (1.2) or (1.3):

$$[[L(x), L(y)], L(a)] = L(x(ya) - y(xa)), \tag{1.2}$$

$$x(y(ab)) - y(x(ab)) + b(y(xa)) - b(x(ya)) + a(x(yb)) - a(y(xb)) = 0. \tag{1.3}$$

The identity (1.2) explains the terminology: the space $L(V)$ is a Lie triple system. From this it follows that $\mathrm{Der}(V, A) \oplus L(V)$ is a symmetric Lie algebra.

Definition II.1.4. Given a Lie triple algebra (V,A) with unit element e, the Lie algebra

$$\mathfrak{str}(V) := \mathfrak{str}(V,A) := \mathrm{Der}(V,A) \oplus L(V) \subset \mathfrak{gl}(V)$$

is called the *structure algebra of* (V,A). If $G \subset \mathrm{Gl}(V)$ is the analytic subgroup with Lie algebra $\mathfrak{str}(V)$, then the orbit

$$\Omega := G \cdot e \subset V$$

is called the *associated cone*. ■

Remark II.1.5. Since $L(V) \to V$, $L(v) \to L(v)e = v$ is surjective, the associated cone of a unital Lie triple algebra is open in V, and since $e^{\mathbb{R}L(e)} = e^{\mathbb{R}}\,\mathrm{id}_V \subset G$, it is indeed a cone. The Lie algebra of the stabilizer H of e is $\mathrm{Der}(V,A)$, and thus Ω is a locally symmetric space belonging to the LTS $L(V)$. However, in general we cannot lift the the corresponding involution from the Lie algebra to the group G, and thus Ω is in general not globally symmetric. Roughly speaking, one can therefore say that unital Lie triple algebras correspond bijectively to a local version of prehomogeneous symmetric spaces. ■

We will see in the next section that for *Jordan algebras* a global version of a similar statement holds (Th. II.2.13).

2. Quadratic prehomogeneous symmetric spaces

2.1. The quadratic representation. For a prehomogeneous symmetric space with open symmetric orbit Ω, the quadratic map $Q : \Omega \to \mathrm{Gl}(V)$ is a representation of Ω in the sense of Def. I.5.1. Recall that it is equivariant w.r.t. the group of automorphisms of Ω, hence in particular w.r.t. G: $Q(gx) = gQ(x)\sigma(g)^{-1}$ for all $g \in G$, $x \in \Omega$. In the following we denote by W the vector space V together with the G-action by $g.w = \sigma(g)(w)$. Then we can consider Q as a G-equivariant map

$$Q : V \supset \Omega \to \mathrm{Hom}(W,V) = W^* \otimes V, \quad x \mapsto Q(x).$$

If $g = t\,\mathrm{id}_V \in G$ is a scalar with $t > 0$, then $\sigma(g) = t^{-1}\,\mathrm{id}_V$, and thus

$$Q(tx) = t\,\mathrm{id}_V \circ Q(x) \circ t\,\mathrm{id}_V = t^2 Q(x);$$

i.e. Q is homogeneous of degree two.

Definition II.2.1. A prehomogeneous symmetric space (G,σ,V,e) is called *quadratic* if the quadratic map

$$Q : V \supset \Omega \to G \subset \mathrm{Hom}(W,V), \quad g \cdot e \mapsto g\sigma(g)^{-1}$$

is quadratic polynomial, i.e. it extends to a homogeneous quadratic polynomial called the *quadratic representation* and denoted by $Q : V \to \mathrm{Hom}(W,V)$. ■

Assume that Q is as in the definition. This means that $Q(x,y) := Q(x+y) - Q(x) - Q(y)$ defines a symmetric bilinear map $Q : V \times V \to V$. Then the first differential of the quadratic map $Q : V \to \mathrm{Hom}(W,V)$ is given by

$$\mathrm{D}Q : V \to \mathrm{Hom}(V, \mathrm{Hom}(W,V)), \quad x \mapsto \mathrm{D}Q(x) = Q(x,\cdot)$$

This is a G-equivariant linear map. The second differential

$$\mathrm{D}^2Q : V \to \mathrm{Hom}(S^2V, \mathrm{Hom}(W,V)), \quad x \mapsto \mathrm{D}^2Q(x) = Q(\cdot,\cdot)$$

is a constant G-equivariant map (where $S^2(V) \subset V \otimes V$ is the second symmetric power of V), and Taylor's formula reads

$$Q(x) = \frac{1}{2}\mathrm{D}^2Q(0)\cdot(x\otimes x) = \frac{1}{2}Q(x,x).$$

We let

$$T : V\otimes W \otimes V \to V, \quad x\otimes y\otimes z \mapsto Q(x,z)y.$$

Then by definition $Q(x)y = \frac{1}{2}T(x,y,x)$.

Lemma II.2.2. *If Q is the quadratic representation of a quadratic prehomogeneous symmetric space, then the "fundamental formula"*

$$Q(Q(x)y) = Q(x)Q(y)Q(x) \tag{2.1}$$

holds for all $x, y \in V$.

Proof. By Lemma I.5.7 (i), this formula holds for $x, y \in \Omega$. Since it is a polynomial formula verified on an open set, it holds on all of V. ∎

Lemma II.2.3. *The differential of $Q : V \to \mathrm{Hom}(W,V)$ at the base point,*

$$\mathrm{D}Q(e) : V = T_e\Omega \to \mathrm{End}(V)$$

is identified with $2L$, where $L(v)x = vx$ is as in the preceding section.

Proof. Since $v = L(v)e$ for all $v \in V$, we have

$$\mathrm{D}Q(e)v = \frac{d}{dt}|_{t=0}Q(\exp(tL(V))e) = \frac{d}{dt}|_{t=0}\exp(2tL(v)) = 2L(v),$$

where we used that $\sigma(\exp(tL(v))^{-1} = \exp(tL(v))$. ∎

On the other hand, $(\mathrm{D}Q(e))v = Q(e,v)$, whence for all $v \in V$,

$$Q(e,v) = 2L(v). \tag{2.2}$$

Proposition II.2.4. *With the notation used above, the formulas*

$$Q(x,z) = 2(L(x)L(z) + L(z)L(x) - L(xz)),$$
$$Q(x) = 2L(x)^2 - L(x^2)$$

hold.

Proof. As already remarked above,

$$Q : V\otimes V \supset S^2(V) \to \mathrm{Hom}(W,V),$$

is a G-equivariant map. Because it is linear, it is also $\mathfrak{g}$-equivariant. This means that for all $X \in \mathfrak{g}$ and $v, w \in V$

$$Q(Xv\otimes w + v\otimes Xw) = X\circ Q(v,w) - Q(v,w)\circ\dot\sigma(X). \tag{2.3}$$

Using (2.2), we let $X := L(u) = Q(e\otimes u) \in \mathfrak{q}$ with $u \in V$. Then $\dot\sigma(X) = -X$, and (2.3) yields

$$Q(uv\otimes w + v\otimes uw) = L(u)Q(v\otimes w) + Q(v\otimes w)L(u). \tag{2.4}$$

Evaluating for $v = e$ and using (2.2), this implies

$$Q(u\otimes w) + 2L(uw) = 2L(u)L(w) + 2L(w)L(u),$$

whence the claim. ∎

The preceding proposition can also be written in the form

$$\frac{1}{2}T(x,y,z) = x(yz) + z(yx) - (xz)y = ([L(x), L(y)] + L(xy))z. \tag{2.5}$$

Definition II.2.5. A *Jordan algebra* is a commutative algebra (V, A) in which the identity

$$\text{(J2)} \qquad x(x^2 y) = x^2(xy)$$

(or, equivalently, $[L(x), L(x^2)] = 0$) holds. ■

By an elementary calculation one can show that every Jordan algebra is a Lie triple algebra (cf. [FK94, Prop. II.4.1]). The converse is not true (cf. [Pe67]). The next theorem relates quadratic prehomogeneous symmetric spaces to unital Jordan algebras.

Theorem II.2.6. *Let (V, G, σ, e) be a prehomogeneous symmetric space with associated Lie triple algebra A_e. Then the following are equivalent:*

(i) *The space (V, G, σ, e) is quadratic.*

(ii) *The map*

$$\widetilde{A} : \Omega \to \mathrm{Hom}(V \otimes V, V), \quad g \cdot e \mapsto \sigma(g) \cdot A_e := \sigma(g) \circ A_e \circ (\sigma(g)^{-1} \otimes \sigma(g)^{-1})$$

is linear (i.e. it is given by restricting a linear map to Ω).

(iii) *The algebra A_e is a Jordan algebra.*

If (i) – (iii) *hold, then the fundamental formula* (2.1) *and the identity*

$$\text{(JT2)} \qquad T(T(x,y,u),v,w) - T(u,T(y,x,v),w) + T(u,v,T(x,y,w)) = T(x,y,T(u,v,w))$$

are verified.

Proof. We prove that (i) implies (iii). Let (V, G, σ, e) be quadratic. We have already proved (2.1). In order to prove (J2) and (JT2), we write $T(x,y)z := T(x,y,z)$; then

$$T : V \otimes W \to \mathrm{End}(V), \quad v \otimes w \mapsto (x \mapsto T(v,w,x))$$

is $\mathfrak{g}$-equivariant. By (2.5) we have the explicit formula

$$T(x,y) = 2([L(x), L(y)] + L(xy)) \tag{2.6}$$

which implies that $T(V \otimes W) \subset \mathfrak{g}$. We choose $X = T(x,y) \in \mathfrak{g}$ and use the $\mathfrak{g}$-equivariance of T, noting that $\dot{\sigma}(T(x,y)) = (2[L(x), L(y)] - L(xy)) = -T(y,x)$:

$$T(T(x,y)u \otimes v - u \otimes T(y,x)v) = [T(x,y), T(u,v)]. \tag{2.7}$$

When applied to an element $w \in V$, this is precisely (JT2). We choose $y = e$; then since $T(x,e) = 2L(x)$, (2.7) yields

$$T(xu \otimes v) - T(u \otimes xv) = [L(x), T(u,v)]. \tag{2.8}$$

Using again the explicit formula (2.6) for T, we obtain an equation between elements of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Since the sum is direct, this is equivalent to two equations, one between operators from $\mathfrak{q}$ and the other between operators from $\mathfrak{h}$. The former is nothing but the identity defining a Lie triple algebra (Eqn. (1.2)). The latter reads

$$[L(xu), L(v)] + [L(vx), L(u)] + [L(uv), L(x)] = 0. \tag{2.9}$$

Putting $u = v = x$, we get the Jordan identity (J2).

For the proof that (iii) implies (i), the preceding arguments can essentially be all reversed: we polarize (J2) and get (2.9). Defining now $T(x,y)$ by the right hand side of Eqn. (2.6) and q by $q(x,y)z := T(x,z,y)$, we get (2.8) from (2.9) and from the defining relation of a Lie triple algebra. This implies (2.7), that is, the $\mathfrak{g}$-equivariance of T and q. It follows that q has the same G_o-equivariance as Q, and since $q(e) = Q(e) = \mathrm{id}_V$, we get $Q(x) = q(x)$ for all $x \in \Omega$. Since q is by definition a quadratic polynomial, (V, G, σ, e) is quadratic.

We prove that (ii) implies (iii). Note that $\widetilde{A}$ is linear if and only if the G-equivariant map

$$S : V \times \Omega^W \to \mathrm{End}(V), \quad (v,x) \mapsto \widetilde{A}(x)(v \otimes \cdot)$$

is bilinear; here Ω^W is the open orbit Ω with the action of G induced by W. If this is the case, then S is $\mathfrak{g}$-equivariant, and specializing to $X = L(v) \in \mathfrak{q} \subset \mathfrak{g}$, we see as above that $S = t$ is given by formula (2.6). Now the arguments given above apply, leading to (JT2) and (J2).

Conversely, the arguments proving that (iii) implies (i) show that (iii) also implies (ii). ■

Remark II.2.7. (1) Let A_p be the Lie triple algebra associated to Ω by the construction of the preceding section when choosing $p \in \Omega$ as base point. Then it is easily verified that $A_{g \cdot e} = g \cdot A_e$ for $g \in G$. Using Prop. I.2.1, one can show that $p \mapsto A_p$, considered as a tensor field of type (2,1), is the difference between the canonical flat connection of V and the canonical connection of $\Omega = G/H$.

(2) The reader may wonder whether it can be seen directly that properties (i) and (ii) of the preceding theorem are equivalent. However, this it is not so easy because the "initial values" given by (i) and (ii) are different: in terms of the trilinear map T, condition (i) gives us the initial value $T(e, \cdot, v) = 2L(v)$ (cf. Eqn. (2.2)), whereas (ii) gives $T(v, e, \cdot) = 2L(v)$. It seems thus that both (i) and (ii) reflect important properties of Jordan algebras. Under our geometric view point, (i) is a global or "integral" property intimately related to the fundamental formula (2.1), whereas (ii) is rather an infinitesimal or "differential" property intimately related to the identity (JT2) which will play a crucial role in the next chapters. ■

Definition II.2.8. A *polynomial representation* of a quadratic prehomogeneous symmetric space (V, G, σ, e) is a representation (in the sense of Def. I.5.1)

$$\pi : \Omega \to \mathrm{Gl}(E)$$

of the symmetric space $\Omega = G.e$ such that π is given by restricting a homogeneous polynomial map without constant term

$$\widetilde{\pi} : V \to \mathrm{End}(E)$$

to Ω. ■

In this sense the quadratic representation Q of Ω is a polynomial representation (homogeneous of degree two), and $\det \circ Q$ is a polynomial representation of degree $2 \dim V$. It is easily verified that a *linear representation* is a homomorphism $V \to \mathrm{End}(E)$ of Jordan algebras.

2.2. Jordan inverse, multiplication map and structure group. We assume that (V, G, σ, e) is a quadratic prehomogeneous symmetric space and derive some important consequences of the "quadratic property".

Proposition II.2.9.

(i) *The symmetry* $s_e : \Omega \to \Omega$, $ge \mapsto \sigma(g)e$ *extends to a birational map* j *of* V *given by*

$$j(x) = Q(x)^{-1} x.$$

Its differential is given by $\mathrm{D}j(x) = -Q(x)^{-1}$.

(ii) *The multiplication map* μ *of the symmetric space* Ω *is given for* $x, y \in \Omega$ *by*

$$\mu(x,y) = Q(x)j(y) = Q(x)Q(y)^{-1}y,$$

and it extends to a multiplication map on the open dense subset $V' := \{x \in V \mid \operatorname{Det} Q(x) \neq 0\}$ *of* V.

Proof. (i) By definition of s_e and Q, we have for all $x = g.e \in \Omega$, $s_e(x) = \sigma(g)e = \sigma(g)g^{-1}g.e = Q(x)^{-1}x$; therefore s_e extends rationally to the map j which is non-singular on the open dense set $\{x \in V \mid \det Q(x) \neq 0\}$.

The symmetry s_e is a G-equivariant map $\Omega \to \Omega^W$. Therefore its differential is a G-equivariant map $\mathrm{D}(s_e) : \Omega \to \operatorname{Hom}(V, W)$ such that $(\mathrm{D}s_e)(e) = -\operatorname{id}_V$. Let $x = g.e$. Then

$$(\mathrm{D}s_e)(x) = (\mathrm{D}s_e)(g.e) = \sigma(g) \circ (\mathrm{D}s_e)(e) \circ g^{-1} = -\sigma(g)g^{-1} = -Q(x)^{-1}.$$

By rationality of both sides, this extends to the equation $\mathrm{D}j(x) = -Q(x)^{-1}$.

(ii) By definition of the multiplication map (Section I.4), for all $x, y \in \Omega$,

$$\mu(x,y) = s_x y = s_x s_e s_e y = Q(x) s_e y = Q(x)j(y).$$

If $x, y \in V'$, then (using formula (2.1))

$$\det Q(Q(x)j(y)) = \det Q(Q(x)Q(y)^{-1}y) = \det Q(x)^2 \det Q(y)^{-1} \neq 0.$$

Thus $\mu : V' \times V' \to V'$ is well-defined, and by rationality the properties (M1) – (M4) extend from Ω to V'. ■

The arguments of the preceding proof show that also $\{x \in V \mid \operatorname{Det} Q(x) = 1\}$ is a symmetric space with multiplication map given by (ii).

Corollary II.2.10. *With the multiplication map from the preceding proposition and the atlas given by* V, V' *is an algebraic symmetric space in the sense of* Def. I.4.4 (3). ■

Note that V' is in general non-connected. For example, if $V = M(n, \mathbb{R})$, then $V' = \mathrm{Gl}(n, \mathbb{R})$ has two connected components, namely the elements with positive and those with negative determinant. In this case $Q(X)Y = XYX$ (see Section II.3.1), and the formula for μ reduces to $\mu(X,Y) = Q(X)Y^{-1} = XY^{-1}X$ (already known from Ex. I.4.2).

Definition II.2.11. If $\det Q(x) \neq 0$, then we let $x^{-1} := j(x) = Q(x)^{-1}x$. The birational map j is called the *Jordan inverse.* The group of linear automorphisms of μ,

$$\operatorname{Aut}(\mu) \cap \mathrm{Gl}(V) = \{g \in \mathrm{Gl}(V) \mid \forall x, y \in V' : \mu(gx, gy) = g\mu(x,y)\},$$

is called the *structure group* of the Jordan algebra (V, A) and is denoted by $\operatorname{Str}(V)$ or $\operatorname{Str}(V, A)$. ■

In the example $V = M(n, \mathbb{R})$ (and more generally for Jordan algebras derived from associative algebras, cf. 3.1), the Jordan inverse coincides with the usual inverse.

Proposition II.2.12.

(i) *The structure group is equal to each of the following groups:*

$$\{g \in \mathrm{Gl}(V) \mid \exists g^\sharp \in \mathrm{Gl}(V) : \forall x \in V : Q(gx)g^\sharp = gQ(x)\}, \tag{2.11}$$

$$\{g \in \mathrm{Gl}(V) |\, jgj \in \mathrm{Gl}(V)\}, \tag{2.12}$$

and the equation $g^\sharp = jgj$ *holds.*

(ii) *The data* $(V, \mathrm{Str}(V), \sharp, e)$ *define a quadratic prehomogeneous symmetric space.*

(iii) *The Lie algebra of* $\mathrm{Str}(V)$ *is*

$$\mathfrak{str}(V) := \mathfrak{str}(V, A) = \mathrm{Der}(V, A) \oplus L(V).$$

Proof. (i) A diffeomorphism g is an automorphism of μ if and only if for all $x \in V'$, $s_{gx} \circ g = g \circ s_x$. With the explicit formula $s_x = Q(x) \circ j$ from Prop. II.2.9, this is equivalent to

$$Q(gx) \circ j \circ g = g \circ Q(x) \circ j,$$

which can be written

$$Q(gx) \circ jgj = g \circ Q(x). \tag{2.13}$$

Since $Q(x)$ and $Q(gx)$ are invertible linear maps, it follows that jgj is a linear map if g is linear. Thus $\mathrm{Str}(V)$ is a subgroup of the group defined by (2.12), and putting $g^\sharp := jgj$, we see that it is also a subgroup of the group defined by (2.11).

Conversely, if g satisfies the condition from (2.11), then

$$\mu(gx, gy) = Q(gx)Q(gy)^{-1}gy = gQ(x)(g^\sharp)^{-1}g^\sharp Q(y)^{-1}g^{-1}gy = g\mu(x, y),$$

and thus g belongs to $\mathrm{Str}(V)$. If g satisfies the condition from (2.12), then the differential $\mathrm{D}j : V' \to \mathrm{End}(V)$ has the following equivariance property: for all g belonging to the group defined by (2.12),

$$\mathrm{D}j(gx) = jgj \circ \mathrm{D}j(x) \circ g^{-1}.$$

(This follows from the fact that $j : V' \to W'$ is equivariant, where W' is the space V' with the action by $g.x := jgj(x)$.) With the explicit formula $\mathrm{D}j(x) = -Q(x)^{-1}$ from Proposition II.2.9 (i), we get (2.13), i.e. $g \in \mathrm{Aut}(\mu)$.

(ii) The orbit $\mathrm{Str}(T).e$ is open in V since it contains $\exp(L(V)).e$. It is symmetric because $\mathrm{Str}(T) \subset \mathrm{Aut}(\mu)$ by definition and $\sharp$ is given by conjugation with the symmetry $s_e = j$ (part (i)).

(iii) From Eqn. (2.11) it follows that the Lie algebra of $\mathrm{Str}(V)$ is the space

$$\mathfrak{l} := \{X \in \mathfrak{gl}(V) |\, \exists X^\sharp \in \mathfrak{gl}(V) : \forall v \in V : Q(Xv \otimes v + v \otimes Xv) = X \circ Q(v, v) - Q(v, v) \circ X^\sharp\},$$

and $X \mapsto X^\sharp$ is the differential of $g \mapsto g^\sharp$ at the unit element. The equivariance property of Q already used in the proof of Prop. II.2.3 (cf. Eqn. (2.3)) shows that $\mathrm{Der}(V) \oplus L(V) \subset \mathfrak{l}$. On the other hand, let $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{q}$ be the decomposition w.r.t. $X \mapsto X^\sharp$. Then $L(V) \subset \mathfrak{q}$, and for reasons of dimension we have equality. Next, $\mathrm{Der}(V) \subset \mathfrak{h}$, and again equality holds since the arguments proving Lemma II.1.2 apply. ■

Finally we will prove that the functor from quadratic prehomogeneous symmetric spaces to unital Jordan algebras is surjective. If one starts with a Jordan algebra V with product xy and unit element e, then one defines the *quadratic representation of* (V, A) by $Q(x) := 2L(x)^2 - L(x^2)$, where $L(x)y = xy$, and the *associated triple product* is defined by $T(x, y, z) = 2(x(yz) - y(xz) + (xy)z)$. The *structure group of* (V, A) is defined by Eqn. (2.11). In this context it is easily seen that this is a group and $\sharp$ is an automorphism, but it is less evident that it is involutive.

Theorem II.2.13. *If V is a Jordan algebra with unit element e, then*

$$(V, \mathrm{Str}(V), \sharp, e)$$

is a quadratic prehomogeneous symmetric space whose associated algebra is the algebra we started with.

Proof. As already mentioned, any Jordan algebra is a Lie triple algebra. Thus we can associate to it a prehomogeneous locally symmetric space in the sense of Remark II.1.5. All arguments from the proof of Th. II.2.6 apply locally, showing that this space is actually "locally quadratic". But, as seen in Proposition II.2.9, then the local symmetry s_e extends to a birational map j of V, and our locally symmetric space is actually a connected component of V' and is global with multiplication map μ. Now the theorem follows by the same arguments which have been used in the proof of part (ii) of the preceding proposition. ∎

2.3. Power associativity. For $k \in \mathbb{N}$ we define the *powers of x in the Jordan algebra* (V, A_e) by

$$x^k := L(x)^k e, \quad x^{-k} := j(x^k),$$

the latter provided that $\det Q(x) \neq 0$.

Lemma II.2.14. *The powers in the symmetric space Ω (*Def. I.5.6*) and the powers in the Jordan algebra (V, A_e) coincide.*

Proof. We have to prove that $Q(x)^k e = L(x)^{2k} e$ and $Q(x)^k x = L(x)^{2k} x$ for all $k \in \mathbb{N}$. Using the formula $Q(x) = 2L(x)^2 - L(x^2)$, we have for $k = 1$: $Q(x)e = 2x^2 - x^2 = x^2 = L(x)^2 e$, $Q(x)x = 2x^3 - x^2 x = x^3 = L(x)^2 x$. Since $Q(x)$ and $L(x)$ commute, the claim now follows by induction. ∎

An algebra V with product denoted by $x \cdot y$ is called *power-associative* if

$$x^k \cdot x^l = x^{k+l}$$

for all $k, l \in \mathbb{N}$, where as above $x^k = L(x)^k e$.

Theorem II.2.15. *A unital Lie triple algebra is a Jordan algebra if and only if it is power associative.*

Proof. (1) Let V be a power associative Lie triple algebra with unit e. For $v \in V$ we define

$$\mathrm{Exp}(v) := \exp(L(v)) \cdot e = \sum_k \frac{v^k}{k!}$$

(this is the exponential map $\mathrm{Exp} : V = T_e\Omega \to \Omega \subset V$ associated to the locally symmetric space Ω; cf. Remark II.1.5.) From the fact that V is power associative one deduces as for the usual exponential map that for all $v \in V$ and $s, t \in \mathbb{R}$,

$$\mathrm{Exp}(tv) \cdot \mathrm{Exp}(sv) = \mathrm{Exp}((s+t)v).$$

Thus for $x = \mathrm{Exp}(v) = \exp(L(v)) \cdot e \in \Omega$, we have

$$Q(x)e = Q(\exp(L(v)) \cdot e)e = \exp(L(v))Q(e)\exp(L(v))e = \exp(2L(v))e = \mathrm{Exp}(2v) = x^2.$$

This is a quadratic polynomial in x. Now let $y = g \cdot e \in \Omega$ with $g \in G$; then

$$Q(x)y = Q(x)g \cdot e = \sigma(g)Q(\sigma(g)^{-1}x)e = \sigma(g)(\sigma(g)^{-1}x)^2$$

is again quadratic in x. Since Ω is open in V, it follows that $x \mapsto Q(x)$ is quadratic, and according to Th. II.2.6, V is a Jordan algebra.

(2) Now we assume that V is a Jordan algebra with unit element e. Let $R \subset V$ be the $\mathbb{R}$-linear span of the powers $e = x^0, x, x^2, \ldots$ and let $R' = R \cap V'$ be the set of invertible elements in R. Since the powers in V coincide with the powers in Ω, Lemma I.5.7 (i) yields $\mu(x^n, x^m) = x^{2n-m}$, and we deduce by rationality of μ that $\mu(R', R') \subset R'$. Thus R' is a subsymmetric space of V'. From the construction of the algebra associated to a prehomogeneous symmetric space (Section II.1) it follows that R is a subalgebra of V. We will prove the following lemma showing that for this subspace the formula for μ from Prop. II.2.9 can be simplified.

Lemma II.2.16. *For all* $a, b \in R'$,

$$\mu(a, b) = L(a^2)b^{-1} = a^2 \cdot b^{-1}.$$

Proof. We say that an element $y \in R$ *generates* R if R is the $\mathbb{R}$-linear span of the powers $e, y, y^2, \ldots$. Since this can be expressed by algebraic equations (using determinants), it follows that the set of generating elements is open dense in R.

We fix a generating element y and prove first by induction that for all $m \in \mathbb{N}$, $y^2 \cdot y^m = y^{m+2}$. For $m = 1$ this is clear, and assuming that the equation holds for $m - 1$, we get by (J2)

$$y^{m+2} = L(y)y^{m+1} = L(y)L(y^2)y^{m-1} = L(y^2)L(y)y^{m-1} = y^2 \cdot y^m.$$

We have proved that R is stable under $L(y)^2$ and under $L(y^2)$, and that both operators coincide on R. Since $Q(y) = 2L(y)^2 - L(y^2)$, we conclude that R is stable under $Q(y)$ and

$$Q(y)|_R = L(y^2)|_R = L(y)^2|_R.$$

If y is invertible, then $Q(y)$ is invertible, and thus also $Q(y)^{-1}y = y^{-1} \in R$.

Thus we get for all invertible generating elements $a, b \in R$ that

$$\mu(a, b) = Q(a)b^{-1} = L(a^2)b^{-1} = a^2 \cdot b^{-1}.$$

By density, the equation holds for all $a, b \in R'$. ∎

Now we finish the proof of the theorem. Assuming that x is invertible, we deduce from Lemma I.5.7 and the preceding lemma that for all $n, m \in \mathbb{Z}$,

$$x^{2n-m} = \mu(x^n, x^m) = (x^n)^2 \cdot x^{-m} = x^{2n} \cdot x^{-m}.$$

Choosing $-m \in \mathbb{N}$, we obtain power associativity for the case that one of the exponents is even. The general case is reduced to this one since on an open neighbourhood of e every element $x = \exp(X) \cdot e$ can be written $x = y^2$ where $y = \exp(\frac{X}{2}) \cdot e$; then $x^m x^n = y^{2n} y^{2m} = y^{2m+2n} = x^{m+n}$. Now power associativity, being an algebraic identity, extends from an open set to all of V. ∎

Note that in the proof of part (2) we have shown more than anounced, namely that $R' \subset V'$ is a subsymmetric space; it is an abelian group w.r.t. the product $(a, b) \mapsto a \cdot b$, and its multiplication map μ comes from this group structure via the usual formula $\mu(a, b) = ab^{-1}a = a^2 b^{-1}$.

3. Examples

3.1. Associative algebras. Let V be an associative algebra with unit element e and denote by $V^\times \subset V$ the group of invertible elements. The group $G = V^\times \times V^\times$ acts on V by $((g,h),v) \mapsto gvh^{-1}$. The G-orbit $G.e$ is equal to $V^\times$ and is therefore open in V.

Lemma II.3.1. *The prehomogeneous symmetric space just defined is quadratic, and the associated Jordan algebra A_e is given by*

$$A_e(X,Y) = \frac{1}{2}(XY + YX).$$

Proof. The open orbit $G.e$ is symmetric w.r.t. the involution $\sigma((g,h)) = (h,g)$ of G. Let $X \in V^\times$. Then $X = g.e$ with $g = (X,e) \in G$, and for all $Y \in V$,

$$Q(X)Y = g\sigma(g)^{-1}.Y = ((X,e)(e,X^{-1})).Y = (X,X^{-1}).Y = XYX.$$

Thus Q extends to a quadratic polynomial on V. Polarizing Q, we get the triple product

$$T(X,Y,Z) = (Q(X+Z) - Q(X) - Q(Z))Y = XYZ + ZYX$$

and the Jordan product

$$A_e(X,Y) = \frac{1}{2}T(X,e,Y) = \frac{1}{2}(XY+YX). \qquad \blacksquare$$

Specializing the lemma to the associative matrix algebras $M(n,\mathbb{F})$ $(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$, we see that the general linear groups $\mathrm{Gl}(n,\mathbb{F})$ are (real) quadratic prehomogeneous symmetric spaces; for $\mathbb{F} = \mathbb{C}$ we get a complex quadratic prehomogeneous symmetric space. Another example is the associative algebra V of upper triangular matrices. Its group of invertible elements is the group $\exp(V)$ of invertible upper triangular matrices. The corresponding Jordan algebra is nilpotent in the sense that $L(V)$ is a nilpotent LTS.

3.2. Subspaces.

Lemma II.3.2. *Let U be a subalgebra of a Jordan algebra V containing the unit element e. Then the prehomogeneous symmetric space belonging to U is given by restricting the corresponding data from V, and if Ω_U denotes the connected open symmetric orbit associated to U, then*

$$\Omega_U = (\Omega_V \cap U)_e.$$

Proof. Note first that, if $x \in U$ is invertible in V, then $j(x) \in U$. In fact, U is stable under $Q(x) = 2L(x)^2 - L(x^2)$ since it is a subalgebra. If x is invertible, then $Q(x)$ is a bijection of U and hence $j(x) = Q(x)^{-1}x \in U$. In this sense we can restrict j and the multiplication map μ to U resp. to $U \times U$.

Next, let G_U be the group generated by $\exp(L(U))$; then $\Omega_U = G_U.e \subset \Omega$ and clearly $\Omega_U \subset U$. By connectedness, $\Omega_U \subset (\Omega_V \cap U)_e$. For the other inclusion, recall from Prop. II.2.9 that Ω_U is a connected component of $U' = \{x \in U \mid \det Q_U(x) \neq 0\}$, and similarly for Ω. Now, if $x \in U$ and $\det Q(x) \neq 0$, then $\det Q_U(x) \neq 0$ since $Q(x)$ can be written as a block-matrix $\left(\begin{smallmatrix} Q_U(x) & * \\ 0 & * \end{smallmatrix}\right)$. Thus $x \in (V' \cap U)$ implies $x \in U'$, and the other inclusion of the claim follows. $\blacksquare$

We may call the orbit $\Omega_U \subset \Omega$ a *subspace* of the prohomogeneous symmetric space given by V. For example, the algebra $U = \mathrm{Sym}(n,\mathbb{R})$ is a subalgebra of $M(n,\mathbb{R})$. It is the fixed point set of the automorphism $X \mapsto X^t$ of $M(n,\mathbb{R})$. The open symmetric orbit associated to U is the cone $\mathrm{Gl}(n,\mathbb{R})/\mathrm{O}(n)$ of symmetric positive definite matrices. More generally, we consider the automorphism "adjoint" of $M(n,\mathbb{F})$ assigning to X its adjoint $X^* = A^{-1}\varepsilon(X^t)A$ (cf. Section I.6.1); then the fixed point space is the space of A-symmetric (resp. Hermitian) operators and the open orbits are of the form $\mathrm{Gl}(n,\mathbb{F})/\mathrm{O}(A,\mathbb{F})$, resp. $\mathrm{Gl}(n,\mathbb{F})/\mathrm{U}(A,\varepsilon,\mathbb{F})$, cf. Section I.6.2. See Tables XII.1.1. and XII.1.2 for a complete list of normal forms.

3.3. Scalar extended quadratic hypersurfaces. Let $V = \mathbb{F}^n$ be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ equipped with a non-degenerate symmetric bilinear form b and $G \subset \mathrm{Gl}(V)$ the subgroup generated by the group of isometries $\mathrm{O}(b)$ and the non-zero multiples of the identity. Choose a base point e such that $b(e,e) = 1$. Let F be the orthogonal reflection with respect to $\mathbb{F}e$. Then the stabilizer G_e is open in the fixed point group of the involution $\sigma(\lambda g) = \lambda^{-1}FgF$, $(g \in \mathrm{O}(b), \lambda \in \mathbb{F}^*)$, and therefore $G.e$ is a symmetric space.

Lemma II.3.3. *The data (V, G, σ, e) define a quadratic prehomogeneous symmetric space. The associated Jordan product is given by the formula*

$$A_e(x,y) = b(x,e)y + b(y,e)x - b(x,y)e.$$

Proof. The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is given explicitly by $\mathfrak{h} = \mathfrak{o}(b|_{e^\perp \times e^\perp})$ and

$$\mathfrak{q} = \mathbb{F}\,\mathrm{id}_V \oplus \mathfrak{q}_1, \quad \mathfrak{q}_1 = \{X \in \mathfrak{o}(b) |\, FXF = -X\}.$$

Direct verification shows that

$$L : \mathbb{F}e \oplus e^\perp \to \mathbb{F}\,\mathrm{id}_v \oplus \mathfrak{q}_1, \quad \lambda e + w \mapsto \lambda + e^* \otimes w - w^* \otimes e$$

(where $(v^* \otimes u)(x) := b(v,x)u$) is an inverse of $\mathrm{ev}_e : \mathfrak{q} \to \mathbb{F}^n$, $X \mapsto X(e)$. This implies that $G.e \subset V$ is open. Moreover, letting $z := x - b(x,e)e \in e^\perp$, this formula yields

$$L(x)y = L(b(x,e)e + z)y = b(x,e)y + b(y,e)z - b(z,y)e = b(x,e)y + b(y,e)x - b(x,y)e.$$

In order to prove that this is a Jordan product, note that

$$x^2 := A_e(x,x) = 2b(x,e)x - b(x,x)e,$$

and thus $L(x^2) = 2b(x,e)L(x) - b(x,x)\,\mathrm{id}_V$ commutes with $L(x)$, whence the identity (J2). ■

If $\mathbb{F} = \mathbb{R}$ and b is positive definite, $G.e$ is the complement of the origin, and if b has signature $(1,3)$, then the connected component of e of $G.e$ is the *open Lorentz-cone*. The normal forms are $b(x,y) = x^t I_{p,q} y$, $\Omega = (\mathrm{SO}(p,q) \times \mathbb{R}^+)/\,\mathrm{SO}(p-1,q)$.

The Jordan algebra just constructed can be interpreted as a subalgebra of the Clifford algebra (cf. Section I.6.6) associated to $V_1 = e^\perp$:

Lemma II.3.4. *Let V_1 be a real or complex vector space equipped with a bilinear form given by symmetric non-singular matrix A. On $V := \mathbb{F}1 \oplus V_1 \subset \mathrm{Cl}(A,\mathbb{F})$ we define a bilinear form b by the block-matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -A \end{smallmatrix}\right)$. Then V is a Jordan subalgebra of $\mathrm{Cl}(A,\mathbb{F})$ with Jordan product for $x, y \in V$ given by*

$$xy + yx = b(x,1)y + b(y,1)x - b(x,y)1.$$

Proof. If $x, y \in V_1$, then the formula is true by the defining relation

$$xy + yx = x^t Ay \cdot 1$$

of the Clifford algebra. If $x = 1$, then the right hand side reduces to y, in accordance to the fact that 1 is the unit element of the Clifford algebra. ■

Note that in combination with Lemma II.3.2, the preceding Lemma yields an independent proof of Lemma II.3.3.

4. Symmetric submanifolds and Helwig spaces

Definition II.4.1. Let $M = G.e \subset V$ be a symmetric orbit (cf. Def. II.1.1). We say that M is an *(extrinsic) symmetric submanifold of* V if there exists a *linear* map $S_e : V \to V$ such that S_e extends the symmetry s_e of M in the following sense:
(S1) $S_e^2 = \mathrm{id}_V$,
(S2) $S_e|M = s_e$,
(S3) the tangent space T_eM is equal to the -1-eigenspace V^- of S_e. ■

It follows immediately that for all $p \in M$ the symmetry s_p has an extension with properties (S1) – (S3). The properties (S1) – (S3) are not all independent. In fact, if M generates V as a vector space, then (S1) follows from (S2). The inclusion $T_eM \subset V^-$ follows already from (S2) since linearity of S_e implies that $S_e|_{T_eM} = T_e(s_e) = -\mathrm{id}_{T_eM}$. The term "symmetric submanifold" is justified by the observation that the canonical connection of M is induced from the connection ∇^0 of V via $(\nabla_X Y)_p = \mathrm{pr}_p^-(\nabla_X^0 Y)_p$, where pr_p^- is the projection onto the -1-eigenspace of the linear map S_p.

Example II.4.2. (a) The sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is a symmetric submanifold of $\mathbb{R}^n$: as seen in Section I.6.6, the symmetries of S^n are induced by linear maps.
(b) The groups $\mathrm{O}(A,\mathbb{F})$ and $\mathrm{U}(A,\varepsilon,\mathbb{F})$ are symmetric submanifolds of $M(n,\mathbb{F})$ with $e = \mathbf{1}_n$ and $S_e(X) = X^*$.
(c) The spaces $M_{p,q}$ of elements of order 2 in $\mathrm{Gl}(n,\mathbb{F})$ (Section I.6.3) are symmetric submanifolds of $M(n,\mathbb{F})$ with $e = I_{p,q}$ and $S_e(X) = I_{p,q}XI_{p,q}$ (for (S3) cf. Eqn. (6.26)). Similarly, the spaces of elements of order 2 in $\mathrm{O}(A,\mathbb{F})$ are symmetric submanifolds of $\mathrm{Sym}(A,\mathbb{F})$. In particular, the Grassmannian $\mathrm{Gr}_{p,n}(\mathbb{R})$ is a symmetric submanifold of $\mathrm{Sym}(n,\mathbb{R})$. Similarly for the spaces of Lagrangians (Section I.6.4).
(d) The spaces of complex structures (Section I.6.4) are symmetric submanifolds of $M(n,\mathbb{F})$, resp. of $\mathrm{Asym}(A,\mathbb{F})$ with $e = J$ and $S_e(X) = JXJ^{-1}$. ■

Lemma II.4.3. *Let V be a Jordan algebra with unit e and α an involutive automorphism of V. Then the space*

$$M := \{x \in V \mid \mathrm{Det}\, P(x) \neq 0, x^{-1} = \alpha(x)\}_e$$

is a symmetric submanifold of V, homogeneous under the group

$$G := \mathrm{Str}(V)_o^{(j\alpha)_*} = \{g \in \mathrm{Str}(V) \mid g^\sharp = \alpha \circ g \circ \alpha^{-1}\}_o.$$

Proof. If $\Omega = \mathrm{Str}(V)_o.e$ is the open symmetric orbit associated to V, then $j\alpha$ is an involutive automorphism of the symmetric space Ω. By definition, M is a connected component of the $j\alpha$-fixed subspace $\Omega^{j\alpha}$ of Ω and thus inherits a natural symmetric space structure. Since j is the symmetry of Ω w.r.t. e, $j|_M = \alpha|_M$ is the symmetry of M w.r.t. e. Thus with $S_e := \alpha$ the properties (S1) – (S3) are clear. From the fact that $\mathrm{Str}(V)_o$ contains the group $G(\Omega)_o$ it follows that G contains the group $G(M)_o$ and is thus transitive on M. ■

The symmetric submanifolds of unital Jordan algebras defined by the preceding lemma will be called *Helwig spaces* since K.H. Helwig was the first who studied such spaces (cf. [Hw70]).

Example II.4.4. The symmetric submanifolds from Example II.4.2 are in fact Helwig spaces: in case (a) take the Jordan algebra defined in Section II.3.3; in case (b) take the Jordan algebra $M(n, \mathbb{F})$ with $\alpha(X) = X^*$; in case (c) take the Jordan algebra $M(n, \mathbb{F})$ resp. $\mathrm{Sym}(A, \mathbb{F})$ with Jordan product $(X, Y) \mapsto \frac{1}{2}(XI_{p,q}Y + YI_{p,q}X)$ and α conjugation by $I_{p,q}$. Case (d) is treated similarly, but one has to turn $\mathrm{Asym}(A, \mathbb{F})$ first into a Jordan algebra. ■

In particular, the spheres and all projective spaces are Helwig-spaces. This holds also for the exceptional octonionic projective plane (cf. [Hw70]); thus all compact symmetric spaces of rank 1 are Helwig spaces.

Problem II.4.5. We have defined Helwig spaces by a construction and not by a geometric concept. What is the characteristic property of the Helwig spaces among the symmetric submanifolds? ■

This problem will turn up again in Chapter XI.4. Let us put it into a more specific form: let $M \subset V$ be a symmetric submanifold and denote by $V = V^+ \oplus V^-$ the eigenspace decomposition w.r.t. S_e. Following the construction from Section II.1, we denote by

$$L : V^- \to \mathfrak{q}, \quad u \mapsto L(u)$$

the inverse of the evaluation map

$$\mathfrak{q} \to V^-, \quad X \mapsto Xe$$

which is bijective by (S3). Here $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the usual decomposition w.r.t. conjugation by S_e. Thus we get a bilinear map

$$\mathfrak{q} \times \mathfrak{q} \to \mathrm{End}(V)^{S_e}, \quad (u, v) \mapsto L(u)L(v).$$

The eigenspaces $V^\pm$ of S_e are stable under the operators $L(u)L(v)$, and thus we get a trilinear map

$$V^- \times V^- \times V^-, \quad (u, v, w) \mapsto uvw := L(u)L(v)L(w)e = L(u)L(v)w.$$

Note that

$$uvw - uwv = L(u)[L(v), L(w)]e = 0$$

since $\mathfrak{h}e = 0$, and therefore the curvature tensor at the base point is given by

$$-R_e(u, v)w = [[L(u), L(v)], L(w)]e = [L(u), L(v)]L(w)e = uvw - vuw.$$

In the Riemannian case, D. Ferus interpretes the term uvw essentially as the "second fundamental form" and proves that

$$T(x, y, z) := R(x, y)z + zxy = yxz - xyz + zxy$$

is a Jordan triple system, i.e. it satisfies the identity (JT2) ([Fe80, Lemma 2]). This result generalizes to the non-degenerate case, but it seems not to be known whether it holds in the general framework defined here:

Problem II.4.6. (a) Give a necessary and sufficient condition on (M, V) for T to be a Jordan triple system.
(b) Under which additional conditions on (M, V) can we define a Jordan algebra structure on V^+ and on V such that $V = V^- \oplus V^+$ is the decomposition of a Jordan algebra under an involution? ■

Notes for Chapter II.

II.1. Prehomogeneous symmetric spaces and their relation with Lie triple algebras have first been considered by Vinberg ([Vi63]) and Koecher (cf. the notion of "ω-Bereich" in [BK66]) and later by Shima [Shi75]. Our presentation is based on [Be94]; cf. also [Be98b].

II.2. The notion of quadratic prehomogeneous symmetric spaces is motivated by [Lo69a, p.67 – 72] (cf. also [Lo67, Satz 1.4]) where essentially the implication (iii) $\Rightarrow$ (i) of Th. II.2.6 is proved and the formula for the multiplication map (Prop. II.2.9) has been noted. The equivalence of (ii) and (iii) in Th. II.2.6 is taken from [Be94], cf. also [Be98b, Th.1.2.2]. The definition of the structure group via Eqns. (II.2.11) and (II.2.12) goes back to Koecher (cf. [BK66, Section II.5]); in order to simplify our notation we write $g^\sharp$ where in [BK66] the notation $(g^\sharp)^{-1}$ is used. Power associativity is a well-known property of Jordan algebras, cf. [BK66, IV., Satz 1.5], [FK94, Prop. II.1.2]; the proof given here is a "geometric version" of the one from [FK94]. Conversely, the fact that a power associative Lie triple algebra is a Jordan algebra can be found in the literature, cf. [Pe67].

II.3. See Section XII.1 for the complete classification of simple real Jordan algebras and for the determination of their structure groups.

II.4. Our definition of a symmetric submanifold is a group theoretic version of the differential geometric definition given by D.Ferus ([Fe80]) in the Riemannian case and by H. Naitoh ([Na84]) in the pseudo-Riemannian case. In theses cases one obtains, as remarked in the text, a bijection of the corresponding objects with certain non-degenerate Jordan triple systems. To our knowledge, the relation between the symmetric submanifolds and the Helwig-spaces has never been systematically studied. In [Hw70], K.H. Helwig describes many interesting geometric features of his spaces by using the imbedding. See the papers by R. Iordanescu [Io90], [Io98] for an extensive bibliography of related topics.

Chapter III: The Jordan-Lie functor

A *complexification of a symmetric space* is a symmetric space with invariant complex structure containing the given space as the subspace fixed under a complex conjugation (an antiholomorphic automorphism of order 2). This notion of complexification is broader than the usual one. As an example consider the Poincaré disc $D = \mathrm{SU}(1,1)/\mathrm{SO}(2)$. It has a $\mathrm{SU}(1,1)$-invariant complex structure. The space fixed under the complex conjugation $z \mapsto \overline{z}$ is the interval $]-1,1[$ which, as a symmetric space, is isomorphic to $\mathbb{R}$. In our sense D is a complexification of $\mathbb{R}$. On the other hand, the symmetric space $\mathbb{C}$ is also a complexification of $\mathbb{R}$. Thus a symmetric space can have several complexifications which are not even locally isomorphic ($\mathbb{C}$ is flat and D is not flat).

The preceding example illustrates already the two main types of complexifications: *straight complexifications* generalize the complexification $\mathbb{C}$, and *twisted complexifications* generalize the complexification D of $\mathbb{R}$. The latter might also be called a *hermitification*, and in the literature the former are usually just called "complexifications"; they correspond to the complexification of Lie algebras and Lie groups, and one often writes just $M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ in order to denote the straight complexification of $M = G/H$. Algebraically, they correspond to a complexification of the multiplication map μ similarly as in the case of Lie groups w.r.t. the group multiplication.

The behavior of twisted complexifications is completely different from the straight case. In this chapter we give their formal definition (Section 2) and investigate their formal properties (Sections 2 and 4); in Chapters VI–XI their geometric theory is developed. A central role in the study of formal properties of complexifications is played by the *structure tensor*

$$T(X,Y)Z := -\frac{1}{2}(R(X,Y)Z + \mathcal{J}\, R(X, \mathcal{J}\, Y)Z)$$

constructed from the curvature R and the invariant almost complex structure $\mathcal{J}$ corresponding to the structure of a complex manifold. In the straight case, we have $T = 0$, whereas the twisted case is characterized by the important relation

$$T(X,Y) - T(Y,X) = -R(X,Y). \tag{!}$$

It turns out that then T satisfies the algebraic relations of a *Jordan triple system* (JTS). The main result of this chapter is that twisted complexifications correspond bijectively to Jordan triple systems related via (!) to the curvature tensor; we call them *Jordan extensions of* R, and the correspondence $T \mapsto R$ is called the *algebraic Jordan-Lie functor*. Our "formal method" has the advantage that the bijection just mentioned is also an equivalence of categories (Th. III.4.7) and that it immediately carries over to the "para-complex" case already mentioned in the introduction (Section 0.3). Thus one goal mentioned in the introduction, the construction of the "Jordan functor for Jordan triple systems", is reached.

However, a natural question remains, namely: *how many inequivalent twisted complexifications does a symmetric space have, if any?* In more algebraic terms: *how many Jordan-extensions does the curvature admit?* Or *how can we determine the fibers of the algebraic Jordan-Lie functor?* These are much harder problems; they remain open in general. Only in the irreducible case one can give a preliminary answer by classification, see the following chapter.

1. Complexifications of symmetric spaces

Definition III.1.1. An *almost complex structure* on a manifold M is a tensor field $\mathcal{J}$ of type $(1,1)$ (i.e., $\mathcal{J} = (\mathcal{J}_p)_{p\in M}$ where $\mathcal{J}_p$ is an endomorphism of the tangent space T_pM) such that $\mathcal{J}_p^2 = -\operatorname{id}_{T_pM}$ for all $p \in M$. A smooth map $\varphi : M \to M'$ between manifolds with almost complex structures $\mathcal{J}$ and $\mathcal{J}'$ is called *almost holomorphic* if for all $p \in M$,

$$T_p\varphi \circ \mathcal{J}_p = \mathcal{J}'_{\varphi(p)} \circ T_p\varphi.$$

It is called *almost anti-holomorphic* if $T_p\varphi \circ \mathcal{J}_p = -\mathcal{J}'_{\varphi(p)} \circ T_p\varphi$ holds for all $p \in M$. ∎

For example, if M is a *complex manifold* (i.e. it has a complex atlas), then every tangent space T_pM of the underlying real manifold has a natural structure of a complex vector space. If we let $\mathcal{J}_p$ be the multiplication by i in T_pM, then $\mathcal{J}$ is an almost complex structure, and the almost holomorphic maps are precisely the holomorphic ones w.r.t. the given complex atlas.

Definition III.1.2.

(1) An *invariant almost complex structure* on a symmetric space M is an almost complex structure $\mathcal{J}$ on M which is invariant under the displacement group $G(M)$ (i.e. $G(M)$ acts almost holomorphically). *Homomorphisms* of symmetric spaces with invariant almost complex structure are almost-holomorphic homomorphisms of symmetric spaces.

(2) A *global complexification of a symmetric space* M is a symmetric space N together with an invariant almost complex structure $\mathcal{J}$ and a conjugation τ (i.e. an almost anti-holomorphic automorphism with $\tau^2 = \operatorname{id}_N$) such that $\tau(o) = o$ and the symmetric space M is isomorphic to the τ-fixed space N^τ (or to a union of connected components of N^τ). In this situation we say that M (or τ) is a *real form* of N.

(3) A *complexification of a germ M^o of a symmetric space* is defined as a germ N^o of a symmetric space with invariant almost complex structure and germ τ of a conjugation such that $M^o \cong (N^o)^\tau$, and again M^o is called a *real form* of N^o.

(4) A *homomorphism of complexifications* is a homomorphism of germs of symmetric spaces with an extension to an almost-complex homomorphism of the complexifications. ∎

Strictly speaking, we should use the term "almost-complexification" instead of "complexification". But (cf. Ch.VI, Appendix A) invariant almost complex structures on symmetric spaces are *integrable*, and thus complexifications as considered here are indeed complexifications in the sense of manifolds, i.e. N is a complex manifold defining $\mathcal{J}$ as explained above, and almost (anti-) holomorphic maps are actually (anti-) holomorphic. These facts, however, will not be needed in this chapter. We now give the infinitesimal version of the preceding definitions:

Definition III.1.3.

(1) An *invariant complex structure* on a LTS $\mathfrak{q}$ is a complex structure J on the vector space $\mathfrak{q}$ such that for all $X, Y, Z \in \mathfrak{q}$,

$$[X, Y, JZ] = J[X, Y, Z].$$

Homomorphisms of LTS with invariant complex structure are complex linear homomorphisms of LTS.

(2) A *complexification* of a LTS $\mathfrak{q}$ is a LTS $\mathfrak{p}$ with an invariant complex structure J and a conjugation τ (i.e. a complex antilinear automorphism with $\tau^2 = \mathrm{id}_{\mathfrak{p}}$) such that the LTS $\mathfrak{q}$ is isomorphic the τ-fixed LTS $\mathfrak{p}^\tau$.

(3) A *homomorphism of complexifications of LTS* is a homomorphism of LTS together with a $\mathbb{C}$-linear extension to a homomorphism of the complexified LTS. ■

Proposition III.1.4.

(i) *The category of germs of symmetric spaces with invariant almost complex structure is equivalent to the category of Lie triple systems with invariant complex structure.*

(ii) *The category of local complexifications of germs of symmetric spaces is equivalent to the category of complexifications of Lie triple systems.*

Proof. (i) If $\mathcal{J}$ is invariant on $M = G/H$, then $\mathcal{J}_o$ is invariant under H and thus also under $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$, i.e. $U \circ \mathcal{J}_o = \mathcal{J}_o \circ U$ for all $U \in \mathfrak{h}$. If we identify T_oM with $\mathfrak{q}$ in the usual way, then $\mathcal{J}_o$ is identified with a complex structure J on $\mathfrak{q}$ having the stated property.

Conversely, any such structure defines an $\mathfrak{h}$- and H_o-invariant complex structure $\mathcal{J}_o$ on T_oM. By forward transport with elements of G it defines an almost complex structure on some neighborhood of o which is an invariant almost complex structure on the germ M^o.

Finally, the differential of an almost holomorphic homomorphism at the base point is a complex linear homomorphism of LTS. Conversely, let $\alpha : \mathfrak{q} \to \mathfrak{q}'$ be a $\mathbb{C}$-linear homomorphism of LTS and $\varphi : M \to M'$ be the corresponding homomorphism of germs symmetric spaces. We have to show that it is almost holomorphic. By assumption, its differential $\dot{\varphi} = \alpha$ is $\mathbb{C}$-linear. Let $p = g(o) \in M$ with $g \in G(M)$. According to Prop. I.3.5. there exists $g' \in G(M')$ such that $g' \circ \varphi = \varphi \circ g$. Thus

$$T_p\varphi = T_{o'}g' \circ T_o\varphi \circ (T_og)^{-1}$$

is a composition of $\mathbb{C}$-linear maps and thus itself $\mathbb{C}$-linear.

(ii) Clearly, under the equivalence (i), real forms of germs correspond bijectively to real forms of Lie triple systems. ■

Proposition III.1.5. *For any Lie triple product $R : \mathfrak{q} \times \mathfrak{q} \times \mathfrak{q} \to \mathfrak{q}$, the $\mathbb{C}$-trilinear extension*

$$R_{\mathbb{C}} : \mathfrak{q}_{\mathbb{C}} \times \mathfrak{q}_{\mathbb{C}} \times \mathfrak{q}_{\mathbb{C}} \to \mathfrak{q}_{\mathbb{C}}$$

is a complex Lie triple product.

Proof. Let $\mathfrak{g} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$ be the standard-imbedding of $\mathfrak{q}$ and σ its involution (Def. I.1.4). Then the $\mathbb{C}$-linear extension of σ is an involution of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$, and the triple commutator on its -1-eigenspace $\mathfrak{q}_{\mathbb{C}}$ is just the $\mathbb{C}$-trilinear extension of the triple commutator in $\mathfrak{q}$. This is again a Lie triple product. ■

Definition III.1.6. The complexification $M^o_{\mathbb{C}}$ of the germ M^o belonging to the complexification $\mathfrak{q}_{\mathbb{C}}$ of $\mathfrak{q}$ defined in the previous proposition is called the *straight complexification of M^o*.■

The preceding proof shows that, if M^o is the germ of the symmetric space G/H, then $M^o_{\mathbb{C}}$ is the germ of

$$M_{\mathbb{C}} := G_{\mathbb{C}}/H_{\mathbb{C}},$$

where $G_{\mathbb{C}}$ is the connected simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and $H_{\mathbb{C}} = G^\sigma_{\mathbb{C}}$ the group fixed under the unique holomorphic involution whose differential at the origin is the complexification of σ. It is clear, however, that $M_{\mathbb{C}}$ is in general *not* a global complexification of $M = G/H$. If one wants global complexifications one has to add further topological or algebraic assumptions; we return to this point later (cf. Prop. X.3.6). Let us mention here just that for the symmetric spaces described in Section I.6 we can easily construct global straight complexifications (cf. I.6.5).

Lemma III.1.7. *Let $\mathfrak{q}$ be a LTS and $A \in \mathrm{End}(\mathfrak{q})$ such that $[X,Y,AZ] = A[X,Y,Z]$ for all $X,Y,Z \in \mathfrak{q}$. Then the Lie triple product is A-linear in the first two variables if and only if for all $X,Y,Z \in \mathfrak{q}$,*

$$[AX,Y,Z] = [X,AY,Z]. \tag{1.1}$$

Proof. Let us assume that (1.1) holds. Then we get from the Jacobi-identity

$$0 = [AX,Y,Z] + [AZ,X,Y] + A[Y,Z,X].$$

We add up the three equations obtained by cyclic permutation of X,Y,Z after having multiplied the second by -1. Again by the Jacobi-identity, this gives

$$0 = 2[AX,Y,Z] - 2A[X,Y,Z],$$

whence $A[X,Y,Z] = [AX,Y,Z] = [X,AY,Z]$, proving that the LTS is A-trilinear. The converse is trivial. ■

Proposition III.1.8. *Under the correspondence of* Th. I.2.9 (ii)*, Lie triple systems $\mathfrak{q}$ with $\mathbb{C}$-trilinear triple product correspond bijectively to germs of symmetric spaces M^o with invariant complex structure $\mathcal{J}$ such that the curvature R satisfies*

$$R(X,\mathcal{J}Y)Z = R(\mathcal{J}Y,X)Z \tag{1.2}$$

for all germs of vector fields X,Y,Z on M^o.

Proof. Recall (Cor. I.2.5) that the curvature R at o is equivalent to the LTS $\mathfrak{q}$ of M^o. Therefore complex LTS correspond bijectively to spaces with almost complex structures such that R is $\mathcal{J}$-trilinear. By the preceding lemma, this is equivalent to invariance combined with (1.2). ■

Definition III.1.9. An invariant almost complex structure $\mathcal{J}$ on a symmetric space is called *straight* if (1.2) holds. ■

The preceding proposition shows that a germ M^o of a symmetric space has one and only one complexification $(N^o,\mathcal{J},\tau)$ such that $\mathcal{J}$ is straight for N, namely the straight complexification $M^o_{\mathbb{C}}$ (Def. III.1.6).

2. Twisted complex symmetric spaces and Hermitian JTS

2.1. Twisted complex symmetric spaces.

Definition III.2.1. A *twisted complex symmetric space* is a symmetric space M with an invariant almost complex structure $\mathcal{J}$ such that for all $X,Y,Z \in \mathfrak{X}(M)$

$$R(\mathcal{J}X,Y)Z = -R(X,\mathcal{J}Y)Z \tag{2.1}$$

where R is the curvature tensor of M. An invariant complex structure J on a LTS $\mathfrak{q}$ is called *twisted* if for all $X,Y,Z \in \mathfrak{q}$

$$[JX,Y,Z] = -[X,JY,Z]$$

holds. ■

It is clear that under the correspondence of Prop. III.1.4 germs of symmetric spaces with twisted invariant almost complex structure correspond bijectively to LTS with twisted invariant complex structures. If $R = 0$, then J may be straight and twisted at the same time, but otherwise this is impossible. However, we will see later examples of spaces having *different* invariant almost complex structures, one of it being straight and the other twisted.

Proposition III.2.2. *If J is an invariant complex structure on a LTS $\mathfrak{q}$, then the following are equivalent:*

(i) *J is twisted.*

(ii) *J is an automorphism of $\mathfrak{q}$, i.e.*

$$\forall X, Y, Z \in \mathfrak{q}: \quad [JX, JY, JZ] = J[X, Y, Z].$$

(iii) *J is a derivation of $\mathfrak{q}$, i.e.*

$$\forall X, Y, Z \in \mathfrak{q}: \quad J[X, Y, Z] = [JX, Y, Z] + [X, JY, Z] + [X, Y, JZ].$$

Proof. In view of the invariance, the equivalence of (i) and (iii) is clear. Using in addition that $J^2 = -\operatorname{id}_{\mathfrak{q}}$, the equivalence of (i) and (ii) follows immediately. ∎

Since $e^{tJ} = \cos(t)\operatorname{id}_{\mathfrak{q}} + \sin(t)J$ and thus $J = e^{\frac{\pi}{2}J}$ and the exponential of a derivation is an automorphism, we obtain another proof of the implication (iii) $\Rightarrow$ (ii).

2.2. Structure tensor and Hermitian Jordan triple systems. Let $M = G/H$ be a symmetric space with curvature tensor R and invariant almost complex structure $\mathcal{J}$. For any $p \in M$, the space $\operatorname{End}(T_pM)$ of endomorphisms of the tangent space T_pM splits into a direct sum of complex-linear and complex-antilinear (w.r.t. $\mathcal{J}_p$) elements. The endomorphism $R(X,Y)$ of $\mathfrak{X}(M)$ is by assumption complex-linear, but the endomorphism $R(X,\cdot)Z$ in general decomposes into two terms. We are thus lead to define the *structure tensor* T of $\mathcal{J}$ by

$$T(X,Y)Z = -\frac{1}{2}(R(X,Y)Z - \mathcal{J}\,R(X,\mathcal{J}^{-1}Y)Z) = -\frac{1}{2}(R(X,Y)Z + \mathcal{J}\,R(X,\mathcal{J}\,Y)Z).$$

Sometimes we will write $T(X,Y,Z)$ instead of $T(X,Y)Z$. Since R and $\mathcal{J}$ are G-invariant, T is a G-invariant tensor field of the same type as the curvature tensor. We write J for $\mathcal{J}_o$. Then

$$T_o(X,Y)Z = \frac{1}{2}([X,Y,Z] - J[X,J^{-1}Y,Z]) = \frac{1}{2}([X,Y,Z] + J[X,JY,Z])$$

is an H-equivariant map $\otimes^3\mathfrak{q} \to \mathfrak{q}$, called the *structure tensor of J*. Since T_o and R_o are, by invariance, equivalent to T and R, we will suppress the index o when there is no risk of confusion.

Proposition III.2.3.

(i) *The structure tensor vanishes if and only if J is straight.*

(ii) *The Lie algebra $\mathfrak{g}$ acts as a Lie algebra of derivations of T, and $\mathfrak{h}$ acts as a Lie algebra of derivations of T_o.*

(iii) *For all vector fields X, Y, Z on M,*

$$T(X,Y,Z) = T(Z,Y,X).$$

(iv) *For all vector fields X, Y, Z on M,*

$$T(X,Y,\mathcal{J}\,Z) = -T(X,\mathcal{J}\,Y,Z) = T(\mathcal{J}\,X,Y,Z) = \mathcal{J}\;T(X,Y,Z).$$

(v) *J is twisted if and only if for all vector fields X,Y,Z on M,*

$$T(X,Y)Z - T(Y,X)Z = -R(X,Y)Z,$$

or, equivalently, $T_o(X,Y)Z - T_o(Y,X)Z = [X,Y,Z]$ *for* $X,Y,Z \in \mathfrak{q}$.

Proof. (i) Clearly, $T=0$ iff R is $\mathcal{J}$-trilinear, and (ii) is the infinitesimal version of the invariance of T under G, resp. under H.

(iii) Using the Jacobi-identity, we get

$$\begin{aligned}-2(T(X,Y,Z)-T(Z,Y,X)) &= R(X,Y,Z)-R(Z,Y,X)+\\ &\qquad \mathcal{J}(R(X,\mathcal{J}\,Y,Z)-R(Z,\mathcal{J}\,Y,X))\\ &= -R(Z,X,Y-\mathcal{J}\,R(Z,X,\mathcal{J}\,Y)=0.\end{aligned}$$

(iv) $T(X,Y)\,\mathcal{J}\,Z = \mathcal{J}\;T(X,Y)Z$ since $\mathcal{J}$ is invariant;

$$T(X,\mathcal{J}\,Y)Z = -\frac{1}{2}(R(X,\mathcal{J}\,Y,Z)-\mathcal{J}\,R(X,Y,Z)) = -\,\mathcal{J}\;T(X,Y)Z;$$

$$T(\mathcal{J}\,X,Y)Z = T(Z,Y)\,\mathcal{J}\,Z = \mathcal{J}\;T(Z,Y)X = \mathcal{J}\;T(X,Y)Z,$$

using twice part (iii).

(v) The tensor

$$\begin{aligned}T(Y,X)Z-T(X,Y)Z-R(X,Y,Z) &= \frac{1}{2}\,\mathcal{J}(R(X,\mathcal{J}\,Y,Z)-R(Y,\mathcal{J}\,X,Z))\\ &= \frac{1}{2}\,\mathcal{J}(R(X,\mathcal{J}\,Y,Z)+R(\mathcal{J}\,X,Y,Z))\end{aligned}$$

clearly vanishes if and only if $\mathcal{J}$ is twisted. ■

Proposition III.2.4. *Let $T=T_o$ be the structure tensor, evaluated at the base point, of an invariant twisted almost complex structure $\mathcal{J}$. Then the following holds for $u,v,w,x,y,z \in T_oM$:*

(JT1) $T(x,y,z) = T(z,y,x)$

(JT2) $T(u,v)T(x,y,z) = T(T(u,v)x,y,z) - T(x,T(v,u)y,z) + T(x,y,T(u,v)z)$.

Proof. The property (JT1) is nothing but Prop. III.2.3 (iii).

We now establish the property (JT2). Since $[\mathfrak{q},\mathfrak{q}] \subset \mathfrak{h}$, Prop. III.2.3 (ii) implies that $R_o(X,Y)$ is a derivation of T_o. If D is a derivation of T_o, then JD is a *skew-derivation* of T_o in the following sense: using III.2.3 (iv) we get

$$\begin{aligned}JD\;T_o(X,Y,Z) &= J(T_o(DX,Y,Z)+T_o(X,DY,Z)+T_o(X,Y,DZ))\\ &= T_o(JD\,X,Y,Z)-T_o(X,JD\,Y,Z)+T_o(X,Y,JD\,Z).\end{aligned}$$

Thus $2T_o(X,Y) \in \mathrm{End}(\mathfrak{q})$ is the sum of the derivation $R_o(X,Y) \in \mathfrak{h}$ and the skew-derivation $J\,R_o(X,JY) \in \mathrm{End}(\mathfrak{q})$. If J is twisted, we obtain for the difference of these two elements

$$\frac{1}{2}(R_o(X,Y)-JR_o(X,JY)) = T_o(Y,X).$$

Using this, we can write (dropping the indices o)

$$\begin{aligned}T(X,Y)\cdot T(U,V,W) &= -\frac{1}{2}(R(X,Y)\cdot T(U,V,W)+J\,R(X,JY)\cdot T(U,V,W))\\ &= -\frac{1}{2}(T\big((R(X,Y)+J\,R(X,JY))U,V,W\big)+\\ &\qquad T\big(U,(R(X,Y)-J\,R(X,JY))V,W\big)+\\ &\qquad T\big(U,V,(R(X,Y)+J\,R(X,JY))W\big))\\ &= T\big(T(X,Y)U,V,W\big)-T\big(U,T(Y,X)V,W\big)+T\big(U,V,T(X,Y)W\big).\end{aligned}$$

This is the identity (JT2). ■

Definition III.2.5. A *Jordan triple system* (V,T) (abbreviated JTS) is a vector space V together with a trilinear map $T : V \times V \times V \to V$ verifying (JT1) and (JT2). A *Hermitian JTS* is a JTS (V,T) together with a complex structure J on V such that for all $u,v,w \in V$,

$$T(Ju,v,w) = -T(u,Jv,w) = T(u,v,Jw) = J\,T(u,v,w).$$

Homomorphisms of Jordan triple systems are linear maps compatible with the triple products; homomorphisms of Hermitian JTS are complex-linear homomorphisms of JTS. ■

We have shown that the structure tensor T of a twisted complex symmetric space is a field of Hermitian Jordan triple products (Prop. III.2.4 and III.2.3 (iv)). Using the following lemma, we will prove the converse.

Lemma III.2.6. *Let T be a Jordan triple product defined on a vector space $\mathfrak{q}$. Then*

$$\begin{aligned}[X,Y,Z]_T &:= -R_T(X,Y)Z \\ &:= T(X,Y)Z - T(Y,X)Z = T(X,Y)Z - T(Z,X)Y\end{aligned}$$

defines a Lie triple product on $\mathfrak{q}$.

Proof. Clearly R_T is antisymmetric in the first two variables and satisfies the Jacobi-identity (LT2); cf. Def.I.1.1. The identity (JT2) implies that $R_T(X,Y)$ is a derivation of T and therefore also of R_T, whence (LT3). ■

We call R_T the LTS *associated to the JTS T*, and if $R = R_T$, then we say that T is a *Jordan-extension of the LTS R*. It is clear that JTS-homomorphisms induce homomorphisms of associated LTS's. Therefore we call $T \mapsto R_T$ the *(algebraic) Jordan-Lie functor.* The Jordan-Lie functor is not bijective, but it does induce bijections of some important sub-categories.

Proposition III.2.7. *The Jordan-Lie functor $T \mapsto R_T$ yields an equivalence of the category of Hermitian JTS's and the category of LTS's with invariant twisted complex structure. Its inverse is given by the formula*

$$T(X,Y)Z := -\frac{1}{2}(R(X,Y)Z - JR(X,J^{-1}Y)Z).$$

Proof. If T is Hermitian, then J is a twisted invariant complex structure for R_T:

$$\begin{aligned}&R_T(X,Y)JZ = T(Y,X,JZ) - T(X,Y,JZ) = JR_T(X,Y)Z;\\ &-R_T(JX,Y) = T(JX,Y) - T(Y,JX) = -(T(X,JY) - T(JY,X) = R_T(X,JY).\end{aligned}$$

Conversely, if J is a twisted invariant complex structure for R, then T is a JTS (Prop.III.2.4) which is Hermitian (Prop. III.2.3 (iv)). Thus both functors are well-defined. It is easily checked that they are inverses of each other: if J is twisted invariant for R, then $R(X,JY)$ is symmetric in X and Y, and antisymmetrizing T in the first two variables, we get back R. If (T,J) is Hermitian, then

$$\begin{aligned}-R_T(X,Y)Z - JR_T(X,JY)Z &= T(X,Y,Z) - T(Y,X,Z) + \\ &\qquad J(T(X,JY,Z) - T(JY,X,Z)) \\ &= 2T(X,Y,Z).\end{aligned}$$

Finally, homomorphisms in both categories are the same since compatibility with R and J is equivalent to compatibility with T and J. ■

We summarize the preceding results:

Theorem III.2.8. *The following three categories are equivalent:*

(1) *tcLTS: twisted complex Lie triple systems,*

(2) *hJTS: Hermitian Jordan triple systems,*

(3) *tcSS: germs of twisted complex symmetric spaces.*

The functor tcSS $\to$ tcLTS is given by evaluating curvature tensor and almost complex structure at the base point, and the functor tcSS $\to$ hJTS by evaluating structure tensor and almost complex structure at the base point.

Proof. The equivalence of (1) and (2) has been established in the previous proposition, and the equivalence of (1) and (3) follows from Prop. III.1.4. (i). ■

In Section IV.1 the most important examples of twisted complex symmetric spaces are given, among them the space of complex structures on $\mathbb{R}^{2n}$.

3. Invariant polarizations, graded Lie algebras and Jordan pairs

3.1. Polarizations. Most of what has been said so far remains valid for tensor fields having the property $\mathcal{J}^2 = \mathrm{id}$ instead of $\mathcal{J}^2 = -\mathrm{id}$. They arise for example by the formula $\mathcal{J} = \mathcal{J}_1 \mathcal{J}_2$ if we have two commuting almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$. If $\mathcal{J}_1$ is twisted and $\mathcal{J}_2$ is straight, then $\mathcal{J}$ will be twisted in the sense of the following definition.

Definition III.3.1. A *polarization of (the tangent bundle of) a manifold M* is a tensor field $\mathcal{J}$ of type (1,1) such that $\mathcal{J}_p^2 = \mathrm{id}_{T_pM}$ for all $p \in M$. It is called a *paracomplex structure* if the -1-eigenspace of $\mathcal{J}_p$ ($p \in M$) has the same dimension as the $+1$-eigenspace of $\mathcal{J}_p$. A polarization $\mathcal{J}$ on a symmetric space M is called *invariant* if $\nabla \mathcal{J} = 0$ (equivalently, if $g \cdot \mathcal{J} = \mathcal{J}$ for all g in the group of displacements), and it is called *straight* if

$$R(\mathcal{J} X, Y) = R(X, \mathcal{J} Y),$$

and *twisted* if

$$R(\mathcal{J} X, Y) = -R(X, \mathcal{J} Y)$$

holds for all $X, Y \in \mathfrak{X}(M)$. A *twisted polarized symmetric space* is a symmetric space together with an invariant twisted polarization. ■

Evaluating at the base point, we obtain the corresponding infinitesimal objects:

Definition III.3.2. An *invariant polarization* on a LTS $(\mathfrak{q}, R)$ is an endomorphism J such that $J^2 = \mathrm{id}_q$ which is invariant in the sense that $[X, Y, JZ] = J[X, Y, Z]$. We call J *straight* if it is invariant and $[JX, Y, Z] = [X, JY, Z]$ and *twisted* if it is invariant and $[JX, Y, Z] = -[X, JY, Z]$ ($X, Y, Z \in \mathfrak{q}$). A *polarized LTS* is a LTS together with a twisted polarization. ■

As in Prop. III.1.4 it is shown that germs of symmetric spaces with invariant polarizations are equivalent to LTS with invariant polarizations, and this correspondence clearly is compatible with the respective notions of "straight" and "twisted".

Lemma III.3.3. *Lie triple systems with straight invariant polarizations are precisely the direct products of two Lie triple systems.*

Proof. As in the proof of Prop. III.1.8 it is seen that J is a straight invariant polarization on $(\mathfrak{q}, R)$ iff R is J-trilinear. Therefore, if $\mathfrak{q}^k$ ($k = 1, -1$) are the eigenspaces of J,

$$[\mathfrak{q}^i, \mathfrak{q}^j, \mathfrak{q}^k] \subset (\mathfrak{q}^i \cap \mathfrak{q}^j \cap \mathfrak{q}^k),$$

which immediately implies that $\mathfrak{q}^1$ and $\mathfrak{q}^{-1}$ are ideals of $(\mathfrak{q}, R)$; thus $\mathfrak{q}$ is a direct product. Conversely, given a direct product, let J be the linear map which is one on the first factor and minus one on the second; this clearly is a straight invariant polarization. ■

Lemma III.3.4. *Let J be an invariant polarization on the LTS $(\mathfrak{q}, R)$. Then the following are equivalent:*

(i) *J is twisted.*

(ii) *J is a derivation of R.*

(iii) *If $\mathfrak{m}^{\pm}$ are the ± 1-eigenspaces of J, then $R(\mathfrak{m}^+, \mathfrak{m}^+) = 0$ and $R(\mathfrak{m}^-, \mathfrak{m}^-) = 0$.*

Proof. $(i) \Leftrightarrow (ii)$: If J is invariant, the conditions $[JX, Y, Z] + [X, JY, Z] = 0$ and

$$J[X, Y, Z] = [JX, Y, Z] + [X, Y, JZ] + [X, Y, JZ]$$

are clearly equivalent.

$(i) \Leftrightarrow (iii)$: If J is twisted, then we have for all $X, Y \in \mathfrak{m}^+$:

$$R(X, Y) = R(X, JY) = -R(JX, Y) = -R(X, Y),$$

and therefore $R(\mathfrak{m}^+, \mathfrak{m}^+) = 0$; similarly for $\mathfrak{m}^-$. Conversely, if $R(\mathfrak{m}^+, \mathfrak{m}^+) = 0 = R(\mathfrak{m}^-, \mathfrak{m}^-)$, then the equality $R(X, JY) = -R(JX, Y)$ is verified for $X, Y \in \mathfrak{m}^+$ and for $X, Y \in \mathfrak{m}^-$. For $X \in \mathfrak{m}^-$, $Y \in \mathfrak{m}^+$, the equality $R(X, JY) = -R(JX, Y)$ holds by definition of the eigenspaces; it is therefore verified in all cases. ■

As in the previous section, we define the *structure tensor of the invariant polarization $\mathcal{J}$* by

$$\begin{aligned} T(X, Y)Z &= -\frac{1}{2}(R(X, Y)Z - \mathcal{J}\, R(X, \mathcal{J}^{-1} Y)Z) \\ &= -\frac{1}{2}(R(X, Y)Z - \mathcal{J}\, R(X, \mathcal{J}\, Y)Z). \end{aligned}$$

Then Propositions III.2.3 and III.2.4 and their proofs carry over to the polarized case without any changes, except that we have to replace "Hermitian Jordan triple systems" by *"polarized Jordan triple systems"* which are defined in a completely analoguous way:

Definition III.3.5. A *polarized JTS* (V, T, J) is a JTS (V, T) together with an endomorphism J with $J^2 = \mathrm{id}_V$ such that

$$T(Ju, v, w) = -T(u, Jv, w) = T(u, v, Jw) = JT(u, v, w)$$

for all $u, v, w \in V$. Homomorphisms of polarized JTS are homomorphisms of JTS which are compatible with the polarizations. ■

The following statement is proved as Prop. III.2.7:

Proposition III.3.6. *The Jordan-Lie functor $T \mapsto R_T$ yields an equivalence of the category of polarized JTS's and the category of LTS's with invariant twisted polarization. Its inverse is given by the formula*

$$T(X, Y)Z := -\frac{1}{2}(R(X, Y)Z - JR(X, J^{-1}Y)Z).$$

■

3.2. Graded Lie algebras and Jordan pairs. Before returning to the general problem of complexifications, we present two concepts corresponding to the concept of twisted polarized symmetric space given above: first the concept of a 3-*graded Lie algebra* which is equivalent to the standard imbedding of a polarized LTS, and second the concept of a *Jordan pair* which is equivalent to the notion of polarized JTS defined above.

Definition III.3.7. A Lie algebra $\mathfrak{g}$ together with a direct sum decomposition

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$$

is called $\mathbb{Z}$*-graded* if $[\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}$ holds for all $j,k \in \mathbb{Z}$. It is called 3*-graded* if $\mathfrak{g}_j = 0$ for $j \notin \{-1,0,1\}$. ■

Proposition III.3.8. *The standard imbedding of a LTS yields a bijection of twisted polarized LTS and* 3*-graded Lie algebras* $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *with* $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$.

Proof. Let $(\mathfrak{q}, J)$ be a polarized LTS, denote by $\mathfrak{q} = \mathfrak{m}^1 \oplus \mathfrak{m}^{-1}$ the eigenspace-decomposition of J and let $\mathfrak{m}^0 := \mathfrak{h} = [\mathfrak{q}, \mathfrak{q}]$. Then

$$\mathfrak{g} = \mathfrak{m}^{-1} \oplus \mathfrak{m}^0 \oplus \mathfrak{m}^1$$

is a 3-grading of the standard-imbedding $\mathfrak{g}$. In fact, since J is invariant, we have the relations

$$[\mathfrak{m}^0, \mathfrak{m}^{\pm}] \subset \mathfrak{m}^{\pm}$$

(where we abbreviate $\mathfrak{m}^{\pm} := \mathfrak{m}^{\pm 1}$), and since J is twisted, the conditions $[\mathfrak{m}^+, \mathfrak{m}^+] = 0$ and $[\mathfrak{m}^-, \mathfrak{m}^-] = 0$ hold according to Lemma III.3.4. Finally, $[\mathfrak{m}^+, \mathfrak{m}^-] = \mathfrak{m}^0$ holds by definition. This proves that $\mathfrak{g}$ is 3-graded.

Conversely, given a 3-graded Lie algebra, we let $\mathfrak{q} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ and $\mathfrak{h} := [\mathfrak{g}_{-1}, \mathfrak{g}_1] \subset \mathfrak{g}_0$. Then because of the grading $\mathfrak{q}$ is a Lie triple system on which $\mathfrak{h}$ acts by derivations; therefore $\mathfrak{g}$ is the standard-imbedding of $\mathfrak{q}$. We define J to be 1 on $\mathfrak{g}_1$ and -1 on $\mathfrak{g}_{-1}$; then J is an invariant polarization because $\mathfrak{h}$ preserves the eigenspaces. Lemma III.3.4 now shows that J is twisted.

Finally, both constructions are clearly inverse to each other. ■

If we consider the $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ and the bilinear composition $\mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0$ given by the Lie bracket, then we obtain an object which has been introduced by K. Meyberg in [Mey70] ("verbundenes Paar"):

Definition III.3.9. Let $\mathfrak{h}$ be a Lie algebra, $\mathfrak{m}^+$ and $\mathfrak{m}^-$ $\mathfrak{h}$-modules and $S : \mathfrak{m}^+ \otimes \mathfrak{m}^- \to \mathfrak{h}$ linear. These data are called a *connected pair* if they verify the following conditions:

(P1) S is $\mathfrak{h}$-equivariant,

(P2) for all $a', a \in \mathfrak{m}^+$, $b \in \mathfrak{m}^-$: $S(a,b)a' = S(a',b)a$,

(P3) for all $b, b' \in \mathfrak{m}^-$, $a \in \mathfrak{m}^+$, $S(a,b)b' = S(a,b')b$. ■

Proposition III.3.10. *There is a canonical bijection between connected pairs and* 3*-graded Lie algebras.*

Proof. If $\mathfrak{g}$ is a 3-graded Lie algebra, then $\mathfrak{m}^{\pm} := \mathfrak{g}_{\pm 1}$, $\mathfrak{h} := \mathfrak{g}_0$ and $S : \mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0$, $(X,Y) \mapsto [X,Y]$ are a connected pair. Conversely, given a connected pair, we define on the direct sum $\mathfrak{g} := \mathfrak{m}^+ \oplus \mathfrak{h} \oplus \mathfrak{m}^-$ a Lie bracket in the obvious way such that the two constructions are inverse of each other. ■

Clearly equivariant maps of the standard imbedding correspond to *homomorphisms of connected pairs* defined to be linear maps $\varphi_{\pm} : \mathfrak{m}^{\pm} \to (\mathfrak{m}^{\pm})'$, $\varphi_0 : \mathfrak{h} \to \mathfrak{h}'$ satisfying the compatibility condition $S' \circ (\varphi_+ \otimes \varphi_-) = \varphi_0 \circ S$. However, as mentioned in Section I.3, the standard imbedding does in general not depend functorially on the LTS; thus the category of connected pairs (3-graded Lie algebras) is not equivalent to the category of twisted polarized LTS. But if we retain only the symmetric bilinear maps

$$\mathfrak{m}^+ \otimes \mathfrak{m}^+ \to \mathrm{Hom}(\mathfrak{m}^-, \mathfrak{m}^+), \quad (X,Y) \mapsto S(X,\cdot)Y,$$
$$\mathfrak{m}^- \otimes \mathfrak{m}^- \to \mathrm{Hom}(\mathfrak{m}^+, \mathfrak{m}^-), \quad (X,Y) \mapsto S(\cdot,X)Y$$

together with their algebraic properties following from (P1) – (P3), then we arrive at the concept of a *Jordan pair* which is indeed equivalent to the one of twisted polarized LTS.

Definition III.3.11. A *Jordan pair* is a pair (V^+, V^-) of vector spaces together with trilinear maps

$$T^{\pm} : V^{\pm} \times V^{\mp} \times V^{\pm} \to V^{\pm}, \quad (u,v,w) \mapsto T^{\pm}(u,v,w) =: T^{\pm}(u,v)w$$

which are symmetric in the outer variables and satisfy the identity

$$T^{\pm}(u,v)T^{\pm}(x,y,z) = T^{\pm}(T^{\pm}(u,v)x,y,z) - T^{\pm}(x,T^{\pm}(v,u)y,z) + T^{\pm}(x,y,T^{\pm}(u,v)z)$$

for $u,x,z \in V^{\pm}$, $v,y \in V^{\mp}$. ■

(We have used here the identity JP14 from [Lo75] for the definition. Since we use a base field of characteristic zero, this is equivalent to the definition in [Lo75]; cf. loc. cit. p. 15.) It is clear that the defining identity of a Jordan pair is just the restriction of the defining identity (JT2) of a JTS to the products of eigenspaces $V^{\pm}$ of the polarization J of a polarized JTS T. Conversely, given a Jordan pair, we let $V := V^+ \oplus V^-$ and define T in the obvious way such that $J = \mathrm{id}_{V^+} \oplus - \mathrm{id}_{V^-}$ becomes an invariant polarization. This defines an equivalence of the categories of polarized Jordan triple systems and Jordan pairs (cf. [Lo75, p.9/10]). We summarize the results of this section:

Theorem III.3.12. *The following four categories are equivalent:*

(1) *pLTS: polarized Lie triple systems,*

(2) *pJTS: polarized Jordan triple systems,*

(3) *JP: Jordan pairs,*

(4) *tpSS: germs of symmetric spaces with invariant twisted polarization.*

The functor tpSS $\to$ pLTS is given by evaluating curvature tensor and polarization at the base point, and the functor tpSS $\to$ pJTS by evaluating structure tensor and polarization at the base point.

Proof. The equivalence of the categories (1), (2) and (4) is proved as in Th.III.2.8, and the equivalence of (2) and (3) has been discussed above. ■

The most important example of a symmetric space with twisted invariant polarization is the space of polarizations of $\mathbb{R}^n$, cf. Section IV.1.

4. Jordan-extensions and the geometric Jordan-Lie functor

4.1. Symmetric spaces with twist. Recall that we have defined *real forms* of (germs of) symmetric spaces with invariant almost complex structure (Def. III.1.2). Replacing almost complex structures by polarizations, we get the definition of *para-real forms* and conversely of *para-complexifications* of (germs of) symmetric spaces and of Lie triple systems (cf. Def. III.1.3). Note that, if a polarized LTS $(\mathfrak{q}, J)$ admits a para-real form (an involution τ anticommuting with J), then τ yields a bijection of the $+1$ eigenspace of J onto the -1-eigenspace, and therefore J is actually a paracomplex structure (cf. Def. III.3.1).

Lemma III.4.1. *Let τ be a (para-) real form of a germ M of a symmetric space with invariant (para-) complex structure $\mathcal{J}$. Then τ is an automorphism of the structure tensor $T(X,Y)Z = -\frac{1}{2}(R(X,Y)Z - \mathcal{J}\, R(X, \mathcal{J}^{-1} Y)Z)$, i.e. $\tau_* T = T$. In particular, the restriction of T to M^τ is a well-defined tensor field on the (para-) real form M^τ.*

Proof. We prove the corresponding infinitesimal statement. Let τ be a (para-) conjugation of the invariant (para-) complex structure J on the LTS $\mathfrak{q}$. Then, using that τ is an automorphism of the LTS and anticommutes with J,

$$\begin{aligned}\tau(T(X,Y,Z)) &= \frac{1}{2}(\tau[X,Y,Z] - \tau(J\,[X,J^{-1}Y,Z]))\\ &= \frac{1}{2}([\tau X,\tau Y,\tau Z] - J\,[\tau X,J^{-1}\tau Y,\tau Z])\\ &= T(\tau X,\tau Y,\tau Z).\end{aligned}$$

Now the local version follows since $\tau_*T - T$ is a $\mathfrak{g}^\tau$-invariant tensor field (where $\mathfrak{g}^\tau$ is the standard imbedding of $\mathfrak{q}^\tau$) vanishing at the origin. Therefore $\tau_*T = T$ on the real form M^τ. ∎

In the straight case the preceding lemma is useless because then the structure tensor vanishes. In the twisted case the situation is completely different: we will see that all local information about the twisted complexification is already encoded in the restriction of the structure tensor to M^τ.

Lemma III.4.2. *If $\mathcal{J}$ is an invariant twisted (para-) complex structure, then the restriction of the structure tensor to M^τ is an invariant tensor field of Jordan triple products verifying the relation*

$$T(X,Y) - T(Y,X) = -R(X,Y) \tag{4.1}$$

for all vector fields X,Y on M^τ.

Proof. According to Prop. III.2.3 and III.2.4 the structure tensor of M has the properties of the claim. Since R and T, by the preceding lemma, can be restricted to M^τ, the algebraic relations (JT1), (JT2) and (4.1) carry over to the real form. ∎

Definition III.4.3. A *Jordan-extension* of the curvature tensor R of a symmetric space M is an invariant tensor field T which is a field of Jordan triple systems, (i.e. a tensor field of type (3,1) satisfying the identities (JT1) and (JT2) from Def. III.2.5 and Eqn. (4.1). A *(germ of) a symmetric space with twist* (M,T) is a (germ of a) symmetric space together with a Jordan-extension T of the curvature tensor R. *Homomorphisms of (germs of) symmetric spaces with twist* are homomorphisms of symmetric spaces which are compatible with the respective Jordan-extensions. ∎

Summarizing, we can say that every (para-) real form of a twisted (para-) complex symmetric space is a symmetric space with twist. Next we are going to prove the converse of this statement.

4.2. The equivalence of Jordan-extensions and twisted (para-) complexifications. The following theorem is the main result of this section.

Theorem III.4.4. (i) *Let $(\mathfrak{q},R)$ be a Lie triple system. Then the following objects are in one-to-one correspondence:*

(1) *twisted complexifications of R,*
(2) *twisted para-complexifications of R,*
(3) *Jordan-extensions T of R.*

(ii) *Let M be a germ of a symmetric space. Then the following objects are in one-to-one correspondence:*

(1') *twisted complexifications of M,*
(2') *twisted para-complexifications of M,*
(3') *Jordan-extensions T of the curvature tensor R of M.*

Proof. The claims (i) and (ii) are equivalent since we have already seen that the objects (k) and (k') are in one-to-one correspondence under the bijection of Lie triple systems and germs of symmetric spaces.

The correspondences (1') $\to$ (3') and (2') $\to$ (3') are given by restricting the structure tensor of the (para-) complexified space to the real space we started with. The main step is now to construct the infinitesimal version (3) $\to$ (1) of the correspondence (3') $\to$ (1'). In order to do this, we need some basic results on Jordan triple systems.

Lemma III.4.5. *Let T be a Jordan triple product on a vector space V and α an endomorphism of V with the property*

$$\forall x, y, z \in V: \quad T(\alpha x, y, \alpha z) = \alpha T(x, \alpha y, z). \tag{4.2}$$

Then the formula

$$T^{(\alpha)}(x, y, z) := T(x, \alpha y, z)$$

defines a Jordan triple product $T^{(\alpha)}$ on V.

Proof. The identity (JT2) for T with v and y replaced by αv and αy, respectively, yields

$$T(T(u, \alpha v)x, \alpha y, z) - T(x, T(\alpha v, u)\alpha y, z) + T(x, \alpha y, T(u, \alpha v)z) = T(u, \alpha v)T(x, \alpha y, z).$$

We apply (4.2) to the middle triple product in the middle term and obtain

$$T(T(u, \alpha v)x, \alpha y, z) - T(x, \alpha T(v, \alpha u)y, z) + T(x, \alpha y, T(u, \alpha v)z) = T(u, \alpha v)T(x, \alpha y, z).$$

This is precisely the identity (JT2) for $T^{(\alpha)}$. Since $T^{(\alpha)}$ clearly satisfies (JT1), it is a Jordan triple product. ∎

If $A : V \to \mathrm{Hom}(V \otimes V, V)$, $x \mapsto T(\cdot, x, \cdot)$ is injective, then the arguments from the above proof can be reversed, showing that (4.2) is necessary for $T^{(\alpha)}$ to define a JTS. We call $T^{(\alpha)}$ the *α-modification of T*, and the set of all α's satisfying (4.2) will be called the *structure variety of T*, denoted by $\mathrm{Svar}(T) \subset \mathrm{End}(V)$. Note that any involution of T belongs to the structure variety. An important application of the previous lemma is the existence of a canonical *(para-) Hermitian complexification* in the category of JTS's:

Proposition III.4.6. *Let (V, T) be a Jordan triple system. We imbed V into $\widetilde{V} := V \oplus V$ as first factor and equip $\widetilde{V}$ with its canonical (para-) complex structure $J(x, y) = (-y, x)$ (resp. $I(x, y) = (y, x)$). Then there exist four Jordan triple systems $T_{\mathbb{C}}$, T_d, $T_{h\mathbb{C}}$ and $T_{ph\mathbb{C}}$ on $\widetilde{V}$ extending T and uniquely determined by the following properties:*

(i) *$T_{\mathbb{C}}$ is complex with respect to J.*
(ii) *T_d is I-linear in all three variables.*
(iii) *$T_{h\mathbb{C}}$ is Hermitian with respect to J.*
(iv) *$T_{ph\mathbb{C}}$ is polarized with respect to I.*

Proof. Uniqueness is immediate from the definitions. We now prove existence: one verifies that the $\mathbb{C}$-trilinear extension $T_{\mathbb{C}}$ of T is again a JTS. (In order to do this one may express (JT1) and (JT2) by commutative diagrams of linear maps of some tensor products and then apply the ordinary complexification functor in the category of vector spaces.) Thus (i) is proved. In order to prove (iii), one verifies that complex conjugation $\tau(x, y) = (x, -y)$ is an automorphism of $T_{\mathbb{C}}$. Since it is an involution, it satisfies Eqn. (4.2), and therefore

$$T_{h\mathbb{C}}(u, v, w) := (T_{\mathbb{C}})^{(\tau)}(u, v, w) = T_{\mathbb{C}}(u, \overline{v}, w)$$

again is a JTS. It clearly is Hermitian and extends T. Now we prove (ii): let $W^\pm$ be the eigenspaces of I; then W^+ is the diagonal and W^- the antidiagonal, and $V \to W^\pm$, $v \mapsto \frac{1}{2}(v, \pm v)$ are vector space isomorphisms which we use as identifications in order to define the JTS $T_d := T \oplus T$ on $W^+ \oplus W^-$. It extends T and is I-trilinear. This proves (ii). The para-conjugation $\tau(x,y) = (x,-y)$ is an involution of $T_d = T \oplus T$; in fact, in the coordinates $W^+ \oplus W^-$ it is just exchange of the two factors. The previous lemma implies that $T_{ph\mathbb{C}} := (T_d)^{(\tau)}$ is a JTS. It clearly is polarized by I and extends T, thus (iv) is proved. ■

Now we can finish the proof of Th. III.4.4. Let $T_{h\mathbb{C}}$ be as in the preceding proposition and define $\widetilde{R}(X,Y) := T_{h\mathbb{C}}(X,Y) - T_{h\mathbb{C}}(X,Y)$. Then $\widetilde{R}$ is a twisted complex LTS since $T_{h\mathbb{C}}$ is a Hermitian JTS (Prop.III.2.7). Further, restriction of $\widetilde{R}$ to $\mathfrak{q}$ yields R since the restriction of $T_{h\mathbb{C}}$ to $\mathfrak{q}$ is T, and $R_T = R$ by assumption. Therefore $\widetilde{R}$ is a twisted complexification of R.

The correspondence (3) $\to$ (2) is constructed in the same way, using this time the para-Hermitian complexification $T_{ph\mathbb{C}}$ of T. Finally, it is clear that the given constructions are inverses of each other. ■

The correspondences from the preceding theorem are functorial:

Theorem III.4.7. *The following categories are equivalent:*

(1) *(real finite dimensional) Jordan triple systems.*

(2) *germs of symmetric spaces with twist.*

(3) *germs of symmetric spaces with local twisted complexification.*

(4) *germs of symmetric spaces with local twisted para-complexification.*

Proof. We show first that the constructions given in Prop. III.4.6 are functorial. If $\varphi : V \to V'$ is a homomorphism of JTS T and T', then we complexify the commutative diagram

$$\begin{array}{ccc} V \times V \times V & \xrightarrow{T} & V \\ \times^3\varphi \downarrow & & \downarrow \varphi \\ V' \times V' \times V' & \xrightarrow{T'} & V', \end{array}$$

and see that $\varphi_{\mathbb{C}}$ is a homomorphism from $T_{\mathbb{C}}$ to $T'_{\mathbb{C}}$. From this it follows that

$$\varphi_{\mathbb{C}} T_{h\mathbb{C}}(u,v,w) = \varphi_{\mathbb{C}} T_{\mathbb{C}}(u,\overline{v},w) = T'_{\mathbb{C}}(\varphi_{\mathbb{C}}u, \varphi_{\mathbb{C}}\overline{v}, \varphi_{\mathbb{C}}w) = T_{h\mathbb{C}}(\varphi_{\mathbb{C}}u, \varphi_{\mathbb{C}}v, \varphi_{\mathbb{C}}w)$$

since $\varphi\overline{v} = \overline{\varphi v}$. Thus $\varphi_{\mathbb{C}}$ is a homomorphism from $T_{h\mathbb{C}}$ to $T'_{h\mathbb{C}}$. Similarly for the para-complexifications.

Next we use the fact that the Jordan-Lie functor $T \mapsto R_T$ is functorial, i.e. we obtain homomorphisms of the twisted (para-) complex LTS equivalent to $T_{h\mathbb{C}}$ resp. $T_{ph\mathbb{C}}$.

Applying now Prop. III.1.4 (ii), we can lift these homomorphisms on a local level, i.e. we have shown that (1) $\to$ (3) and (1) $\to$ (4) are functorial.

It is clear that restriction to a real form is functorial; thus (3) $\to$ (2) and (4) $\to$ (2) are functorial. It is also clear that evaluating at the base point is a functorial construction; thus (2) $\to$ (1) is functorial, and the theorem is proved. ■

Definition III.4.8. The *geometric Jordan-Lie functor* is the forgetful-functor from the category of (germs of) symmetric spaces with twist to the category of (germs of) symmetric spaces. The geometric Jordan-Lie functor for germs of symmetric spaces can equivalently be characterized as the forgetful-functor from germs of symmetric spaces with twisted (para-) complexification to germs of symmetric spaces by forgetting the (para-) complexification. ■

The relation with the algebraic Jordan-Lie functor is given by the commutative diagram (0.7) (Introduction). In Chapter IX we will return to the problem which symmetric spaces with twist admit a *global* (para-) complexification.

4.3. The complexification functors. We write $\mathbb{C}$, d, $h\mathbb{C}$ and $ph\mathbb{C}$ for the functors JTS $\to$ JTS defined in Prop. III.4.6; we call them the functors of *complexification*, *doubling*, *Hermitian complexification* and *para-Hermitian complexification* in the category of Jordan triple systems.

Theorem III.4.9. *The category of germs of symmetric spaces with twist is stable under the functors $\mathbb{C}$, d, $h\mathbb{C}$ and $ph\mathbb{C}$. Any two of the four functors commute, i.e. the opposite composition yields functors isomorphic to those obtained by the usual composition. Further,*

(i) $\mathbb{C}\circ\mathbb{C} = d\circ\mathbb{C}$, $h\mathbb{C}\circ h\mathbb{C} = d\circ h\mathbb{C}$, $ph\mathbb{C}\circ ph\mathbb{C} = d\circ ph\mathbb{C}$,

(ii) $\mathbb{C}\circ h\mathbb{C} \cong \mathbb{C}\circ ph\mathbb{C} \cong h\mathbb{C}\circ ph\mathbb{C}$,

(iii) $ph\mathbb{C}\circ h\mathbb{C}\circ\mathbb{C} \cong d\circ h\mathbb{C}\circ\mathbb{C}$.

Proof. Since the functors $\mathbb{C}$, d, $h\mathbb{C}$ and $ph\mathbb{C}$ are functors inside the category of Jordan triple systems, they are by Th. III.4.7 also functors inside the category of germs of symmetric spaces with twist.

The "commutation relations" (i)–(iii) are verified by using the "multiplication table" for complex and para-complex structures: the product of two commuting complex structures is a para-complex structure, the product of two commuting twisted structures is straight, and so on.■

Note that exactly one interesting new functor arises by composition: it is a sort of *double complexification-functor* $\mathbb{C}\circ h\mathbb{C} \cong \mathbb{C}\circ ph\mathbb{C} \cong h\mathbb{C}\circ ph\mathbb{C}$ in the category of symmetric spaces with twist. It assigns to a space another one having four times its real dimension and carrying two commuting almost complex structures. We thus have four non-trivial complexification-functors $\mathbb{C}$, $h\mathbb{C}$, $ph\mathbb{C}$ and $\mathbb{C}\circ h\mathbb{C}$ which are related by (i) – (iii). The situation may be visualized by the *complexification diagram* of a symmetric space with twist T:

$$\begin{array}{ccccc} & \nearrow & M_{\mathbb{C}} & \searrow & \\ M & \to & M_{h\mathbb{C}} & \to & (M_{h\mathbb{C}})_{\mathbb{C}} \\ & \searrow & M_{ph\mathbb{C}} & \nearrow & \end{array}$$

In the next Chapter we will give many examples of complexification diagrams. – Recall from Section I.1.4 that to every (germ of a) symmetric space M one associates a (germ of a) *c-dual symmetric space* M^c whose associated LTS is $-R$ if R is the LTS of M. Therefore we define:

Definition III.4.10. The *c-dual* of a LTS R, resp. of a JTS T, is the LTS $-R$, resp. the JTS $-T$. ■

If (R,J) is straight complex, then

$$(J\cdot R)(X,Y,Z) = JR(Z^{-1}X, J^{-1}Y, J^{-1}Z) = J^{-2}R(X,Y,Z) = (-R)(X,Y,Z),$$

i.e. the action of J in $\mathrm{Hom}(\otimes^3 V, V)$ induces an isomorphism $R\mapsto -R$. In other words, straight complex LTS are self c-dual. The same holds for twisted polarized LTS, but not for twisted complex LTS (for the latter, $J\cdot R = R$). Similarly, straight complex and twisted polarized JTS are self c-dual, but Hermitian JTS are in general not. As a consequence, we have the relations

$$(M_{\mathbb{C}})^c \cong M_{\mathbb{C}}, \quad (M_{ph\mathbb{C}})^c \cong M_{ph\mathbb{C}}.$$

Thus, when applying the c-dual functor to a complexification diagram, only the first two spaces in the middle line change.

Notes for Chapter III.

III.2. It is known that almost complex structures related to *Hermitian symmetric spaces* are twisted (see Prop. V.2.2); cf. e.g. [Lo69b] where invariant twisted complex structures are called *anticomplex*. The correspondence between such structures and Hermitian Jordan triple systems has already been remarked by W. Kaup (cf. [Kau93]).

III.3. The *para-Hermitian symmetric spaces* (see Def. V.2.1) introduced by P. Libermann and investigated by M. Kozai and S. Kaneyuki ([KanKo85]) are twisted polarized (see Prop. V.2.2). The approach of Kozai and Kaneyuki, including the classification (cf. Tables XII.4.1, XII.4.2), is via graded Lie algebras. Motivated by M. Koechers investigations on graded Lie algebras related to Jordan algebras [Koe68], K. Meyberg in [Mey70] was led to the notion "verbundenes Paar" which is a precursor of the algebraically more satisfactory notion of a Jordan pair introduced by O. Loos ([Lo75]).

III.4. Various special cases of symmetric spaces with twist have already been investigated in the research literature; however, it seems that the present work is the first systematic attempt to clarify their geometric and algebraic nature – see Section 0.3 (introduction).

The algebraic Jordan-Lie functor (Lemma III.2.6) has first been noticed by K. Meyberg (cf. [Ne85]) and then been investigated by E. Neher, see Notes to Chapter IV. A special case of the geometric Jordan-Lie functor is considered in [Lo85], and the important equation (4.1) has been noted in [Lo77, Th. 2.10 (d)].

Chapter IV: The classical spaces

In this chapter we present a fairly exhaustive collection of examples of "classical" symmetric spaces with twist, and we explain the basic principles of their classification. The classification itself is contained in the tables given in Chapter XII. As noted there (Remark XII.4.8), comparison of the classification of simple real Lie triple systems (carried out by M. Berger, cf. [B57]) and the classification of simple real Jordan triple systems (carried out by E. Neher, cf. [Ne80], [Ne81], and implicitly in another way by B.O. Makarevič, cf. [Ma73]) leads to the following two remarkable observations:

A. (Existence.) *Every non-exceptional irreducible symmetric space has (possibly after a central extension) a Jordan-extension.* (A central extension is needed e.g. in the group case of $\mathrm{Sl}(n,\mathbb{R})$ which does not admit any Jordan extension, but the central extension $\mathrm{Gl}(n,\mathbb{R})$ does.)

B. (Uniqueness.) *The number of (local) Jordan extensions of an irreducible non-exceptional symmetric space is either* $0, 1, 2$ *or* 3*; in most cases it is* 1.

Summarizing, we can say that the Jordan-Lie functor seems to be fairly close to being injective and surjective. This has already been remarked by E. Neher, cf. [Ne85], where also the exceptional spaces are considered.

The organizing principle both for our presentation of examples and for the classification is the *complexification diagram*

$$\begin{array}{ccccccc} & & \nearrow & (4)\quad M_{\mathbb{C}} & \searrow & & \\ (5) & M & \to & (3)\quad M_{h\mathbb{C}} & \to & (1) & (M_{h\mathbb{C}})_{\mathbb{C}} \\ & & \searrow & (2)\quad M_{ph\mathbb{C}} & \nearrow & & \end{array}$$

introduced in the preceding chapter. Objects of type (1), i.e. spaces having both invariant straight and twisted complex structures, are most easily classified, next come objects of type (2) (spaces having an invariant twisted para-complex structure), up to type (5) (having no invariant complex or para-complex structure) which is the most difficult. Only spaces of type (5) (i.e. which are not of type (1) – (4)) have a "generic" complexification diagram; by this we mean a diagram containing no direct products. In contrast, spaces of type (1) have a trivial complexification diagram: it contains only direct products. The types (2) – (4) show an intermediate behaviour.

In Section 1 we work out the most important examples: the *general linear groups*, *Grassmannians* and *spaces of Lagrangian type*, among them the *orthogonal, symplectic* and *unitary groups*. In all of these cases, the twisted complexification $M_{h\mathbb{C}}$ is given by a *space of complex structures*, i.e. a certain symmetric space of operators X whose square is minus the identity. The most famous example of these is the *Siegel-space* (cf. Section I.6.3). Similarly, the twisted para-complexifications are given by certain *spaces of polarizations*. The last series of classical spaces, containing the *spheres* and *hyperbolic spaces*, shows a different behavior.

1. Examples

All symmetric spaces given in Sections I.6.1 – I.6.4 are symmetric spaces with twist. In fact, once we have seen that $\mathrm{Gl}(n,\mathbb{F})$ admits a twist, we only have to verify that the Lie triple systems $\mathfrak{q}$ described in Section I.6 are in fact sub-JTS of the JTS defined on the space $M(n,\mathbb{F}) = \mathfrak{gl}(n,\mathbb{F})$. Since the spaces $\mathfrak{q}$ are always given as spaces fixed under one or two involutions, it is enough to check that these involutions are also JTS-automorphisms of $\mathfrak{gl}(n,\mathbb{F})$. This is straightforward. However, this observation only tells us that the spaces in quesion do admit twisted (para-) complexifications, but not how they are actually realized. We are going to describe them for the most important examples.

1.1. The general linear groups.

Proposition IV.1.1.

(i) *The space* $M = \mathrm{Gl}(2n,\mathbb{R})/\mathrm{Gl}(n,\mathbb{C})$ *of complex structures on* $\mathbb{R}^{2n}$ *is a twisted complex symmetric space.*

(ii) *The space* $M_{p,q} = \mathrm{Gl}(p+q,\mathbb{R})/(\mathrm{Gl}(p,\mathbb{R})\times\mathrm{Gl}(q,\mathbb{R}))$ *of polarizations on* $\mathbb{R}^{p+q}$ *with signature* (p,q) *is a twisted polarized symmetric space.*

(iii) *The group* $\mathrm{Gl}(n,\mathbb{R})$ *is a real form of* M *and a para-real form of* $M_{n,n}$. *The complexification diagram of* $\mathrm{Gl}(n,\mathbb{R})$ *is:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Gl}(n,\mathbb{C}) & \searrow & \\ \mathrm{Gl}(n,\mathbb{R}) & \rightarrow & \mathrm{Gl}(2n,\mathbb{R})/\mathrm{Gl}(n,\mathbb{C}) & \rightarrow & \mathrm{Gl}(2n,\mathbb{C})/(\mathrm{Gl}(n,\mathbb{C})\times\mathrm{Gl}(n,\mathbb{C})). \\ & \searrow & \mathrm{Gl}(2n,\mathbb{R})/(\mathrm{Gl}(n,\mathbb{R})\times\mathrm{Gl}(n,\mathbb{R})) & \nearrow & \end{array}$$

The corresponding Jordan extension of the LTS $\mathfrak{gl}(n,\mathbb{R})$ *is given by*

$$T(X,Y,Z) = XYZ + ZYX.$$

Proof. (i) Recall from Section I.6.3, Eqns. (I.6.33) and (I.6.35), the realization of the space $\mathrm{Gl}(2n,\mathbb{R})/\mathrm{Gl}(n,\mathbb{C})$ in $M(2n,\mathbb{R})$ and of its LTS $\mathfrak{q} = \{X \in M(2n,\mathbb{R}) |\, XJ = -JX\}$ with Lie triple product $[X,Y,Z] = [[X,Y],Z]$. The space $\mathfrak{q}$ is stable under left multiplication by the matrix J, and the map

$$l_J : \mathfrak{q} \to \mathfrak{q}, \quad X \mapsto JX$$

is an invariant twisted complex structure on $\mathfrak{q}$: in fact, if X and Y anticommute with J, then $[X,JY] = -[JX,Y]$, and moreover $[X,Y]$ commutes with J, whence $[X,Y,JZ] = J[X,Y,Z]$. We have proved that $\mathfrak{q}$ is twisted complex. The corresponding structure tensor is

$$T(X,Y,Z) = \frac{1}{2}([X,Y,Z] + J[X,JY,Z]) = XYZ + ZYX. \tag{1.1}$$

(ii) The proof follows the same pattern as the proof of part (i), using the realizations (I.6.24) and (I.6.26) of the space of polarizations, resp. of its LTS. The twisted polarized structure is given by left multiplication with $I_{p,q}$ on $\mathfrak{q}$, and the structure tensor is given by the same formula as above.

(iii) Let $\overline{z} = I_{n,n}z$ be the standard complex conjugation of $\mathbb{R}^{2n} = \mathbb{C}^n$. Then $\tau(X) := -I_{n,n}X\,I_{n,n}$ is an involutive automorphism of the space M of complex structures on $\mathbb{R}^{2n}$ fixing the base point J. The τ-fixed subspace of M is

$$\begin{aligned} M^\tau &= \{X \in \mathrm{Gl}(2n,\mathbb{R}) \mid X^2 = -\mathbf{1}_{2n}, I_{n,n}X = -X\,I_{n,n}\} \\ &= \{\begin{pmatrix} 0 & g \\ -g^{-1} & 0 \end{pmatrix}\} \mid g \in \mathrm{Gl}(n,\mathbb{R})\}. \end{aligned} \tag{1.2}$$

This space is isomorphic to $\mathrm{Gl}(n,\mathbb{R})$ via the map $\mathrm{Gl}(n,\mathbb{R}) \to M^\tau$, $g \mapsto \left(\begin{smallmatrix} 0 & g \\ -g^{-1} & 0 \end{smallmatrix}\right)$. The LTS $\mathfrak{gl}(n,\mathbb{R})$ is imbedded into the LTS $\mathfrak{q}$ of M via $X \mapsto \left(\begin{smallmatrix} 0 & X \\ X & 0 \end{smallmatrix}\right)$, and the structure tensor is given just by restricting (1.1) to this space.

Similarly, in the situation of part (ii) assume $p = q = n$ and consider the map $\tau(X) := JXJ = -JXJ^{-1}$. It is an involutive automorphism of $M_{n,n}$ stabilizing the base point $I_{n,n}$. The τ-fixed subspace is

$$\begin{aligned} M^\tau_{n,n} &= \{X \in \mathrm{Gl}(2n,\mathbb{R}) \mid JXJ = X, X^2 = \mathbf{1}_{2n}, \mathrm{sgn}(X) = (n,n)\} \\ &= \{\begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix} \mid g \in \mathrm{Gl}(n,\mathbb{R})\}; \end{aligned} \tag{1.3}$$

again this is seen to be isomorphic with $\mathrm{Gl}(n,\mathbb{R})$, and again the Jordan extension of $\mathfrak{gl}(n,\mathbb{R})$ is given by (1.1). ∎

In a similar way as above one can prove that the space of complex structures in $\mathrm{Gl}(n,\mathbb{H})$ is twisted complex and that $\mathrm{Gl}(n,\mathbb{H})$ is a real form:
(1.4) *Complexification diagram of* $\mathrm{Gl}(n,\mathbb{H})$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Gl}(2n,\mathbb{C}) & \searrow & \\ \mathrm{Gl}(n,\mathbb{H}) & \to & \mathrm{Gl}(2n,\mathbb{H})/\,\mathrm{Gl}(2n,\mathbb{C}) & \to & \mathrm{Gl}(4n,\mathbb{C})/(\mathrm{Gl}(2n,\mathbb{C}) \times \mathrm{Gl}(2n,\mathbb{C})) \\ & \searrow & \mathrm{Gl}(2n,\mathbb{H})/(\mathrm{Gl}(n,\mathbb{H}) \times \mathrm{Gl}(n,\mathbb{H})) & \nearrow & \end{array}$$

with structure tensor given again by formula (1.1). For $\mathrm{Gl}(n,\mathbb{C})$, twisted complexification and twisted para-complexification are both isomorphic to the space $\mathrm{Gl}(2n,\mathbb{C})/\ (\mathrm{Gl}(n,\mathbb{C}) \times \mathrm{Gl}(n,\mathbb{C}))$; again the structure tensor is given by Eqn. (1.1).

1.2. Grassmannians and spaces of Grassmannian type.

Proposition IV.1.2. *The complex Grassmannians $\mathrm{Gr}_{p,p+q}(\mathbb{C})$ are twisted complex symmetric spaces, and the real Grassmannians $\mathrm{Gr}_{p,p+q}(\mathbb{R})$ are real forms. They are para-real forms of the twisted polarized space $M_{p,q}$ defined in the preceding proposition. In other words, the complexification diagram of the real Grassmannians $M = \mathrm{O}(n)/(\mathrm{O}(p) \times \mathrm{O}(q))$ is*

$$\begin{array}{ccccc} & \nearrow & \mathrm{O}(n,\mathbb{C})/(\mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(q,\mathbb{C})) & \searrow & \\ M & \to & \mathrm{U}(n)/(\mathrm{U}(p) \times \mathrm{U}(q)) & \to & \mathrm{Gl}(n,\mathbb{C})/(\mathrm{Gl}(p,\mathbb{C}) \times \mathrm{Gl}(q,\mathbb{C})). \\ & \searrow & \mathrm{Gl}(n,\mathbb{R})/(\mathrm{Gl}(p,\mathbb{R}) \times \mathrm{Gl}(q,\mathbb{R})) & \nearrow & \end{array}$$

The corresponding Jordan triple system of the real Grassmannian is isomorphic to the space $\mathfrak{q} = M(p,q;\mathbb{R})$ with the triple product

$$T(X,Y,Z) = -(XY^tZ + ZY^tX).$$

Proof. Recall from Section I.6.2 that we realize the complex Grassmannian as

$$\mathrm{Gr}_{p,p+q}(\mathbb{C}) = \{g \in \mathrm{U}(p+q) \mid g^2 = \mathbf{1}, \mathrm{sgn}(g) = (p,q)\}$$

with base point $I_{p,q}$ and LTS

$$\begin{aligned}\mathfrak{q} &= \{X \in M(p+q,\mathbb{C}) |\, I_{p,q}X = -XI_{p,q},\, X = \overline{X}^t\} \\ &= \{X_a = \begin{pmatrix} 0 & a \\ \overline{a}^t & 0 \end{pmatrix} |\, a \in M(p,q;\mathbb{C})\}.\end{aligned}$$

The space $\mathfrak{q}$ is stable under that map $X \mapsto iI_{p,q}X$ (that is, $X_a \mapsto X_{ia}$), and as in the proof of Prop. IV.1.1 (i) it is easily verified that this is an invariant twisted complex structure on $\mathfrak{q}$. The structure tensor can be calculated from the structure tensor T' of the complex version of the space $M_{p,q}$ from Prop. IV.1.1 (ii):

$$T(X,Y,Z) = iT'(X,iY,Z) = -T'(X,Y,Z);$$

this implies $T(X_a,X_b,X_c) = X_{T(a,b,c)}$ with T as in the claim. Ordinary complex conjugation is a complex conjugation of $\mathrm{Gr}_{p,p+q}(\mathbb{C})$ w.r.t. $\mathrm{Gr}_{p,p+q}(\mathbb{R})$.

Similarly one proves that the real Grassmannian is a para-real form of the space $M_{p,q}$ from Prop. IV.1.1 (ii) (the para-conjugation is $X \mapsto X^*$). ∎

If p and q are even, the complex Grassmannians have another family of real forms, namely the quaternionic Grassmannians.

(1.5) *Complexification diagram of quaternionic Grassmannians* $M = \mathrm{Sp}(n)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Sp}(n,\mathbb{C})/(\mathrm{Sp}(p,\mathbb{C})\times\mathrm{Sp}(q,\mathbb{C})) & \searrow & \\ M & \to & \mathrm{U}(2n)/(\mathrm{U}(2p)\times\mathrm{U}(2q)) & \to & \mathrm{Gl}(2n,\mathbb{C})/(\mathrm{Gl}(2p,\mathbb{C})\times\mathrm{Gl}(2q,\mathbb{C})). \\ & \searrow & \mathrm{Gl}(n,\mathbb{H})/(\mathrm{Gl}(p,\mathbb{H})\times\mathrm{Gl}(q,\mathbb{H})) & \nearrow & \end{array}$$

The twisted para-complexification of the *complex* Grassmannian is isomorphic to its straight complexification, that is, to $\mathrm{Gl}(n,\mathbb{C})/(\mathrm{Gl}(p,\mathbb{C})\times\mathrm{Gl}(q,\mathbb{C}))$, and its twisted complexification is just a direct product with itself. The complexification diagrams of the spaces $\mathrm{O}(l+j,k+i)/(\mathrm{O}(l,k)\times\mathrm{O}(j,i))$, $\mathrm{U}(l+j,k+i)/(\mathrm{U}(l,k)\times\mathrm{U}(j,i))$ and $\mathrm{Sp}(l+j,k+i)/(\mathrm{Sp}(l,k)\times\mathrm{Sp}(j,i))$ are similar to those of the corresponding Grassmannians, and also the following are of the same type:

(1.6) *Complexification diagram of* $M = \mathrm{Sp}(p+q,\mathbb{R})/(\mathrm{Sp}(p,\mathbb{R})\times\mathrm{Sp}(p,\mathbb{R}))$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Sp}(p+q,\mathbb{C})/(\mathrm{Sp}(p,\mathbb{C})\times\mathrm{Sp}(p,\mathbb{C})) & \searrow & \\ M & \to & \mathrm{U}(n,n)/(\mathrm{U}(p,p)\times\mathrm{U}(q,q)) & \to & \mathrm{Gl}(2n,\mathbb{C})/(\mathrm{Gl}(2p,\mathbb{C})\times\mathrm{Gl}(2q,\mathbb{C})) \\ & \searrow & \mathrm{Gl}(2n,\mathbb{R})/(\mathrm{Gl}(2p,\mathbb{R})\times\mathrm{Gl}(2q,\mathbb{R})) & \nearrow & \end{array}$$

(1.7) *Complexification diagram of* $M = \mathrm{O}^*(2n)/(\mathrm{O}^*(2p)\times\mathrm{O}^*(2q))$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{O}(2n,\mathbb{C})/(\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(p,\mathbb{C}) & \searrow & \\ M & \to & \mathrm{U}(n,n)/(\mathrm{U}(p,p)\times\mathrm{U}(q,q)) & \to & \mathrm{Gl}(2n,\mathbb{C})/(\mathrm{Gl}(2p,\mathbb{C})\times\mathrm{Gl}(2q,\mathbb{C})) \\ & \searrow & \mathrm{Gl}(2n,\mathbb{R})/(\mathrm{Gl}(2p,\mathbb{R})\times\mathrm{Gl}(2q,\mathbb{R})) & \nearrow & \end{array}$$

1.3. Spaces of Lagrangian type; orthogonal, symplectic and unitary groups.

Proposition IV.1.3.

(i) *The orthogonal group* $\mathrm{O}(A,\mathbb{R})$ *admits a twisted complexification given by the space* $M_J = \mathrm{O}(\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right), \mathbb{R})/\mathrm{U}(A,\mathbb{C})$.

(ii) *The twisted para-complexification of* $\mathrm{O}(A,\mathbb{R})$ *is given by* $\mathrm{O}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right), \mathbb{R})/\,\mathrm{Gl}(n,\mathbb{R})$. *Thus the complexification diagram of* $\mathrm{O}(A,\mathbb{R})$ *is:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{O}(A,\mathbb{C}) & \searrow & \\ \mathrm{O}(A,\mathbb{R}) & \rightarrow & \mathrm{O}(\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right), \mathbb{R})/\mathrm{U}(A,\mathbb{C}) & \rightarrow & \mathrm{O}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right), \mathbb{C})/\,\mathrm{Gl}(n,\mathbb{C}). \\ & \searrow & \mathrm{O}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right), \mathbb{R})/\,\mathrm{Gl}(n,\mathbb{R}) & \nearrow & \end{array}$$

The corresponding JTS is the space $\mathrm{Asym}(A,\mathbb{R})$ *with the triple product* $T(X,Y,Z) = -(XYZ + ZYX)$.

Proof. (i) Recall from Section I.6.3, Eqn. (I.6.42) that M_J is a twisted complex symmetric space which can be realized as the connected component of the space of complex structures in $\mathrm{O}(\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right), \mathbb{R})$ containing the base point J. As in the proof of Prop. IV.1.1 (iii) we consider the conjugation $\tau(X) := -I_{n,n} X\, I_{n,n}$ of M_J. The space of fixed points is

$$M_J^\tau = \{ \begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix} \mid g \in \mathrm{O}(A,\mathbb{R})\}.$$

As a symmetric space, this is the space $\mathrm{O}(A,\mathbb{R})$.

(ii) As remarked in Section I.6.4, the space defined by Eqn. (I.6.51) is the subspace of the space defined by Eqn. (I.6.48) under the automorphism $\tau(X) = D^{-1} X^t D$. It is easily verified that τ is a para-conjugation. Thus it follows from Eqn. (I.6.56) that $\mathrm{O}(A,\mathbb{R})$ is a para-real form of $\mathrm{O}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right), \mathbb{R})/\,\mathrm{Gl}(n,\mathbb{R})$. The associated JTS is given by restricting the JTS from Prop. IV.1.1 to the space $\mathfrak{q}$ given by Eqn. (I.6.52). ∎

(1.8) *Complexification diagram of* $\mathrm{O}(p,q)$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{O}(n,\mathbb{C}) & \searrow & \\ \mathrm{O}(p,q) & \rightarrow & \mathrm{O}(2p,2q)/\mathrm{U}(p,q) & \rightarrow & \mathrm{O}(2n,\mathbb{C})/\,\mathrm{Gl}(n,\mathbb{C}) \\ & \searrow & \mathrm{O}(n,n)/\,\mathrm{Gl}(n,\mathbb{R}) & \nearrow & \end{array}$$

(1.9) *Complexification diagram of* $\mathrm{Sp}(n,\mathbb{R})$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Sp}(n,\mathbb{C}) & \searrow & \\ \mathrm{Sp}(n,\mathbb{R}) & \rightarrow & \mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n,n) & \rightarrow & \mathrm{Sp}(2n,\mathbb{C})/\,\mathrm{Gl}(2n,\mathbb{C}) \\ & \searrow & \mathrm{Sp}(2n,\mathbb{R})/\,\mathrm{Gl}(2n,\mathbb{R}) & \nearrow & \end{array}$$

The corresponding diagrams for $\mathrm{O}(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{C})$ are obtained by a straight complexification of the above diagrams. The complexification diagrams of unitary groups are obtained in a similar way as in Prop. IV.1.3. The results are:

(1.10) *Complexification diagram of* $\mathrm{U}(p,q)$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Gl}(n,\mathbb{C}) & \searrow & \\ \mathrm{U}(p,q) & \rightarrow & \mathrm{U}(2p,2q)/(\mathrm{U}(p,q)\times\mathrm{U}(p,q)) & \rightarrow & \mathrm{Gl}(2n,\mathbb{C})/(\mathrm{Gl}(n,\mathbb{C})\times\mathrm{Gl}(n,\mathbb{C})) \\ & \searrow & \mathrm{U}(n,n)/\,\mathrm{Gl}(n,\mathbb{C}) & \nearrow & \end{array}$$

(1.11) *Complexification diagram of* $\mathrm{Sp}(p,q)$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Sp}(n,\mathbb{C}) & \searrow & \\ \mathrm{Sp}(p,q) & \rightarrow & \mathrm{Sp}(2p,2q)/\mathrm{U}(2p,2q) & \rightarrow & \mathrm{Sp}(2n,\mathbb{C})/\,\mathrm{Gl}(2n,\mathbb{C}) \\ & \searrow & \mathrm{Sp}(n,n)/\,\mathrm{Gl}(n,\mathbb{H}) & \nearrow & \end{array}$$

(1.12) *Complexification diagram of* $\mathrm{SO}^*(2n)$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{SO}(2n,\mathbb{C}) & \searrow & \\ \mathrm{SO}^*(2n) & \to & \mathrm{SO}^*(4n)/\mathrm{U}(n,n) & \to & \mathrm{SO}(4n,\mathbb{C})/\,\mathrm{Gl}(2n,\mathbb{C}) \\ & \searrow & \mathrm{SO}^*(4n)/\,\mathrm{Gl}(n,\mathbb{H}) & \nearrow & \end{array}$$

Remark: what about the groups $\mathrm{Spin}(p,q)$? Classification shows that they only admit locally a twisted complexification (the one of $\mathrm{O}(p,q)$), but not globally.

Proposition IV.1.4.

(i) *The twisted complexification of the space* $M = \mathrm{Lag}(J,\mathbb{R}) \cong \mathrm{U}(n)/\mathrm{O}(n)$ *of Lagrangian subspaces of the skew-symmetric form defined by* J *over* $\mathbb{R}$ *is the space* $\mathrm{Lag}(J,\tau,\mathbb{C}) \cong \mathrm{Sp}(n)/\mathrm{O}(n)$ *of Lagrangian subspaces of the standard skew-Hermitian form defined by* J *over* $\mathbb{C}$.

(ii) *The twisted para-complexification of* M *is the space* $\mathrm{Sp}(n,\mathbb{R})/\,\mathrm{Gl}(n,\mathbb{R})$.

(iii) *The complexification diagram of* $M = \mathrm{U}(p,q)/\mathrm{O}(p,q)$ *is:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Gl}(n,\mathbb{C})/\mathrm{O}(n,\mathbb{C}) & \searrow & \\ \mathrm{U}(p,q)/\mathrm{O}(p,q) & \to & \mathrm{Sp}(p,q)/\mathrm{U}(p,q) & \to & \mathrm{Sp}(n,\mathbb{C})/\,\mathrm{Gl}(n,\mathbb{C}) \\ & \searrow & \mathrm{Sp}(n,\mathbb{R})/\,\mathrm{Gl}(n,\mathbb{R}) & \nearrow & \end{array}$$

The corresponding JTS is the space $\mathrm{Sym}(n,\mathbb{R})$ *with the triple product* $T(X,Y,Z) = -(XI_{p,q}YI_{p,q}Z + ZI_{p,q}YI_{p,q}X)$.

Proof. (i) The fact that $\mathrm{Lag}(A,\tau,\mathbb{C})$ is the twisted complexification of $\mathrm{Lag}(A,\mathbb{R})$ is proved as the corresponding fact for Grassmannians (Prop. IV.1.2; this relation appears already in Diagram (1.8), where $\mathrm{O}(n) \cong \mathrm{Lag}(I_{n,n},\mathbb{R})$, $\mathrm{O}(2n)/\mathrm{U}(n) \cong \mathrm{Lag}(I_{n,n},\tau,\mathbb{C})$).

(ii) In Section I.6.4 we have observed that $\mathrm{Sp}(n,\mathbb{R})/\,\mathrm{Gl}(n,\mathbb{R})$ can be interpreted as a variety of pairs of Lagrangians (cf. Eqn. (I.6.49)) and that the corresponding varieties of Lagrangians are the fixed point spaces under an involution which is easily seen to be a para-conjugation (cf. Eqn. (I.6.51)).

(iii) The case $q = 0$ follows from (i) and (ii), and the general case is verified in a similar way. ■

(1.13) *Complexification diagram of* $M = \mathrm{U}(n,n)/\mathrm{O}^*(2n)$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Gl}(2n,\mathbb{C})/\mathrm{O}(2n,\mathbb{C}) & \searrow & \\ \mathrm{U}(n,n)/\mathrm{O}^*(2n) & \to & \mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n,n) & \to & \mathrm{Sp}(2n,\mathbb{C})/\,\mathrm{Gl}(2n,\mathbb{C}) \\ & \searrow & \mathrm{Sp}(n,n)/\,\mathrm{Gl}(n,\mathbb{H}) & \nearrow & \end{array}$$

(1.14) *Complexification diagram of* $M = \mathrm{U}(2p,2q)/\,\mathrm{Sp}(p,q)$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Gl}(2n,\mathbb{C})/\,\mathrm{Sp}(n,\mathbb{C}) & \searrow & \\ \mathrm{U}(2p,2q)/\,\mathrm{Sp}(p,q) & \to & \mathrm{SO}(4p,4q)/\mathrm{U}(2p,2q) & \to & \mathrm{SO}(4n,\mathbb{C})/\,\mathrm{Gl}(2n,\mathbb{C}) \\ & \searrow & \mathrm{SO}^*(4n)/\,\mathrm{Gl}(n,\mathbb{H}) & \nearrow & \end{array}$$

(1.15) *Complexification diagram of* $M = \mathrm{U}(n,n)/\,\mathrm{Sp}(n,\mathbb{R})$*:*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Gl}(2n,\mathbb{C})/\,\mathrm{Sp}(n,\mathbb{C}) & \searrow & \\ \mathrm{U}(n,n)/\,\mathrm{Sp}(n,\mathbb{R}) & \to & \mathrm{SO}^*(4n)/\mathrm{U}(n,n) & \to & \mathrm{SO}(4n,\mathbb{C})/\,\mathrm{Gl}(2n,\mathbb{C}) \\ & \searrow & \mathrm{SO}(2n,2n)/\,\mathrm{Gl}(2n,\mathbb{R}) & \nearrow & \end{array}$$

Finally, there are two rather special types of spaces which can be interpreted as certain spaces of complex structures and whose complexification diagrams are (cf. Th. XI.5.6 and Table XII.4.5):

(1.16) *Complexification diagram of* $M = \mathrm{SO}^*(2n,\mathbb{R})/\,\mathrm{SO}(n,\mathbb{C})$*:*

$$\begin{array}{cccll} & \nearrow & \mathrm{SO}(2n,\mathbb{C})/(\mathrm{SO}(n,\mathbb{C})\times \mathrm{SO}(n,\mathbb{C})) & \searrow & \\ M & \to & \mathrm{Gl}(n,\mathbb{H})/\,\mathrm{Gl}(n,\mathbb{C}) & \to & \mathrm{Gl}(2n,\mathbb{C})/(\mathrm{Gl}(n,\mathbb{C})\times \mathrm{Gl}(n,\mathbb{C})) \\ & \searrow & \mathrm{U}(n,n)/\,\mathrm{Gl}(n,\mathbb{C}) & \nearrow & \end{array}$$

(1.17) *Complexification diagram of* $M = \mathrm{Sp}(2n,\mathbb{R})/\,\mathrm{Sp}(n,\mathbb{C})$*:*

$$\begin{array}{cccll} & \nearrow & \mathrm{Sp}(2n,\mathbb{C})/(\mathrm{Sp}(n,\mathbb{C})\times \mathrm{Sp}(n,\mathbb{C})) & \searrow & \\ M & \to & \mathrm{Gl}(2n,\mathbb{R})/\,\mathrm{Gl}(n,\mathbb{C}) & \to & \mathrm{Gl}(2n,\mathbb{C})/(\mathrm{Gl}(n,\mathbb{C})\times \mathrm{Gl}(n,\mathbb{C})) \\ & \searrow & \mathrm{U}(n,n)/\,\mathrm{Gl}(n,\mathbb{C}) & \nearrow & \end{array}$$

1.4. c-duality. In principle, the complexification diagrams of the c-dual spaces of the spaces treated above are simply obtained by applying the c-dual functor to the whole diagram. Since straight complex and twisted para-complex spaces are self c-dual, effectively only the twisted complexification changes under duality (cf. remarks in Section III.4.3).

Example: the group cases. Note that $G^c = G_{\mathbb{C}}/G$ for a Lie group G considered as a symmetric space.

$$(\mathrm{Gl}(n,\mathbb{C})/\,\mathrm{Gl}(n,\mathbb{R}))_{h\mathbb{C}} \cong \mathrm{Gl}(n,\mathbb{H})/\,\mathrm{Gl}(n,\mathbb{C})$$

$$(\mathrm{Gl}(2n,\mathbb{C})/\,\mathrm{Gl}(n,\mathbb{H}))_{h\mathbb{C}} \cong \mathrm{Gl}(2n,\mathbb{R})/\,\mathrm{Gl}(n,\mathbb{C})$$

$$(\mathrm{Gl}(n,\mathbb{C})/\mathrm{U}(p,q))_{h\mathbb{C}} \cong \mathrm{U}(n,n)/(\mathrm{U}(p,q)\times \mathrm{U}(p,q))$$

$$(\mathrm{O}(n,\mathbb{C})/\mathrm{O}(p,q))_{h\mathbb{C}} \cong \mathrm{O}(4n)/\mathrm{U}(2p,2q)$$

$$(\mathrm{Sp}(n,\mathbb{C})/\,\mathrm{Sp}(n,\mathbb{R}))_{h\mathbb{C}} \cong \mathrm{Sp}(n,n)/\mathrm{U}(n,n)$$

$$(\mathrm{Sp}(n,\mathbb{C})/\,\mathrm{Sp}(p,q))_{h\mathbb{C}} \cong \mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n,n)$$

$$(\mathrm{SO}(2n,\mathbb{C})/\,\mathrm{SO}^*(2n))_{h\mathbb{C}} \cong \mathrm{SO}(2n,2n)/\mathrm{U}(n,n)$$

1.5. Spheres and hyperbolic spaces.

Proposition IV.1.5.

(i) *The projective completion of the complex sphere* $S^n_{\mathbb{C}}$ *is a symmetric space* $M = \mathrm{SO}(n+2)/(\mathrm{SO}(n)\times \mathrm{SO}(2))$*, and* M *is twisted complex.*

(ii) *The sphere* S^n *is a real form of* M*. The corresponding complexification diagram of* S^n *is*

$$\begin{array}{cccll} & \nearrow & S^n_{\mathbb{C}} & \searrow & \\ S^n & \to & \mathrm{SO}(n+2)/(\mathrm{SO}(n)\times \mathrm{SO}(2)) & \to & \mathrm{SO}(n+2,\mathbb{C})/(\mathrm{SO}(n,\mathbb{C})\times \mathrm{SO}(2,\mathbb{C})). \\ & \searrow & \mathrm{SO}(n+1,1)/(\mathrm{SO}(n)\times \mathrm{SO}(1,1)) & \nearrow & \end{array}$$

The JTS belonging to this complexification diagram of S^n is given by the vector space $\mathbb{R}^n$ with the triple product

$$T(x,y,z) = (x|z)y - (x|y)z - (z|y)x.$$

Proof. (i) We realize M as

$$M = \{[z] \in \mathbb{P}\mathbb{C}^{n+2} | \sum_{j=1}^{n+2} z_j^2 = 0\}.$$

The group $\mathrm{SO}(n+2,\mathbb{C})$ and its maximal compact subgroup $\mathrm{SO}(n+2) = \mathrm{SO}(n+2,\mathbb{C}) \cap \mathrm{SU}(n+2)$ act on M in the usual way. As base point we choose $o = [(0,\dots,0,i,1)^t]$. The vector $(i,1)^t$ is an eigenvector of $\mathrm{SO}(2)$:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i\cos\varphi - \sin\varphi \\ i\sin\varphi + \cos\varphi \end{pmatrix} = e^{i\varphi} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

This implies that the stabilizer of the base point o in $\mathrm{SO}(n+2)$ is the subgroup of matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ with $a \in \mathrm{SO}(n)$, $b \in \mathrm{SO}(2)$; we identify it with $\mathrm{SO}(n) \times \mathrm{SO}(2)$, whence $\mathrm{SO}(n+2).o \cong \mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$. This is a symmetric space; its associated involution is conjugation by the matrix $I_{n,2}$. The corresponding LTS is the space

$$\mathfrak{q} = \{\begin{pmatrix} 0 & -X^t \\ X & 0 \end{pmatrix} | \, X \in M(2,n;\mathbb{R})\} \subset \mathfrak{so}(n+2).$$

It has the same dimension as M; therefore $\mathrm{SO}(n+2).o$ is open in M, and since both sets are compact and connected, they agree.

We identify $\mathfrak{q}$ with $M(2,n;\mathbb{R})$; then the LTS on $\mathfrak{q}$ is the same as the one corresponding to $\mathrm{Gr}_{2,n+2}(\mathbb{R})$, i.e. given by

$$[X,Y,Z] = YX^tZ + ZX^tY - (XY^tZ + ZY^tX) \quad (X,Y,Z \in M(2,n;\mathbb{R})).$$

For all $X,Y \in M(2,n;\mathbb{R})$ the matrix $XY^t - YX^t \in M(2,2;\mathbb{R})$ is skew-symmetric, i.e. it is an element of $\mathfrak{so}(2)$ and therefore commutes with $J = J_2$. Using this, we get

$$[X,Y,JZ] = YX^tJZ + JZX^tY - (XY^tJZ + JZY^tX) = J[X,Y,Z].$$

Next, the identity

$$[JX,JY,JZ] = J[X,Y,Z]$$

is immediately verified using that $J^tJ = \mathbf{1}_2$. These two properties are equivalent to saying that $X \mapsto JX$ defines an invariant twisted complex structure on $\mathfrak{q}$ (cf. Prop. III.2.2). The corresponding structure tensor is

$$\begin{aligned} T(X,Y,Z) &= \frac{1}{2}([X,Y,Z] + J[X,JY,Z]) \\ &= \frac{1}{2}(YX^tZ + ZX^tY - (XY^tZ + ZY^tX) + \\ &\qquad J(JYX^tZ + ZX^tJY + XY^tJZ + ZY^tJX)) \\ &= \frac{1}{2}((ZX^t + JZX^tJ)Y - XY^tZ - JX(JY)^tZ - ZY^tX - JZ(JY)^tJX). \end{aligned}$$

(ii) In order to see that the sphere is a real form of M, we have to use another realization of M which is related to the first one via a simple change of coordinates, namely

$$M' = \{[z] \in \mathbb{P}\mathbb{C}^{n+2} | \sum_{j=1}^{n} z_j^2 + z_{n+1}z_{n+2} = 0\}.$$

(The change of coordinates can be intepreted as a sort of "Cayley transform".) In these coordinates let $\tau(z) = \overline{z}$ be ordinary complex conjugation; then the fixed point space is isomorphic to the real projective quadric

$$\{[x] \in \mathbb{P}\mathbb{R}^{n+2} | \sum_{j=1}^{n} x_j^2 + x_{n+1}x_{n+2}^2 = 0\} \cong S^n.$$

Let us now verify the formula for the associated JTS: returning to the realization from Part (i), the complex conjugation of $\mathfrak{q}$ corresponding to the one of M just defined is given by $X \mapsto \overline{X}$. (Here we identify $\mathbb{R}^2$ with $\mathbb{C}$ and $\mathfrak{q} = M(2, n; \mathbb{R})$ with $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R}^2) \cong \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{C})$; then the twisted complex structure of $\mathfrak{q}$ is identified with $X \mapsto i \circ X$.) The corresponding fixed point space of $\mathfrak{q}$ is $M(1, n; \mathbb{R})$ with the Jordan triple product

$$T(X, Y, Z) = ZX^tY - XY^tZ - ZY^tX.$$

This proves the statement about the twisted complexification of S^n. The statement about the twisted para-complexification is proved formally in the same way by replacing in all arguments concerning $M(2, n; \mathbb{R})$ the field $\mathbb{C}$ by the algebra of para-complex numbers. (The geometric interpretation via the projective quadric does of course not carry over. It would be interesting to have a good geometric interpretation relating S^n to the space $\mathrm{SO}(n+1, 1)/(\mathrm{SO}(n) \times \mathrm{SO}(1, 1))$.)■

The c-dual space of the sphere and of the projective space $\mathbb{R}\mathbb{P}^n$ is the hyperbolic space $M := H^n(\mathbb{R}) = \mathrm{SO}(n, 1)/\mathrm{SO}(n)$. It has two different complexification diagrams obtained by applying the c-dual functor to the diagrams from the preceding proposition and from Prop. IV.1.2:

$$\begin{array}{ccccc} & \nearrow & S^n_{\mathbb{C}} & \searrow & \\ M & \to & \mathrm{SO}(n,2)/(\mathrm{SO}(n) \times \mathrm{SO}(2)) & \to & \mathrm{SO}(n+2, \mathbb{C})/(\mathrm{SO}(n, \mathbb{C}) \times \mathrm{SO}(2, \mathbb{C})). \\ & \searrow & \mathrm{SO}(n+1,1)/(\mathrm{SO}(n+1) \times \mathrm{SO}(1,1)) & \nearrow & \end{array}$$

$$\begin{array}{ccccc} & \nearrow & \mathrm{O}(n+1, \mathbb{C})/(\mathrm{O}(n, \mathbb{C}) \times \mathrm{O}(1, \mathbb{C})) & \searrow & \\ M & \to & \mathrm{U}(n,1)/(\mathrm{U}(n) \times \mathrm{U}(1)) & \to & \mathrm{Gl}(n+1, \mathbb{C})/(\mathrm{Gl}(n, \mathbb{C}) \times \mathrm{Gl}(1, \mathbb{C})). \\ & \searrow & \mathrm{Gl}(n+1, \mathbb{R})/(\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Gl}(1, \mathbb{R})) & \nearrow & \end{array}$$

The first corresponds to the "conformal" JTS $\mathbb{R}^n$ with $T(x, y, z) = (x|y)z + (z|y)x - (x|z)y$, and the second to the "projective" JTS $M(1, n; \mathbb{R})$ with $T(x, y, z) = xy^tz + zy^tx$.

The *complex sphere* $S^n_{\mathbb{C}}$ has three different local twisted complexifications of which two are global: the first one is obtained by straight complexification of the diagram from Prop. IV.1.5, leading to the global twisted complexification $\mathrm{SO}(n+2, \mathbb{C})/(\mathrm{SO}(n, \mathbb{C}) \times \mathrm{SO}(2, \mathbb{C}))$. The second one is local; it comes from the local isomorphism of S^n and $\mathbb{R}\mathbb{P}^n$ and is obtained by straight complexification of the diagram from Prop. IV.1.2, leading to the local twisted complexification $\mathrm{Gl}(n+1, \mathbb{C})/(\mathrm{Gl}(n, \mathbb{C}) \times \mathrm{Gl}(1, \mathbb{C}))$. The third one is again global; it is obtained by applying formally some arguments from the proof of Prop. IV.1.4 to a "projective quaternionic quadric". One obtains the following complexification diagram:

$$
\begin{array}{ccccc}
 & \nearrow & S^n_{\mathbb{C}} \times S^n_{\mathbb{C}} & \searrow & \\
S^n_{\mathbb{C}} & \to & \mathrm{SO}^*(2n+2)/(\mathrm{SO}^*(2n) \times \mathrm{SO}^*(2)) & \to & ... \\
 & \searrow & \mathrm{SO}(n+1,n+1)/(\mathrm{SO}(n,n) \times \mathrm{SO}(1,1)) & \nearrow &
\end{array}
$$

$$
... \quad \mathrm{SO}(2n+2,\mathbb{C})/(\mathrm{SO}(2n,\mathbb{C}) \times \mathrm{SO}(2,\mathbb{C}))
$$

So far we do not have a good geometric interpretation of this complexification diagram. It would be very interesting to know whether it is related to periodicity phenomena in the theory of Clifford algebras.

1.6. The case $\dim M = 1$.

Lemma IV.1.6. *Up to isomorphy, there exist exactly three one-dimensional real Jordan triple systems. They are given by the following triple products on the underlying vector space* $V = \mathbb{R}$*:*

(1) $T^{(+)}(x,y,z) = 2xyz$,

(2) $T^{(0)}(x,y,z) = 0$,

(3) $T^{(-)}(x,y,z) = -2xyz$.

Proof. If $T(1,1,1) = t \in \mathbb{R}$, then T is necessarily given by $T^{(t)}(x,y,z) = txyz$. But since $x \mapsto sx$ for $0 \neq s \in \mathbb{R}$ is an isomorphism from $T^{(t)}$ onto $T^{(s^2 t)}$, there remain only the three isomorphism classes mentioned in the claim. ∎

The complexification diagrams associated to the three one-dimensional real JTS from the preceding lemma are as follows:

$$
\begin{array}{ccccc}
 & \nearrow & \mathbb{C} & \searrow & \\
\mathbb{R} & \to & D = \mathrm{SO}(2,1)/\mathrm{SO}(2) & \to & S^2_{\mathbb{C}} = \mathrm{SO}(3,\mathbb{C})/\mathrm{SO}(2,\mathbb{C}), \\
 & \searrow & \mathrm{SO}(2,1)/\mathrm{SO}(1,1) & \nearrow &
\end{array}
$$

$$
\begin{array}{ccccc}
 & \nearrow & \mathbb{C} & \searrow & \\
\mathbb{R} & \to & \mathbb{C} & \to & \mathbb{C} \times \mathbb{C}, \\
 & \searrow & \mathbb{R} \times \mathbb{R} & \nearrow &
\end{array}
$$

$$
\begin{array}{ccccc}
 & \nearrow & \mathbb{C}^* & \searrow & \\
S^1 & \to & S^2 = \mathrm{SO}(3)/\mathrm{SO}(2) & \to & S^2_{\mathbb{C}} = \mathrm{SO}(3,\mathbb{C})/\mathrm{SO}(2,\mathbb{C}). \\
 & \searrow & \mathrm{SO}(2,1)/\mathrm{SO}(1,1) & \nearrow &
\end{array}
$$

In fact, the first diagram is the one of $\mathrm{Gl}(1,\mathbb{R})_o$ (and of $H^1(\mathbb{R})$), the second one is trivial, and the third one is the one of $\mathbb{RP}^1$. Formally, it also obtained from the diagram of $\mathrm{O}(2)$ (Diagram (1.8)) and noting that the group $\mathrm{O}(4)$ is not simple, but locally isomorphic to $\mathrm{O}(3) \times \mathrm{O}(3)$:

$$
\begin{array}{ccccc}
 & \nearrow & \mathrm{O}(2,\mathbb{C}) & \searrow & \\
\mathrm{O}(2) & \to & \mathrm{O}(4)/\mathrm{U}(2) & \to & \mathrm{O}(4,\mathbb{C})/\mathrm{Gl}(2,\mathbb{C}). \\
 & \searrow & \mathrm{O}(2,2)/\mathrm{Gl}(2,\mathbb{R}) & \nearrow &
\end{array}
$$

2. Principles of classification

2.1. Classification of simple complex Jordan pairs. These objects can be classified in various ways:

(1) Recall from Section III.3 that Jordan pairs correspond to 3-graded Lie algebras. Complex ones correspond to complex 3-graded Lie algebras, and simple Jordan pairs correspond to simple Lie algebras (cf. Section V.3). The classification of simple complex 3-graded Lie algebras is well-known (cf. [KoNa64], [KanKo85]), see Table XII.3.1 (where these Lie algebras are the Lie algebras of "conformal groups").

(2) A Jordan-theoretic classification of simple complex Jordan pairs can be found in [Lo75, p.196], see Table XII.2.1. It turns out that simple (complex) Jordan pairs are always *underlying* a simple (complex) JTS in the following sense:

Definition IV.2.1. If T is a JTS on a vector space V, then the Jordan pair equivalent to the para-Hermitian complexification $T_{ph\mathbb{C}}$ of T (cf. Prop. III.4.6) is called the *underlying Jordan pair of* (T,V). ■

Recall that according to Prop. III.4.6, the underlying Jordan pair of (T,V) is of the form (V,V) where the maps $T^{\pm}$ are obtained from T by choosing the arguments in the appropriate factor. Note that non-isomorphic JTS may have the same underlying Jordan pair. In other words, a twisted polarized JTS may have several non-isomorphic para-real forms. In general, there is no distinguished one among them. In some cases there exists a para-real form associated to a Jordan *algebra* via the formula $T(x,y,z)=2(x(yz)-y(xz)+(xy)z)$; then this para-real form plays a somewhat distinguished role. This is the case for the first four series and the exceptional pair no. 5 in Table XII.2.1.

2.2. Classification of simple real Jordan pairs. The same remarks as above apply. We have to classify real forms of the complex objects; see Tables XII.2.2 and XII.3.1.

2.3. Para-real forms and structure variety. Next we explain how to obtain the classification of JTS from the classification of Jordan pairs. Let $(\mathfrak{q},T)$ be a real JTS and denote by

$$\mathrm{PR}(T_{ph\mathbb{C}})=\{\nu\in \mathrm{Aut}_{\mathbb{R}}(T_{ph\mathbb{C}})|\,\nu^2=\mathrm{id},\nu J=-J\nu\}$$

the variety of para-real forms of the para-complexification $T_{ph\mathbb{C}}$. Two para-real forms are isomorphic iff they are conjugate under the action of the J-linear automorphism group $\mathrm{Aut}_J(T_{ph\mathbb{C}})$ via $g.\nu=g_*(\nu)=g\circ\nu\circ g^{-1}$.

Lemma IV.2.2. *The action of* $\mathrm{Aut}_J(T_{ph\mathbb{C}})$ *is transitive on each connected component of* $\mathrm{PR}(T_{ph\mathbb{C}})$.

Proof. By composition with the standard para-real form τ, we obtain an isomorphism

$$\mathrm{PR}(T_{ph\mathbb{C}})\to\{g\in\mathrm{Aut}_J(T_{ph\mathbb{C}})|\,\tau g\tau=g^{-1}\},\quad \nu\mapsto\nu\tau$$

onto the subspace of symmetric elements of $\mathrm{Aut}_J(T_{ph\mathbb{C}})$ w.r.t. the involution $g\mapsto\tau g\tau$. The given action on $\mathrm{PR}(T_{ph\mathbb{C}})$ is transferred by this isomorphism to the action of $\mathrm{Aut}_J(T_{ph\mathbb{C}})$ by

$$g\cdot\rho=g\rho\tau g^{-1}\tau.$$

Now Lemma I.4.5, applied to the anti-involution $g^*:=\tau g^{-1}\tau$, implies the claim. ■

The previous lemma leaves us with the task of describing the connected components of the variety $\mathrm{PR}(T_{ph\mathbb{C}})$. We first describe the automorphism group already considered above.

Lemma IV.2.3. *If T is a JTS, then the group of J-linear automorphisms of the polarized JTS $T_{ph\mathbb{C}}$ is given by*

$$\mathrm{Aut}_J(T_{ph\mathbb{C}})=\{(g,h)\in\mathrm{Gl}(W^+)\times\mathrm{Gl}(W^-)|\,\forall x,y,z,u,v,w\in V:$$

$$T(gx, hy, gz) = g\,T(x, y, z), \quad T(hu, gv, hw) = h\,T(u, v, w)\},$$

where the eigenspaces $W^\pm$ *of* J *are identified with* V *as in the proof of* Prop. III.4.6.

Proof. An automorphism of $T_{ph\mathbb{C}}$ commuting with J can be written in the form (g_+, g_-) with $g_\pm \in \mathrm{Gl}(W^\pm)$. Using the definition of $T_{ph\mathbb{C}}$, one verifies by a straightforward calculation that an endomorphism of this form is an automorphism if and only if the condition stated in the lemma holds. ■

Recall from Lemma III.4.5 that the α*-modification* of a JTS T is the JTS

$$T^{(\alpha)}(u, v, w) = T(u, \alpha v, w), \tag{2.1}$$

where α belongs to the *structure variety* $\mathrm{Svar}(T)$, i.e. it satisfies Eqn. (III.4.2).

Proposition IV.2.4. *Let* T *be a JTS. Then*

$$\mathrm{Svar}(T) \to \mathrm{PR}(T_{ph\mathbb{C}}), \quad \alpha \mapsto \tau \circ (\alpha, \alpha^{-1})$$

is a bijection. All JTS having the same underlying Jordan pair as T *are of the form* $T^{(\alpha)}$*, where* α *belongs to the structure variety of* T*.*

Proof. Let τ be the para-conjugation of $T_{ph\mathbb{C}}$ w.r.t. V (cf. Prop. III.4.6 and its proof) and let ν be another para-conjugation of $T_{ph\mathbb{C}}$. We write $\nu = \tau\beta$; then β belongs to the group of J-linear automorphisms of $T_{ph\mathbb{C}}$.

Using the preceding lemma, we write $\beta = (\alpha_+, \alpha_-)$. The condition $\mathrm{id} = \nu^2 = \tau\beta\tau\beta$ then reads $\alpha_+ = \alpha_-^{-1}$ since τ is just exchange of W^+ and W^-. Let $\alpha := \alpha_+$. Now the condition on (α, α^{-1}) expressed by the preceding lemma is precisely the condition (III.4.2) from Lemma III.4.5.

The space fixed under ν is

$$\{(x, \alpha x) \,|\, x \in V\},$$

and

$$T_{ph\mathbb{C}}((x, \alpha x), (y, \alpha y), (z, \alpha z)) = (T(x, \alpha y, z), \alpha T(x, \alpha y, z)).$$

It follows that the corresponding para-real form of $T_{ph\mathbb{C}}$ is $T^{(\alpha)}$. ■

Combining Propositions IV.2.2 and IV.2.4, we see that, in order to determine all para-real forms of $T_{ph\mathbb{C}}$, it is enough to classify the connected components of the structure variety of T. For the case that T belongs to a simple Jordan algebra this has been carried out by B.O. Makarevič (the result is given in [Ma73] without proof); see also [Sch85]. The result is given in Tables XII.2.4 – XII.2.6. It turns out that in most cases it is possible to choose as representative α of a connected component of the structure variety an *involution*, i.e. an automorphism of T of order 2; in any case one can choose an element of order at most 4. Note that α is an involution of T iff so is $-\alpha$; in general, there is no canonical choice between them. The para-real forms $T^{(\alpha)}$ and $T^{(-\alpha)} = -T^{(\alpha)}$ are *c-dual* in the sense that the associated LTS are c-dual (cf. Section III.4.3). In general, $T^{(\alpha)}$ and $-T^{(\alpha)}$ are not isomorphic. (They are isomorphic if $T^{(\alpha)}$ is already a complex or a polarized JTS and in some rather special cases.) In the tables we list just one of these two dual modifications. In case T is a JTS associated to a Jordan *algebra*, then if possible we choose α to be an involutive algebra-automorphism; in this case $-\alpha$ is just a JTS-automorphism, but not an algebra-automorphism.

If T is a *complex* JTS, then we distinguish the cases that α is $\mathbb{C}$-linear (Table XII.2.4) and that α is $\mathbb{C}$-conjugate linear (Table XII.2.5). In the latter case $T^{(\alpha)}$ is a Hermitian JTS, and in the former case $T^{(\alpha)}$ is again a complex JTS.

2.4. Classification: Geometric part. Having classified simple real JTS T, it remains to describe (a) the symmetric spaces $M = G/H$ belonging to R_T and (b) the complexification diagram belonging to T. The result is given in Tables XII.4.2 – XII.4.5. The preceding procedure shows how to obtain $T_{h\mathbb{C}}$ and $T_{ph\mathbb{C}}$ for the JTS T classified above, and thus a solution of task (a) yields a solution of task (b). Now, task (a) is equivalent to describe the *standard imbedding* of R_T. Unfortunately, there is no direct and simple algorithm allowing to calculate the standard imbedding of a given LTS. In our situation, we can use the strategy outlined in Section 1 of this chapter: if a LTS is the LTS fixed under one or two involutions of $\mathfrak{gl}(n,\mathbb{F})$, then the standard imbedding will be the algebra fixed under the induced involution of the standard imbedding of $\mathfrak{gl}(n,\mathbb{F})$. However, calculations may become quite messy.

Later (cf. Prop. X.3.7) we will write the complexification diagram of the space $M^{(\alpha)}$ belonging to the JTS $T^{(\alpha)}$ in the following form:

$$\begin{array}{ccccc} & \nearrow & \mathrm{Co}(T_{\mathbb{C}})^{\Theta\alpha_{\mathbb{C}}} / \mathrm{Str}(T)^{\Theta\alpha_*} & \searrow & \\ \mathrm{Co}(T)^{\Theta\alpha_*} / \mathrm{Str}(T)^{\Theta\alpha_*} & \to & \mathrm{Co}(T_{\mathbb{C}})^{\Theta\alpha_*\tau_*} / \mathrm{Str}(T)^{\Theta\alpha_*\tau_*} & \to & \mathrm{Co}(T_{\mathbb{C}}) / \mathrm{Str}(T_{\mathbb{C}}) \\ & \searrow & \mathrm{Co}(T) / \mathrm{Str}(T) & \nearrow & \end{array}$$

where $\Theta\alpha_*$ is an involution of the *conformal group* $\mathrm{Co}(T)$ of T depending on α, and τ_* is induced by complex conjugation of $T_{\mathbb{C}}$ w.r.t. T. In this realization the relation between $T^{(\alpha)}$ and the corresponding homogeneous space becomes more transparent (cf. Section XI.5).

Notes for Chapter IV.

As already mentioned, the algebraic version of the observations A and B has first been made by E. Neher ([Ne85]).

IV.1. The presentation of examples by means of the complexification diagram given here is an extended version of examples given in [Be97b].

IV.2. The classification principle (Section 2.3) is a Jordan theoretic version of the classification principle from [Ri70] and [Ma73]; see also [Sch85]. The classification of simple real JTS by E. Neher ([Ne80], [Ne81]) follows a different pattern: E. Neher classifies first the complex (equivalently, the compact) JTS and then their straight real forms; this strategy has already been used by K.H. Helwig in his classification of simple real Jordan algebras (cf. Notes to Ch.XII).

Chapter V: Non-degenerate spaces

We distinguish the following three "degrees of non-degeneracy" for Lie groups: *existence of invariant forms; semisimplicity; simplicity*; between the first two levels there is the class of *reductive Lie groups* which we will not consider here. In all categories we have met so far (symmetric spaces; prehomogeneous ones; twisted complex ones; twisted polarized ones; symmetric spaces with twist) the same three degrees of non-degeneracy appear; we thus get 15 geometric categories of "regular" spaces. They correspond to 15 algebraic categories which we describe in this chapter. For symmetric spaces, the three degrees of non-degeneracy are geometrically described as follows (Section 1):

- existence of invariant forms: symmetric spaces with an invariant pseudo-Riemannian metric;
- semisimple level: symmetric spaces with non-degenerate Ricci-tensor;
- simple level: symmetric spaces without proper ideal.

(Again, between the first two levels one could mention also the class of *reductive spaces.*) Semisimplicity has some remarkable consequences: homomorphisms of semisimple symmetric spaces are automatically equivariant (Th. V.1.9), and semisimple prehomogenous symmetric spaces are automatically quadratic (Th. V.4.4). For simple symmetric spaces we prove an important structure theorem on the algebra of invariant $(1,1)$-tensor fields (Th. V.1.10) which among other things implies that invariant complex structures on simple symmetric spaces always are either straight or twisted (Cor. V.1.12).

Inside the category of Lie groups we have the important class of *compact Lie groups.* Since it is closely related to negative definiteness of the Killing-form, we may consider the compact Lie groups as "negative objects" in the category of Lie groups. (Of course, this terminology depends on a sign-convention implicitly given by the definition of the Killing-form.) Similarly, we may say that the *compact symmetric spaces* are the "negative objects" in the category of symmetric spaces. In contrast to the situation for groups, there are also "positive objects" in the category of symmetric spaces; these are the *Riemannian symmetric spaces of non-compact type* which are c-dual to the compact ones. We characterize the "positive" and "negative objects" in the category of symmetric spaces with twist by conditions on the trace form (Section 5). In the category of Lie algebras, there are no positive objects, and in the category of Jordan algebras there are no negative objects. Correspondingly, in the category of prehomogeneous symmetric spaces there are no negative objects, but there are positive ones which play an important role. They are called *symmetric cones.*

1. Pseudo-Riemannian symmetric spaces

1.1. Invariant pseudo-metrics. Recall that a *pseudo-metric* g on a manifold M is a smooth assignment of a non-degenerate symmetric bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ to $p \in M$. Morphisms of manifolds which are compatible with pseudo-metrics are called *isometries*.

Definition V.1.1. If $M = G/H$ is a symmetric space, then a pseudo-metric g on M is called G*-invariant* if G acts by isometries. It is called just *invariant* if it is invariant under $G(M)$. This is equivalent to the condition that $\nabla g = 0$ for the canonical connection ∇. A *pseudo-Riemannian symmetric space* is a symmetric space M together with an invariant pseudo-metric. ■

If g is G-invariant, then H belongs to the orthogonal group of g_o and thus $\mathfrak{h}$ acts skew-symmetrically. Thus we get the following infinitesimal concept:

Definition V.1.2. An *invariant symmetric bilinear form* g_o on a LTS $(\mathfrak{q}, R_o)$ is a symmetric bilinear form g_o on $\mathfrak{q}$ such that

$$\forall u,v,x,y \in \mathfrak{q}: \quad g_o(R_o(u,v)x,y) = -g_o(x, R_o(u,v)y).$$ ■

Conversely, for germs or for simply connected spaces an invariant symmetric bilinear form can be transported to any point, and if it is non-degenerate, it defines thus an invariant pseudo-metric.

Lemma V.1.3. *If g_o is a non-degenerate invariant symmetric bilinear form on a LTS $(\mathfrak{q}, R_o)$, then the relation*

$$\forall u,v,x,y \in \mathfrak{q}: \quad g_o(R_o(u,v)x,y) = g_o(R_o(x,y)u,v)$$

holds.

Proof. Cf. [Lo69a, p. 146] or [Hel78, p. 69]. ■

If g is positive definite, then the tensor field $(U,V,X,Y) \mapsto g(R(U,V)X,Y)$ is known as the *Riemannian curvature tensor*.

1.2. Semisimple symmetric spaces. Recall that the *Ricci-form* of a connection ∇ with curvature R is defined as the tensor field Ric given by

$$\mathrm{Ric}(X,Y) := \mathrm{Tr}(R(X,\cdot)Y).$$

If ∇ is the canonical connection of a symmetric space M, then the Ricci-form is invariant under $G(M)$ since so is R; in particular, $[\mathfrak{q},\mathfrak{q}]$ belongs to the orthogonal Lie algebra of Ric_o, i.e. it acts skew-symmetrically. Note further that $R(X,\cdot)Y = R(Y,\cdot)X - R(Y,X)$; therefore Ric is symmetric if and only if $R(X,Y)$ is trace-free for all $X,Y \in \mathfrak{X}(M)$.

Definition V.1.4. A *semisimple symmetric space* is a symmetric space whose Ricci-form is non-degenerate. A LTS $\mathfrak{q}$ is called *semisimple* if the bilinear form $(X,Y) \mapsto \mathrm{Tr}[X,\cdot,Y]$ is non-degenerate. ■

The main features of the structure theory of Lie algebras may be generalized to Lie triple systems. In particular, there is a notion of a *radical of a LTS*, and it can be proved that a LTS is semisimple iff its radical is reduced to zero (cf. [Li52, Th.2.9]).

Lemma V.1.5. *If M is semisimple, then the Ricci-form is symmetric.*

Proof. By the remarks preceding the definition, $R_o(u,v)$ acts skew-symmetrically w.r.t. the Ricci-form at the origin: $R_o(u,v) \in \mathfrak{o}(\mathrm{Ric}_o)$. If Ric_o is non-degenerate, then it follows that $\mathrm{Tr}\, R_o(u,v) = 0$. Again by the preceding remarks, this is equivalent to the symmetry of the Ricci-form. ∎

1.3. Irreducible symmetric spaces.

Definition V.1.6.

(1) An *ideal* of a LTS $\mathfrak{q}$ is the kernel of some LTS-homomorphism $\varphi : \mathfrak{q} \to \mathfrak{q}'$. A LTS is called *simple* if its dimension is greater than one and it contains no ideals other than zero and the whole space.

(2) A symmetric space and a germ of a symmetric space are called *simple* if the associated LTS is simple. ∎

An example of a simple symmetric space is the sphere S^n. There may exist non-trivial homomorphisms from simple symmetric spaces into other spaces, as shows the example of the double covering $S^n \to \mathbb{RP}^n$ (its kernel is the *center* of S^n, cf. [Lo69a]).

Lemma V.1.7. *A subspace $\mathfrak{a}$ of a LTS $\mathfrak{q}$ is an ideal if and only if $[\mathfrak{q},\mathfrak{a},\mathfrak{q}] \subset \mathfrak{a}$.*

Proof. If $\mathfrak{a}$ is an ideal, then the three relations $[\mathfrak{a},\mathfrak{q},\mathfrak{q}] \subset \mathfrak{a}$, $[\mathfrak{q},\mathfrak{a},\mathfrak{q}] \subset \mathfrak{a}$ and $[\mathfrak{q},\mathfrak{q},\mathfrak{a}] \subset \mathfrak{a}$ hold, and conversely, if they hold, then $\mathfrak{q}/\mathfrak{a}$ has an obvious structure of LTS such that $\mathfrak{a}$ is the kernel of the canonical projection.

We have to show that the second relation is already equivalent to all three of them together. It is clear that it is equivalent to the first. It implies the third because of (LT2). ∎

1.4. Behavior of the standard-imbedding.

Proposition V.1.8. *Let $\mathfrak{q}$ be a LTS and $\mathfrak{g} = [\mathfrak{q},\mathfrak{q}] \oplus \mathfrak{q}$ be its standard-imbedding.*

(i) *The LTS $\mathfrak{q}$ is simple if and only if the symmetric Lie algebra $(\mathfrak{g},\sigma)$ is simple (i.e. $\mathfrak{g}$ has no proper σ-invariant ideals); and this is the case if and only if either $\mathfrak{g}$ is simple or $\mathfrak{q}$ is a simple Lie algebra considered as LTS.*

(ii) *The LTS $\mathfrak{q}$ is semisimple if and only if $\mathfrak{g}$ is a semisimple Lie algebra.*

Proof. (i) [Lo69a, p.141] (Note that the standard imbedding of a simple Lie algebra considered as LTS is isomorphic to a direct product of the algebra with itself.)

(ii) [Lo69a, p.142] (There it is also proved that semisimple LTS are precisely the direct sums of simple ones.) ∎

If $\mathfrak{q}$ is a semisimple LTS, then $\mathfrak{q}$ is equal to its derived LTS $\mathfrak{q}'$, and every derivation is inner. This follows easily from part (ii) of the proposition and the corresponding well-known facts on semisimple Lie algebras. We now prove that in the category of semisimple symmetric spaces homomorphisms are always equivariant maps (cf. Def. I.1.5).

Theorem V.1.9. *Let M be a germ of a semisimple symmetric space which is equivalent to the semisimple LTS $\mathfrak{q}$.*

(1) *If $\varphi : \mathfrak{q} \to \mathfrak{q}'$ is a LTS-homomorphism, then there exists a unique Lie-algebra homomorphism $\widetilde{\varphi} : \mathfrak{g} \to \mathfrak{g}'$ of the respective standard-imbeddings such that $\widetilde{\varphi}|_{\mathfrak{q}} = \varphi$.*

(2) *Every homomorphism $\varphi : M \to M'$ of germs of symmetric spaces is equivariant.*

Proof. (1) Decomposing a semisimple LTS in a direct sum of simple ones (cf. [Lo69a, p.142]), we see that images of semisimple LTS are again semisimple and that

$$\widetilde{\mathfrak{g}} := [\varphi(\mathfrak{q}), \varphi(\mathfrak{q})] \oplus \varphi(\mathfrak{q}) \subset \mathfrak{g}'$$

is a semisimple subalgebra. We let $\mathfrak{h} := [\mathfrak{q}, \mathfrak{q}]$, $\widetilde{\mathfrak{h}} := [\varphi(\mathfrak{q}), \varphi(\mathfrak{q})]$ and define for $H = \sum_i [X_i, Y_i] \in \mathfrak{h}$ ($X_i, Y_i \in \mathfrak{q}$):

$$\widetilde{\varphi}(H) := \sum_i [\varphi(X_i), \varphi(Y_i)] \in \widetilde{\mathfrak{h}}.$$

We have to show that $\widetilde{\varphi}(H)$ is well-defined, i.e. if $H = 0$, then $\widetilde{\varphi}(H) = 0$. Assume that $H = 0$, then for all $Z \in \mathfrak{q}$,

$$[\widetilde{\varphi}(H), \varphi(Z)] = \sum_i [\varphi(X_i), \varphi(Y_i), \varphi(Z)] = \varphi(\sum_i [X_i, Y_i, Z]) = \varphi([H, Z]) = 0.$$

Thus $\widetilde{\varphi}(H)$ lies in the center of $\widetilde{\mathfrak{g}}$. Since $\widetilde{\mathfrak{g}}$ is semisimple, it follows that $\widetilde{\varphi}(H) = 0$. Thus $\widetilde{\varphi}$ is well-defined on $\mathfrak{h}$. For $X \in \mathfrak{q}$ we let $\widetilde{\varphi}(X) := \varphi(X)$ and define thus a linear map $\mathfrak{g} \to \widetilde{\mathfrak{g}} \subset \mathfrak{g}'$. Then it is immediately verified that $\widetilde{\varphi}$ is a Lie algebra-homomorphism.

(2): This follows from (1) (cf. remark after Def. I.1.5). ■

1.5. The algebra $\mathrm{End}(\mathfrak{q})^{\mathfrak{h}}$. For a LTS $\mathfrak{q}$ and $\mathfrak{h} := [\mathfrak{q}, \mathfrak{q}] \subset \mathrm{Der}(\mathfrak{q})$, we denote by

$$\mathrm{End}(\mathfrak{q})^{\mathfrak{h}} = \{X \in \mathrm{End}(\mathfrak{q}) | \forall Y \in \mathfrak{h} : [Y, X] = 0\}$$

the algebra of $\mathfrak{h}$-invariants in $\mathrm{End}(\mathfrak{q})$. If $\mathfrak{q}$ is semisimple, we denote by $\mathrm{End}(\mathfrak{q}) = \mathrm{Sym}(\mathfrak{q}) \oplus \mathrm{Asym}(\mathfrak{q})$ the decomposition of $\mathrm{End}(\mathfrak{q})$ into spaces of symmetric and skew-symmetric operators w.r.t. the Ricci-form. Since the Ricci-form is $\mathfrak{h}$-invariant, the projectors onto $\mathrm{Sym}(\mathfrak{q})$ resp. $\mathrm{Asym}(\mathfrak{q})$ are $\mathfrak{h}$-equivariant, and therefore we have a decomposition

$$\mathrm{End}(\mathfrak{q})^{\mathfrak{h}} = \mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}} \oplus \mathrm{Asym}(\mathfrak{q})^{\mathfrak{h}}. \tag{1.1}$$

Theorem V.1.10. *If $\mathfrak{q}$ is a simple LTS, then precisely the following five cases can arise:*

	$\mathrm{End}(\mathfrak{q})^{\mathfrak{h}}$	$\mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}}$	$\mathrm{Asym}(\mathfrak{q})^{\mathfrak{h}}$
1. generic	$\mathbb{R}$	$\mathbb{R}$	0
2. straight complex generic	$\mathbb{C}$	$\mathbb{C}$	0
3. twisted complex generic	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$
4. twisted polarized generic	$\mathbb{R} \times \mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
5. all structures exist	$\mathbb{C} \times \mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

In this table, the isomorphy class of $\mathrm{End}(\mathfrak{q})^{\mathfrak{h}}$ *as an associative algebra is given, and* $\mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}}$ *and* $\mathrm{Asym}(\mathfrak{q})^{\mathfrak{h}}$ *are described as vector spaces (and as sub-JTS of the JTS* $\mathrm{End}(\mathfrak{q})^{\mathfrak{h}}$*). By "generic" we mean that the LTS $\mathfrak{q}$ has no additional straight or twisted structure besides the one mentioned.*

Proof. Step 1. We denote by $g(u, v) = \mathrm{Tr}([u, \cdot, v])$ the Ricci-form of $\mathfrak{q}$. Let $X \in \mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}}$. Using twice Lemma V.1.3, we have for all $a, b, u, v \in \mathfrak{q}$,

$$g(R(Xa, b)u, v) = g(R(u, v)Xa, b) = g(R(u, v)a, Xb) = g(R(a, Xb)u, v),$$

and therefore $R(Xa, b) = R(a, Xb)$ since g is non-degenerate. From Lemma III.1.7 we get

$$R(Xa, b) = R(a, Xb) = X\, R(a, b), \tag{1.2}$$

i.e. the Lie triple product on $\mathfrak{q}$ is X-trilinear. Now we complexify the whole set-up, thus assuring the existence of eigenvalues of X. From the complexified version of condition (1.2) it follows immediately that the eigenspaces of X are ideals of $\mathfrak{q}_{\mathbb{C}}$. Now we have to distinguish two cases:

(a) The LTS $\mathfrak{q}_{\mathbb{C}}$ is simple. Then X has just one eigenvalue $t \in \mathbb{C}$, and the corresponding eigenspace must be all of $\mathfrak{q}$, i.e. $X = t\,\mathrm{id}_{\mathfrak{q}_{\mathbb{C}}}$. It follows that t must have been already real. We have shown that in case (a) $\mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}} = \mathbb{R}\,\mathrm{id}_{\mathfrak{q}}$.

(b) The LTS $\mathfrak{q}_{\mathbb{C}}$ is not simple. As in the case of Lie algebras, this means that $\mathfrak{q}$ is already a simple complex LTS, and $\mathfrak{q}_{\mathbb{C}} \cong \mathfrak{q} \oplus \mathfrak{q}$. Then the arguments from case (a) show that $\mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}} = \mathbb{C}\,\mathrm{id}_{\mathfrak{q}}$.

Step 2. If $\mathrm{Asym}(\mathfrak{q})^{\mathfrak{h}} = 0$, then $\mathrm{End}(\mathfrak{q})^{\mathfrak{h}} = \mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}}$, and we are in cases 1. or 2. of the table. In the first case there exists no invariant almost (para-)complex structure, and in the second case there exist precisely two invariant almost complex structures which are straight.

We now assume that $\mathrm{Asym}(\mathfrak{q})^{\mathfrak{h}} \neq 0$ and let $0 \neq X \in \mathrm{Asym}(\mathfrak{q})^{\mathfrak{h}}$. The arguments used in the beginning of Step 1 show that for all $u, v \in \mathfrak{q}$,

$$R(Xu, v) = -R(u, Xv). \tag{1.3}$$

It follows that $R(Xu,v)w + R(u,Xv)w + R(u,v)Xw = XR(u,v)w$, i.e. X is a derivation of $\mathfrak{q}$. Since $\mathfrak{q}$ is simple, all derivations are inner, i.e. $\mathfrak{h} = \mathrm{Der}(\mathfrak{q})$ (cf. remark after Prop. V.1.8), and thus $X \in \mathfrak{h}$. By assumption, X commutes with $\mathfrak{h}$, therefore $X \in \mathfrak{z}(\mathfrak{h})$. Decomposing $\mathfrak{q}$ into a direct sum $\mathfrak{q} = \oplus_i \mathfrak{q}_i$ of irreducible $\mathfrak{h}$-modules, we see that the restriction of X to $\mathfrak{q}_i$ is either zero or bijective. Then the same is true for the element X^2 of $\mathrm{Sym}(\mathfrak{q})$. But according to Step 1, $X^2 \in \mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}}$ operates as a (real or complex) scalar $t\,\mathrm{id}_{\mathfrak{q}}$ on $\mathfrak{q}$; it follows that either $X = 0$ or $X^2 = t\,\mathrm{id}_{\mathfrak{q}} \neq 0$. The former case being excluded, we conclude that X is bijective, and since it commutes with all elements of $\mathrm{End}(\mathfrak{q})^{\mathfrak{h}}$, the map

$$\mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}} \to \mathrm{Asym}(\mathfrak{q})^{\mathfrak{h}}, \quad Y \mapsto XY = YX$$

is a bijection. Therefore only the cases that both are isomorphic to $\mathbb{R}$ or that both are isomorphic to $\mathbb{C}$ can appear. If both are isomorphic to $\mathbb{R}$, we can rescale X such that $X^2 = \mathrm{id}_{\mathfrak{q}}$ or $X^2 = -\mathrm{id}_{\mathfrak{q}}$. If $X^2 = \mathrm{id}$, then Eqn. (1.3) shows that it is an invariant twisted polarization, and we are in case 4 of the claim. If $X^2 = -\mathrm{id}_{\mathfrak{q}}$, then we are in case 3 of the claim. Finally, if both $\mathrm{Sym}(\mathfrak{q})^{\mathfrak{h}}$ and $\mathrm{Asym}(\mathfrak{q})^{\mathfrak{h}}$ are isomorphic to $\mathbb{C}$, then we can rescale X by a complex scalar such that $X^2 = \mathrm{id}_{\mathfrak{q}}$, and thus X is an invariant twisted polarization on a straight complex LTS, and we are in case 5 of the claim. ■

Corollary V.1.11. *If $\mathfrak{q}$ is a simple LTS, then either $\mathfrak{q}$ is a simple $\mathfrak{h}$-module or it is the direct sum of two irreducible $\mathfrak{h}$-modules which are dual to each other. The latter case arises if and only if $\mathfrak{q}$ is twisted polarized, and then the irreducible $\mathfrak{h}$-modules are the eigenspaces of the invariant polarization.*

Proof. By the converse of Schur's lemma, $\mathfrak{q}$ is irreducible if $\mathrm{End}(\mathfrak{q})^{\mathfrak{h}}$ is a field. This happens precisely in cases $1-3$ of the preceding theorem, and in the other cases the determination of $\mathrm{End}(\mathfrak{q})^{\mathfrak{h}}$ shows that there are precisely two irreducible submodules. They are dual w.r.t. the Ricci-form: in fact, the eigenspaces of the invariant polarization are maximal isotropic complementary subspaces w.r.t. the Ricci-form, and therefore the Ricci-form sets up a duality between them. ■

Corollary V.1.12. *If $\mathfrak{q}$ is a simple LTS, then an invariant complex structure on $\mathfrak{q}$ is either straight or twisted.*

Proof. In cases 1 and 4 of Th. V.1.10, there are no invariant almost complex structures; in cases 2 and 3 there are precisely two, and they are straight in case 2 and twisted in case 3; in case 5 there are four invariant almost complex structures, and two of them are straight and two are twisted. ■

Corollary V.1.13. *If $\mathfrak{q}$ is a simple LTS, then the representation $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{q})$ is never quaternionic.*

Proof. The claim means just that $\mathrm{End}(\mathfrak{q})^{\mathfrak{h}}$ is under the given assumptions never isomorphic to $\mathbb{H}$. This follows immediately from the table in Th. V.1.10. ∎

Corollary V.1.14. *Let M be an irreducible symmetric space.*

(i) *If M is not straight complex, the Ricci-form is (up to a real scalar) the only invariant pseudo-Riemannian metric on M. If M is straight complex, then all invariant pseudo-Riemannian metrics on M are of the form $(X,Y) \mapsto tg(e^{\varphi \mathcal{J}}X,Y)$ with $t,\varphi \in \mathbb{R}$, where g is the Ricci-form and $\mathcal{J}$ the straight invariant almost complex structure.*

(ii) *M admits an invariant symplectic form ω if and only if it is twisted complex or twisted polarized. This form satisfies $\mathrm{d}\omega = 0$ (i.e. M is pseudo-Kähler, resp. para-Kähler).*

Proof. The Ricci-form, being non-degenerate, sets up an $\mathfrak{h}$-equivariant bijection

$$b : \mathrm{End}(\mathfrak{q}) \to \mathrm{Bil}(\mathfrak{q}) := (\mathfrak{q} \otimes \mathfrak{q})^*, \quad X \mapsto \mathrm{Ric}_o(X\cdot, \cdot)$$

such that $\mathrm{Sym}(\mathfrak{q})$ corresponds to symmetric and $\mathrm{Asym}(\mathfrak{q})$ to skewsymmetric bilinear forms on $\mathfrak{q}$. Moreover $b(X)$ is non-degenerate iff X is non-singular. Under this correspondence, part (i) and the first claim of (ii) follow directly from Th. V.1.10. For the last claim, recall (cf. e.g. [Gam91, p. 74]), that for any p-form ω, the exterior differential $\mathrm{d}\omega$ is given by anti-symmetrizing $\nabla\omega$. We apply this to the invariant 2-form ω equivalent to $\mathrm{Ric}(\mathcal{J}\cdot, \cdot)$ where $\mathcal{J}$ is twisted. Then, by invariance, $\nabla\omega = 0$, and therefore the three-form $\mathrm{d}\omega$ obtained by antisymmetrizing this tensor field also vanishes. ∎

2. Pseudo-Hermitian and para-Hermitian symmetric spaces

Definition V.2.1. A *pseudo-Hermitian symmetric space* $(M, \mathcal{J}, g)$ is a symmetric space $M = G/H$ with an invariant pseudo-metric tensor field g and an invariant almost complex structure $\mathcal{J}$ such that $g(\mathcal{J}X,Y) = -g(X,\mathcal{J}Y)$ for all vector fields X,Y on M. A *para-Hermitian symmetric space* is defined similarly, with $\mathcal{J}^2 = -\mathrm{id}$ replaced by $\mathcal{J}^2 = \mathrm{id}$. ∎

Proposition V.2.2.

(1) *A pseudo-Hermitian symmetric space is twisted complex, and a para-Hermitian symmetric space is twisted para-complex.*

(2) *A semisimple twisted complex symmetric space is pseudo-Hermitian symmetric, and a semisimple twisted involutive symmetric space is para-Hermitian.*

Proof. In the following $\mathcal{J}$ denotes an invariant twisted almost complex structure or polarization.

(1) By the same argument as in Step 1 of the proof of Th.V.1.10 we get

$$g(R(\mathcal{J}X,Y)V,W) = g(R(V,W)\mathcal{J}X,Y) = -g(R(V,W)X,\mathcal{J}Y) = -g(R(X,\mathcal{J}Y)V,W),$$

and therefore $R(\mathcal{J}X,Y) = -R(X,\mathcal{J}Y)$ since g is non-degenerate.

(2) If M is semisimple, the Ricci-form $\mathrm{Ric}(X,Y) = \mathrm{tr}\, R(X,\cdot,Y)$ is non-degenerate, and since $\mathcal{J}$ is twisted,

$$\mathrm{Ric}(\mathcal{J}X,Y) = \mathrm{tr}\, R(\mathcal{J}X,\cdot,Y) = -\mathrm{tr}(R(X,\cdot,Y) \circ \mathcal{J}) = -\mathrm{tr}\, R(X,\cdot,\mathcal{J}Y) = -\mathrm{Ric}(X,\mathcal{J}Y). \quad ∎$$

Definition V.2.3.

(1) A form $(\cdot \mid \cdot)$ is called *invariant* w.r.t. a JTS T if

$$(T(x,y)u \mid w) = (u \mid T(y,x)w)$$

holds for all $x, y, u, w \in V$. *Homomorphisms of JTS with an invariant form* are JTS-homomorphisms which are isometries.

(2) A real JTS (T, V) is called *non-degenerate* if the *trace form*

$$b : V \times V \to \mathbb{R}, \quad (u,v) \mapsto \operatorname{tr} T(x,y)$$

is non-degenerate.

(3) *Ideals* in Jordan triple systems are kernels of homomorphisms, and a JTS T is called *simple* if $T \neq 0$ and it contains no ideals other than 0 and the space itself. *(Hermitian) ideals* in a Hermitian JTS (T, J) are J-invariant ideals of T, and T is called *simple (in the category of Hermitian JTS)* if it contains no Hermitian ideal other than 0 and the space itself. ■

Similarly as for Lie algebras and Lie triple systems, non-degeneracy is equivalent to *semi-simplicity* which means that a suitably defined radical vanishes or in turn that one has a decomposition into simple ideals (cf. [Lo75], [Lo77]).

Lemma V.2.4. *The trace form of a JTS (T, V) is an invariant bilinear form. If it is non-degenerate, then it is symmetric.*

Proof. From the defining identity (JT2) of a JTS we get

$$\begin{aligned} b(T(X,Y)U, W) - b(U, T(Y,X)W) &= \operatorname{tr}(T(T(X,Y)U, W) - T(U, T(Y,X)W)) \\ &= \operatorname{tr}[T(X,Y), T(U,W)] = 0. \end{aligned}$$

Thus the trace form is invariant. This implies that $R(X,Y) = T(X,Y) - T(Y,X)$ is skew-symmetric for b, i.e. it belongs to the orthogonal Lie algebra $\mathfrak{o}(b)$. Therefore, if b is non-degenerate, $\operatorname{tr} R(X,Y) = 0$, and since

$$b(X,Y) - b(Y,X) = \operatorname{tr}(T(X,Y) - T(Y,X)) = \operatorname{tr} R(X,Y),$$

it follows that then b is symmetric. ■

Theorem V.2.5.

(1) *The category of germs of pseudo-Hermitian symmetric spaces is equivalent to the category of Hermitian JTS with non-degenerate invariant symmetric bilinear form for which J is orthogonal.*

(2) *The category of germs of semisimple twisted complex symmetric spaces is equivalent to the category of non-degenerate Hermitian JTS.*

(3) *Germs of simple twisted complex symmetric spaces correspond bijectively to simple Hermitian JTS.*

Similar statements hold with "Hermitian" replaced by "para-Hermitian".

Proof. (1) If $(M, \mathcal{J}, g)$ is a pseudo-Hermitian symmetric space, then by definition $\mathcal{J}$ is orthogonal for g, and

$$\begin{aligned} g(T(X,Y)U, V) &= \frac{1}{2}(g(R(X,Y)U, V) + g(\mathcal{J} R(X, \mathcal{J} Y)U, V)) \\ &= \frac{1}{2}(-g(U, R(X,Y)V) + g(U, \mathcal{J} R(X, \mathcal{J} Y)V) \\ &= g(U, T(Y,X)V), \end{aligned}$$

thus g is invariant for T.

Conversely, if $(\cdot \mid \cdot)$ is an invariant form on V for which J is orthogonal, then we can reverse the calculation: $R(X,Y)$ is antisymmetric for $(\cdot \mid \cdot)$, and thus $(\cdot \mid \cdot)$ gives rise to a Hermitian pseudo-metric.

(2) Using that J is twisted, we get

$$\begin{aligned} T(X,Y) = T(\cdot,Y,X) &= -\frac{1}{2}(R(\cdot,Y,X) - J \circ R(\cdot, J^{-1}Y, X)) \\ &= \frac{1}{2}(R(Y,\cdot,X) + J \circ R(Y,\cdot,X) \circ J^{-1}), \end{aligned}$$

and therefore

$$b(X,Y) = \operatorname{tr} R(Y,\cdot,X) = \operatorname{Ric}(Y,X).$$

Thus under the equivalence of categories from Prop. II.2.7, the respective trace forms on the Jordan- und Lie-side correspond to each other, and the claim follows.

(3) Clearly under the equivalence of categories from Prop. II.2.7, ideals on both sides correspond to each other, and the claim follows. ∎

Note that the observation from part (2) of the proof does not hold for general JTS. For example, the one-dimensional JTS $\mathbb{R}$ with $T(x,y,z) = 2xyz$ clearly is non-degenerate, but the corresponding LTS R_T on $\mathbb{R}$ is flat and has therefore vanishing Ricci-form.

Proposition V.2.6. *Every semisimple pseudo-Hermitian symmetric space has a real form.*

Proof. By the general theory of Cartan involutions, the (straight) complexification of a semisimple symmetric space M has a non-compact Riemannian real form M^r. If M is pseudo-Hermitian, then M^r is Hermitian and has, according to [Sa80, Lemma 8.4], a real form. The restriction of the complexification of this real form to M is a real form of M. ∎

3. Pseudo-Riemannian symmetric spaces with twist

If $(M, \mathcal{J}, g)$ is a pseudo-Hermitian symmetric space, it is understood that a *real form* is a complex conjugation of $(M, \mathcal{J})$ which is an isometry for g.

Theorem V.3.1. *The category of real forms of germs of pseudo-Hermitian symmetric spaces is equivalent to the category of Jordan triple systems with non-degenerate invariant symmetric form.*

Proof. From Th. V.2.5 (1) it is clear that the JTS of a real form of a pseudo-Hermitian symmetric space has a non-degenerate invariant symmetric bilinear form. Conversely, if (T,B) is such a JTS, then $T_{h\mathbb{C}}$ together with the Hermitian extension of B given by

$$B_{h\mathbb{C}}(x + Jy, u + Jv) := B(x,u) + B(y,v) + i(B(y,u) - iB(x,v))$$

is a Hermitian JTS with a form having the properties of Th. V.2.5 (1). ∎

Theorem V.3.2. *The category of real forms of germs of semisimple pseudo-Hermitian symmetric spaces is equivalent to the category of non-degenerate Jordan triple systems.*

Proof. We have to prove the following infinitesimal version of the claim: *A JTS T is non-degenerate if and only if its Hermitian complexification $T_{h\mathbb{C}}$ is non-degenerate.*

From the fact that $T_{h\mathbb{C}}$ is Hermitian on the vector space $\mathfrak{q} \oplus J\mathfrak{q}$, it follows that for all $u, v \in \mathfrak{q}$

$$T_{h\mathbb{C}}(u, Jv) \cdot \mathfrak{q} \subset J\mathfrak{q}, \quad T_{h\mathbb{C}}(u, Jv) \cdot J\mathfrak{q} \subset \mathfrak{q};$$

thus $T_{h\mathbb{C}}(u, Jv)$ is given by a matrix of the form $\left(\begin{smallmatrix} 0 & -A \\ A & 0 \end{smallmatrix}\right)$ and hence $\operatorname{tr} T_{h\mathbb{C}}(u, Jv) = 0$. On the other hand, $T_{h\mathbb{C}}(u, v) = T_{h\mathbb{C}}(Ju, Jv)$ is given by a matrix of the form $\left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right)$, and thus $\operatorname{tr} T_{h\mathbb{C}}(u, v) = \operatorname{tr} T_{h\mathbb{C}}(Ju, Jv) = 2 \operatorname{tr} T(u, v)$. These two facts together imply that the trace form of $T_{ph\mathbb{C}}$ is non-degenerate if and only if the trace form of T is non-degenerate. ■

Theorem V.3.3. *The category of germs of symmetric spaces with twist which are irreducible or have an irreducible twisted complexification is equivalent to the category of simple Jordan triple systems.*

Proof. We prove the following infinitesimal version which clearly implies the claim: *If T is a real JTS, the following are equivalent:*

(1) *T is a simple JTS.*

(2) *R_T or $R_{T_{h\mathbb{C}}}$ is a simple LTS.*

(3) *Either R_T is a simple twisted complex JTS or $R_{T_{h\mathbb{C}}}$ is a simple LTS.*

It is clear that (3) implies (2). In order to prove that (2) implies (1), note that an ideal of T is also an ideal of R_T; therefore T is simple if R_T is simple. If $R_{T_{h\mathbb{C}}}$ is simple, then $T_{h\mathbb{C}}$ is simple, and this implies that T is simple.

Before proving that (1) implies (3), let us remark that a *Hermitian* JTS T is simple if and only if R_T is simple. This is an immediate consequence of the equivalence of categories Prop. II.2.7 and the fact that J is in the simple case always an *inner* derivation.

Now let T be a simple JTS and assume that $R_{T_{h\mathbb{C}}}$ is not simple; then by the preceding remark $T_{h\mathbb{C}}$ is not simple. We have to show that then R_T is simple. By assumption there exists a proper $T_{h\mathbb{C}}$-ideal $W \subset V_{\mathbb{C}}$. Using that (V, T) is simple, one deduces that $(V_{\mathbb{C}}, T_{h\mathbb{C}})$ decomposes as a direct sum $W \oplus \tau(W)$ (where τ is the conjugation w.r.t. V). Then T must have been already a Hermitian JTS, isomorphic to W. Again by the above remark, for such systems simplicity of T and R_T is equivalent; thus R_T is simple. ■

The fact that $M_{h\mathbb{C}}$ is simple does *not* imply that M is simple: for example, the twisted complexification of $\mathrm{Gl}(n, \mathbb{R})$ (Prop. IV.1.1) is a simple symmetric space, although $\mathrm{Gl}(n, \mathbb{R})$ has a one-dimensional center. For the sake of completeness we quote the following result due to E. Neher ([Ne85, Th.1.11]) which classifies the cases where T is simple, but R_T is not simple:

Theorem V.3.4. *If T is a simple real JTS, then either R_T is simple or*

(1) *R_T is the direct sum of a simple ideal and a one-dimensional center; this case appears precisely if T is associated to a simple Jordan algebra via $T(x, y, z) = 2(x(yz) - y(xz) + (xy)z)$; or*

(2) *R_T is the direct sum of two simple ideals; this case appears precisely in the cases of a JTS associated to a quadratic form and a non-trivial reflection* (Ch. XII, Tables XII.4.4 type 4.a and XII.4.5 type 4.1.a). ■

4. Semisimple Jordan algebras

Throughout this section, (V, G, σ, e) denotes a prehomogeneous symmetric space with open symmetric orbit $\Omega = G \cdot e \subset V$ and associated Lie triple algebra $A(x \otimes y) = L(x)y = xy$ (cf. Section II.1). One may be tempted to call such a space *pseudo-Riemannian*, *semisimple* or *simple* whenever Ω has the corresponding properties. However, this is misleading for we have seen (Section II.1) that Ω is always a *cone*, and correspondingly $\mathfrak{q} = L(V)$ has always a non-trivial center containg $\mathbb{R}L(e) = \mathbb{R}\,\mathrm{id}_V$; thus there would be no simple objects. Therefore we start with the corresponding notions on the algebraic level which are at a first glance more canonical.

4.1. Semisimple Lie triple algebras.

Definition V.4.1. Let V be a vector space with a commutative algebra structure $A : V \otimes V \to V$ and let $xy := L(x)y := A(x \otimes y)$.

(1) A symmetric bilinear form b on V is called *associative* if for all $x, y, z \in V$,

$$b(xy, z) = b(y, xz).$$

(2) The algebra (V, A) is called *semisimple* if the *trace form of* A

$$b_A(x, y) := \mathrm{tr}\, L(xy)$$

is non-degenerate.

(3) An *ideal* of (V, A) is an $L(V)$-invariant subspace. The algebra (V, A) is called *simple* if V has no proper ideals. ■

Lemma V.4.2. *If (V, A) is a Lie triple algebra, then the trace form is associative.*

Proof. Using the defining identity $L(z(yx) - y(zx)) = [[L(y), L(z)], L(x)]$ of a Lie triple algebra, we get

$$b(xy, z) - b(y, xz) = \mathrm{tr}(L((xy)z - y(xz))) = \mathrm{tr}([[L(y), L(z)], L(x)]) = 0. \quad ■$$

If b is associative, then $L(V)$ acts by symmetric operators and thus $[L(V), L(V)]$ acts by skewsymmetric operators w.r.t. b. In other words, the linear map

$$\widetilde{b} : V \to W^*, \quad v \mapsto b(v, \cdot)$$

is equivariant w.r.t. the algebra $\mathfrak{g} = L(V) \oplus [L(V), L(V)]$, where W is the vector space V with the action of G by $g \cdot w = \sigma(g)(w)$ and the $\mathfrak{g}$-action $X \cdot w = \dot{\sigma}(X)(w)$.

Lemma V.4.3. *A simple Lie triple algebra is semisimple.*

Proof. Note that ideals in V are just $\mathfrak{g}$-submodules of V; thus V is simple iff it is an irreducible $\mathfrak{g}$-module. By the preceding lemma, the map $\widetilde{b}_A : V \to W^*$ is $\mathfrak{g}$-equivariant; hence it is either zero or injective. It is not zero since $b(e, e) = \mathrm{tr}\, L(e) = \dim V$. Therefore it is injective, and b_A is non-degenerate. ■

Note that, in contrast to the situation for Lie algebras, the one-dimensional unital algebra is semisimple.

Theorem V.4.4. *A semisimple Lie triple algebra is a Jordan algebra.*

Proof. Let b be a non-degenerate associative symmetric bilinear form on the Lie triple algebra V with product $xy = L(x)y$. For $u, v, w \in V$ we let

$$J(u, v, w) := [L(u), L(vw)] + [L(w), L(uv)] + [L(v), L(uw)]$$

and prove that $J(u, v, w) = 0$; for $u = v = w = x$ this is the Jordan identity (J2). Using the associativity of b, we have for all $x, y \in V$,

$$b([L(u), L(vw)]x, y) = b(u(x(vw)) - (ux)(vw), y) = b(vw, [L(x), L(y)]u). \tag{4.1}$$

Now we use that $[L(x), L(y)]$ is skew-symmetric w.r.t. b and then that it is a derivation of A and obtain

$$b(vw, [L(x), L(y)]u) = -b([L(x), L(y)]v, wu) - b([L(x), L(y)]w, vu). \tag{4.2}$$

Putting (4.1) and (4.2) together and applying again (4.1) to the right-hand side of the equation thus obtained we get

$$b([L(u), L(vw)]x, y) = -b([L(v), L(wu)]x, y) - b([L(w), L(vu)]x, y),$$

whence $b(J(u, v, w)x, y) = 0$ for all $x, y \in V$ and thus $J(u, v, w) = 0$ by non-degeneracy. ∎

Putting the preceding three statements together, it follows that a simple Lie triple algebra is the same as a simple Jordan algebra. It is easy to show that a *semisimple Jordan algebra* (i.e. a Jordan algebra with non-degenerate trace form) can in a unique way be decomposed into simple ideals (cf. [FK94, Cor. VIII.2.2]).

4.2. Completely reducible prehomogenous symmetric spaces.

Definition V.4.5. A prehomogenous symmetric space (G, σ, V, e) is called *irreducible* if V is an irreducible G-module, and it is called *completely reducible* if V is a completely reducible G-module. ∎

Theorem V.4.6. *Let (G, σ, V, e) be a prehomogeneous symmetric space with associated algebra (V, A_e). Then the following are equivalent:*

(i) *(G, σ, V, e) is completely reducible.*

(ii) *A_e is a semisimple Jordan algebra.*

(iii) *There is a symmetric non-degenerate bilinear form b such that for all $g \in G$, $\sigma(g) = (g^*)^{-1}$ is the inverse of the adjoint w.r.t. b.*

If these properties are verified, then (G, σ, V, e) is quadratic. Furthermore, irreducible spaces correspond to simple Jordan algebras.

Proof. (i) $\Leftrightarrow$ (ii): As we have remarked in the proof of Lemma V.4.2, irreducible $\mathfrak{g}$-modules correspond to simple Lie triple algebras, and these are the same as simple Jordan algebras (consequence of Th. V.4.4). Thus irreducible prehomogeneous symmetric spaces correspond to simple Jordan algebras. By taking direct sums we obtain the claim.

(iii) $\Rightarrow$ (ii): Taking $g = \exp(tX)$ with $X \in L(V)$ and deriving, we see that X is self-adjoint w.r.t. b, i.e. b is associative. Now (ii) follows from Th. V.4.4.

(ii) $\Rightarrow$ (iii): We choose b to be the trace form. Note that $\mathrm{tr}(L(uv)) = \mathrm{tr}(T(u, v))$, where $T(u, v) = [L(u), L(v)] + L(uv)$. Thus by equivariance of T,

$$b(\sigma(g)u, v) = \mathrm{tr}(T(\sigma(g)u, v)) = \mathrm{tr}(\sigma(g)T(u, g^{-1}v)\sigma(g)^{-1}) = b(u, g^{-1}v).$$

Finally, from (ii) it follows by Th. II.2.6 that (G, σ, V, e) is quadratic. The last claim has already been proved above. ∎

4.3. Geometric interpretation of associativity. The preceding results show that the crucial property of the trace form b is its associativity. This property implies that b is invariant under the stabilizer H of the base point e, und thus

$$b_{g.e}(u, v) := b(gu, gv)$$

defines a G-invariant pseudo-metric on the open orbit Ω. However, H-invariance does not imply associativity: for example, in the irreducible case associative forms are essentially unique, whereas changing such a form by an arbitrary choice for the value $b(e,e)$ still defines an H-invariant form. In the following we describe some properties which distinguish associative forms from H-invariant ones.

1. Clearly b is associative if and only if the operator $P(x) = 2L(x)^2 - L(x^2)$ is for all x self-adjoint. This implies that the formula
$$b_x(u,v) := b(P(x)^{-1}u, v)$$
defines an invariant pseudo-metric not only on Ω, but on the whole dense open set V' (cf. Section II.2.2). Composing with the Jordan inverse, we see that
$$\Omega \to \mathrm{Bil}(V) = S^2(V^*), \quad x \mapsto b_{s_e(x)}$$
extends to a quadratic polynomial $V \to S^2(V^*)$. Identifying $(V \otimes V)^*$ with $\mathrm{End}(V)$ via b_e, this polynomial is equivalent to P itself.
2. A *relative invariant* of a prehomogenous symmetric space is a rational function f such that for all $g \in G$ there is a number $\chi(g)$ such that for all $x \in V$,
$$f(g^{-1}x) = \chi(g) f(x).$$
We say that a prehomogenous symmetric is *log-regular* if there is a relative invariant f and a point $z \in \Omega$ such that $D^2(\log f)(z)$ is a non-degenerate bilinear form. Then (if the prehomogenous space is symmetric) $D^2(\log f)(z)$ is associative ([Be94, Lemma 4.6.8]) and corresponds thus to a semisimple Jordan algebra. Conversely, every Jordan algebra is log-regular, the relative invariant being given by $f(x) = \mathrm{Det}\, P(x)$.
3. Finally, if we assume from the beginning that "regular" prehomogenous symmetric spaces should be quadratic, we may consider them as special cases of the general theory of Jordan triple systems. In fact, then
$$T(x,y,z) = 2(x(yz) - y(xz) + (xy)z)$$
is a JTS which extends the LTS $L(V)$. Thus by our general theory, Ω is a symmetric space with twist and admits a Hermitian complexification (which we will later realize as a generalized tube domain, cf. Chapter XI). Note that if a form b is associative for A_e, then it is invariant for T:
$$\begin{aligned} b(T(u,v)x,y) &= b(([L(u),L(v)] + L(uv))x, y) \\ &= b(x, ([L(v),L(u)] + L(vu))y) = b(x, T(v,u)y), \end{aligned}$$
and vice versa since $L(v) = T(v,e) = T(e,v)$. Furthermore, the trace forms of A_e and T are the same since
$$\mathrm{tr}(T(u,v)) = \mathrm{tr}(L(uv)).$$
Summing up, we get

Proposition V.4.6. *Let (G,σ,V,e) be a quadratic prehomogenous symmetric space with open orbit Ω and associated Jordan algebra A_e and JTS T. Then the following are equivalent:*

(i) *The Hermitian complexification of Ω is a pseudo-Hermitian symmetric space.*

(ii) *The Hermitian complexification of Ω is semisimple.*

(iii) *T admits an invariant symmetric non-degenerate bilinear form.*

(iv) *T is non-degenerate.*

(v) *A_e is a semisimple Jordan algebra.*

Proof. By Section V.3, (i) and (iii) are equivalent, as well as (ii) and (iv). Since the trace forms of A_e and T are the same, (v) and (iv) are equivalent. Finally, we have remarked above that (iii) is equivalent to saying that A_e admits an associative non-degenerate symmetric bilinear form. But then A_e can be decomposed into simple ideals and is semisimple, and (v) holds. ∎

Compared to the situation of general Jordan triple systems, the equivalence of (iii) and (iv) is new. This is due to the existence of the unit element in A_e which excludes cases like $T=0$ satisfying (iii) but not (iv).

5. Compact spaces and duality

5.1. Riemannian symmetric spaces. A symmetric space $M = G/H$ is called *Riemannian* if it has a G-invariant Riemannian metric tensor. Their theory is by now classical (cf. [Hel78]). Let us recall without proof some basic facts. Each Riemannian symmetric space can (locally) be decomposed into a direct product of spaces of the following kind:

– Euclidean type: M is flat.
– Compact type: M is compact and semisimple.
– Non-compact type: the c-dual space M^c (cf. Section I.1.4) is of compact type.

In the first case M is a product of vector spaces and tori. The second type is usually defined by negative definiteness and the third type by positive definiteness of the Ricci form. However, this definition depends on the sign-convention used in the definition of the Ricci-form. The following characterization is independent of the sign-convention: recall that the standard-imbedding

$$\mathfrak{q} \to \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$$

is the infinitesimal version of the quadratic map $Q : M \to G(M)$, and the symmetric subspaces

$$G(M)^- := \{g \in G(M) \mid \sigma(g)^{-1} = g\}, \quad G(M)^+ := G(M)^\sigma$$

intersect transversally and orthogonally (w.r.t.the Killing form) at the base point o. The Killing form is (up to a scalar factor) just the Ricci form of the symmetric space $G(M)$. Then M is of compact type if and only if the restrictions of the Ricci form of $G(M)$ to $G(M)^+$ and to $G(M)^-$ are definite and have the *same* sign (i.e. the restrictions are both positive or both negative definite; with the usual sign-convention, they are then both negative), and M is of non-compact type if and only if both restrictions are definite and have *different* signs.

5.2. Duality. Recall that, if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the standard-imbedding of a LTS $\mathfrak{q}$, then the *c-dual LTS* $\mathfrak{q}^c$ is defined by the standard-imbedding $\mathfrak{g}^c = \mathfrak{h} \oplus i\mathfrak{q}$. From this it is immediately seen that $\mathfrak{q}^c$ is of compact type if and only if $\mathfrak{q}$ is of non-compact type.

If $\mathfrak{q}$ comes from a Lie group, then its standard-imbedding is a direct product where both factors are canonically isomorphic. In particular, both restrictions of the Ricci form are canonically isomorphic, and it cannot happen that one of it is positive and the other negative. Thus a Lie group is never of non-compact type. On the other hand, every semisimple Lie group G has a Cartan-involution giving rise to a space $M = G/K$ of non-compact type whose compact dual is U/K with U a compact real form of $G_{\mathbb{C}}$.

5.3. Hermitian symmetric spaces. The class of *Hermitian symmetric spaces* is the intersection of the class of Riemannian symmetric spaces and the class of pseudo-Hermitian symmetric spaces. As mentioned above, such a space has a unique decomposition into a Euclidean space, a space of compact type and of non-compact type. The Euclidean spaces are locally isomorphic to complex vector spaces. Since Ricci form and the trace form of the structure tensor are the same for pseudo-Hermitian symmetric spaces (cf. proof of Th.V.2.5), the spaces of non-compact type correspond bijectively to *positive Hermitian Jordan triple systems* (i.e. the trace form is positive definite) and the spaces of compact type to *negative Hermitian Jordan triple systems.*

5.4. Compact symmetric spaces with twist. When passing to real forms of (pseudo-)Hermitian symmetric spaces, the problem arises that the Ricci form of the real form and the trace form of its structure tensor need no longer be the same.

Definition V.5.1. A JTS T is called *positive (negative)* if its trace form is positive (negative) definite. A symmetric space with twist T is called *of non-compact type* if its JTS T is positive and *of compact type* if its JTS T is negative. ■

Proposition V.5.2. *A symmetric space M with twist T is of compact (resp. of non-compact) type if and only if its twisted complexification is a Hermitian symmetric space of compact (resp. of non-compact) type.*

Proof. The arguments proving Th. V.3.2 show that the trace form of $T_{h\mathbb{C}}$ is negative (positive) iff so is the trace form of T. But since the trace form of $T_{h\mathbb{C}}$ is equal to the Ricci form of $M_{h\mathbb{C}}$, this proves the claim. ■

Note that a symmetric space with twist of the non-compact type need not be semisimple – see example of symmetric cones below. Symmetric spaces with twist of compact type are the same as the class of *symmetric R-spaces* (cf. Sections 0.3 and X.6). Their classification is given in Table XII.3.3.

5.5. Symmetric cones and Euclidean Jordan algebras. If $\Omega = G/H$ is the open symmetric orbit of a prehomogeneous symmetric space (G, σ, V, e), then Ω is never compact (if it were compact, it would be open and closed in V, whence $\Omega = V$. But V is not compact: contradiction). If Ω is quadratic with Jordan-extension T, then $\operatorname{tr} T(e,e) = \operatorname{tr} \mathrm{id}_V = \dim V > 0$, showing again that Ω is never a symmetric space with twist of compact type.

Definition V.5.3. A Jordan algebra is called *Euclidean* if its trace-form is positive definite.■

Proposition V.5.4. *If (G, σ, V, e) is a quadratic prehomogeneous symmetric space, then the following are equivalent:*

(i) *The associated algebra A_e is a Euclidean Jordan algebra.*

(ii) *The twisted complexification $\Omega_{h\mathbb{C}}$ of Ω is a Hermitian symmetric space of the non-compact type.*

Proof. This is an immediate consequence of Prop. V.4.6 and the remarks in Section 5.3. ■

If the conditions of the preceding proposition are verified, then Ω is called a *symmetric cone.*

Example V.5.5. Consider the spaces defined in Section II.3.3: b is a non-degenerate symmetric bilinear form on a vector space V, $e \in V$ with $b(e,e) = 1$, and $\Omega = (\mathrm{O}(b) \times \mathbb{R}^+).e$. Then a short calculation shows that the trace form of the corresponding Jordan algebra is defined by

$$\operatorname{tr}(L(x^2)) = n(2b(x,e)^2 - b(x,x)).$$

Thus, if b has the signature (p,q), then the trace form has the signature $(q+1, p-1)$. Therefore V is Euclidean iff b has the signature $(1, n-1)$. In this case Ω is an open Lorentz cone. ■

The example shows that it is not possible to characterize the symmetric cones as the Riemannian symmetric spaces among the open symmetric orbits Ω. In fact, if b is positive definite, then $\Omega = S^{n-1} \times \mathbb{R}^+$ carries also an invariant Riemannian metric, but it is not a symmetric cone.

The following geometric characterization of symmetric cones is by now classical (cf. [FK94, Ch.I]): they are the *homogeneous* and *self-dual open convex cones* in Euclidean vector spaces.

Using the theory of Jordan algebras, one can also prove that the symmetric cones are precisely the *convex* ones among the open symmetric orbits associated to semisimple Jordan algebras. It would be interesting to have a simple direct argument relating convexity to the positive definiteness of the trace form.

Notes for Chapter V.

V.1. For the material of Sections 1.1 – 1.3, [Lo69a] and [Li52] are standard references. Theorem V.1.9 (1) is due to N. Jacobson ([Jac51, Th.7.3]). Jacobson defines, besides the standard-imbedding, a *universal imbedding* of a LTS which in all cases has the extension-property, and then proves that in the semisimple case the standard-imbedding is already universal. This basic result on symmetric spaces seems not to be very well known. For instance, it would have simplified parts of the exposition in [Sa80] (cf. [Sa80, Prop.I.9.1] where a special case of Th. V.1.9 is proved). Theorem V.1.10 summarizes various results due to S. Koh ([Koh65, Th. 3, 5, 6, 7, 8]); the presentation given here is new. For a different proof of Cor.V.1.11 see [HO96, Lemma 1.3.4].

V.2. and V.3. Theorems V.2.5, V.3.1 and V.3.2 generalize the correspondence between Hermitian symmetric spaces and positive Hermitian JTS already mentioned in the introduction (Section 0.3 (b)). However, our proofs are different from the original ones and are considerably simpler. The correspondence between positive real JTS and real forms of Hermitian symmetric spaces has already been investigated by O. Loos ([Lo77, Ch.11]); from another point of view such spaces have been classified by H.A. Jaffee [Jaf75]. The correspondence between non-degenerate polarized JTS (Jordan pairs) and para-Hermitian symmetric spaces is known by the work of S. Kaneyuki ([KanKo85], [Kan92]).

V.4. The main result on semisimple Lie triple algebras, Th. V.4.4., is due to M. Koecher ([Koe58]; cf. [FK94, Ch.III, Exercise 8]) and E.B. Vinberg ([Vi60]); our presentation follows [Sa80, Lemma I.8.6]. For a different proof cf. [Be94, Prop.4.2.6]. The equivalence of (i) and (ii) in Th. V.4.6 is due to H. Shima ([Shi75]).

V.5. Jordan theoretic aspects of compact symmetric spaces with twist (symmetric R-spaces) have been studied by O. Loos ([Lo85]). For a very detailed exposition of the theory of symmetric cones see [FK94]. The particular role symmetric cones play for *causal symmetric spaces with twist* will be discussed in Chapter XI.3.

Chapter VI: Integration of Jordan structures

In Chapter III we established the equivalence of Jordan triple systems and Jordan pairs with certain geometric objects (symmetric spaces with some additional structure). However, the method used there does not give much insight into the *geometry* of the symmetric spaces in question. If we want to understand the geometry, we have to pass from tensors to the spaces itself. The fundamental tool to be used in this process is the *integrability* of invariant almost complex structures and polarizations on symmetric spaces. The fact that such structures are integrable is well-known and easy to prove (we don't need to invoke the theorem of Newlander and Nirenberg; cf. Appendix VI.A), but it has important consequences.

In three steps we define "integrated versions" of the three basic Jordan theoretic "infinitesimal" concepts: Hermitian Jordan triple systems, Jordan pairs, general Jordan triple systems. In doing this, we follow the model of the theory of symmetric spaces: the multiplication map μ (Def. I.4.4) is the integrated version of the associated LTS which is an infinitesimal concept. The geodesic symmetry $s_x(y) = \mu(x,y)$ can be seen as an extension of the scalar -1 acting on the tangent space T_xM. Similarly, the integration of Jordan structures yields extensions of other non-zero complex or real scalars r. The Jordan-Lie functor is nothing but specialization to the value $r = -1$. Let us explain this in some more detail.

In the first step (Section 1) we characterize the twisted complex symmetric spaces as *circled spaces*: bounded symmetric domains M are well-known to be *circled* in the sense that they are stable under the action $S^1 \times M \to M$, $(u,y) \mapsto u_x(y)$ of the circle group S^1 fixing a given point x which we may choose to be the origin of a Harish-Chandra realization of M, and then the circle group acts by automorphisms of the domain. In other words, for all $u \in S^1$ we have a *multiplication map*

$$M \times M \to M, \quad (x,y) \mapsto u_x(y) =: u(x,y).$$

Its axiomatic properties give rise to our definition of *circled spaces*; they are very similar to those of the multiplication map μ defining a symmetric space. In fact, the underlying symmetric space structure of M is given by $\mu(x,y) = (-1)_x(y)$. In this first step the integrability is not yet used.

Let us skip for the moment the second step and pass to the third one: we want to characterize geometrically *real forms* of circled spaces. Thus we have to select a structure feature of circled spaces which is stable under complex conjugations. The structure tensor introduced in Section III.2 is such a feature; the action of S^1 attached to a point clearly is not. Here the example of the unit disc D is quite helpful: we complexify the action of $S^1 = \mathrm{SO}(2)$ and get the action of $\mathbb{C}^* = \mathrm{SO}(2,\mathbb{C})$ by multiplication with complex scalars, and then restrict attention to the action of the non-compact real form $\mathbb{R}^+$ of $\mathrm{SO}(2,\mathbb{C})$ by multiplication with real positive scalars. It is clear that any real form of D containing the origin is locally stable under this action. This local action defines a vector field E^o on the real form; it is the *Euler operator*, i.e. the vector field having value x at a point x. We will use the concept of "Euler operators on symmetric spaces" in the next chapter; here we focus our attention on the corresponding (germs of) multiplication maps

$$M \times M \to M, \quad (x,y) \mapsto r_x(y)$$

parametrized by *real* scalars r, which in a sense are just the local flows of Euler vector fields. The algebraic properties of such maps are more complicated than those defining circled spaces: for $r \neq \pm 1$, the dilatation by r can never be an automorphism of a curvature tensor and thus r_x cannot be realized as an automorphism of the whole structure in the way explained for circled spaces.

It is the concept of a *Jordan pair* which shows the way we have to go (Section 2): Polarized symmetric spaces have locally the structure of a direct product; this is precisely the point where integrability is used. The two leaves M^+ and M^- through a base point $o \in M$ correspond to the two spaces forming the Jordan pair. If the structure is twisted, then we may think of M^+ and M^- as the eigenspaces of the usual action of the small Lorentz group $\mathrm{O}(1,1)$ on $\mathbb{R}^2$: on M^+ an element of this group acts by a scalar r and on M^- by r^{-1}; the other orbits are hyperbolas. Now we have to keep in mind that to *any* point of M such an action is attached (in the example, M is the one-sheeted hyperboloid and $\mathbb{R}^2$ is the chart of M obtained by stereographic projection w.r.t. an arbitrary point). The properties of the family of multiplication maps thus obtained, summarized in Def. VI.2.4, show a remarkable similarity with the defining properties of a Jordan pair in the sense that the latter are formally obtained by differentiating the former. The one-sheeted hyperboloid is indeed a typical example of such a structure which in a sense generalizes the ruled surfaces known from analytic geometry; therefore we call our spaces *ruled spaces*. By adding the structure of a para-real form, we get a similar integrated version of general real Jordan triple systems (Th. VI.3.1).

Summarizing, we may say that integrability allows to extend in a canonical way multiplication by *real* scalars from tangent spaces to the space itself (at least locally). In other words, we have laws relating a re-scaling on a tangent space to the geometry of the ambient space. If we agree to consider the effects of scaling a primary topic in *conformal geometry*, then it is clear that the concepts presented in this chapter are a step towards understanding Jordan theory as an aspect of a "generalized conformal geometry".

1. Circled spaces

Definition VI.1.1. A smooth manifold M together with a smooth map

$$j : M \times M \to M, \quad (x,y) \mapsto j(x,y) =: j_x(y)$$

is called a *circled space* if, for all $x,y,z \in M$,

(c1) $j(x,x) = x$,

(c2) $(j_x)^4 = \mathrm{id}_M$,

(c3) $j_x(j(y,z)) = j(j_x(y), j_x(z))$,

(c4) The fixed point x of $(j_x)^2$ is isolated. ∎

Theorem VI.1.2. *The (germs of) circled spaces are precisely the (germs of) twisted complex symmetric spaces.*

Proof. Let (M,j) be a circled space. For $x,y \in M$ we let $s_x := (j_x)^2$ and $\mu(x,y) := s_x(y) = j(x, j(x,y))$. Then the properties (M1) – (M4) from Lemma I.4.2 follow immediately from the properties (c1) – (c4), and therefore M carries the structure of a symmetric space (Def. I.4.4). Now let $\mathcal{J}_x := T_x(j_x)$. Then $(\mathcal{J}_x)^2 = T_x(s_x) = -\mathrm{id}_{T_xM}$, i.e. $\mathcal{J} := (\mathcal{J}_x)_{x \in M}$ is an almost complex structure on M. It is invariant since from (c3) we get that the s_x are automorphisms of j: $s_x(j(y,z)) = j(s_x(y), s_x(z))$, which implies that $G(M)$ acts as a group of automorphisms of j and therefore also of $\mathcal{J}$. Finally, it is twisted because from (c3) it follows also that j_x acts as an automorphism of μ ($j_x(\mu(y,z)) = \mu(j_x(y), j_x(z))$), and therefore $\mathcal{J}_x$ is an automorphism of the curvature R_x at x.

Conversely, assume $(M, \nabla, \mathcal{J})$ is a twisted complex symmetric space. Since, for all $x \in M$, $\mathcal{J}_x$ is an automorphism of R_x, it has a unique (local) extension to an automorphism j_x of M (Cor. I.2.10). We let $j(x,y) := j_x(y)$. Then clearly $j(x,x) = x$, i.e. (c1); (c3) is verified since j_x is an automorphism of μ and of $\mathcal{J}$ and therefore of the whole structure; (c2) and (c4) follow since $T_x(j_x)^2 = -\operatorname{id}_{T_x M}$, whence $(j_x)^2 = s_x$ is the geodesic symmetry w.r.t. x. ■

Definition VI.1.1 parallels the algebraic definition of a symmetric space (Def. I.4.4). As for symmetric spaces, one can also give a differential geometric characterization of circled spaces in terms of smooth manifolds M with affine connection ∇ and almost complex structure $\mathcal{J}$ such that

(c1') $\mathcal{J}$ is invariant under ∇ (i.e. $\nabla \mathcal{J} = 0$),

(c2') $\mathcal{J}_p$ extends, for all p, to a local (resp. global) affine map j_p (cf. Def. I.A.1).

Recall that, by definition, $(\nabla \mathcal{J})(X,Y) = \nabla_X(\mathcal{J} Y) - \mathcal{J}(\nabla_X Y)$, and condition (c1') can be written $\nabla_X(\mathcal{J} Y) = \mathcal{J} \nabla_X Y$ for all $X, Y \in \mathfrak{X}(M)$. The proof that the conditions (c1') and (c2') describe precisely the twisted complex (locally) symmetric spaces is easy and will be left to the reader.

Proposition VI.1.3. *If $(M, \nabla, \mathcal{J})$ is a twisted complex symmetric space, then the operator $e^{t\mathcal{J}_p}$ has, for all $p \in M$ and $t \in \mathbb{R}$, a unique affine extension.*

Proof. According to Prop. III.2.2, $\mathcal{J}_p$ is a derivation of R_p. Therefore $e^{t\mathcal{J}_p}$ is an automorphism of R_p for all $t \in \mathbb{R}$. From Cor. I.2.10 it then follows that $e^{t\mathcal{J}_p}$ has an affine extension. ■

Let $u \in S^1 = e^{i\mathbb{R}}$. According to the proposition, for all $x \in M$, there exists a unique affine and almost-holomorphic map u_x with $u_x(x) = x$ and $T_x(u_x) = u \operatorname{id}_{T_x M}$. Let $u(x,y) := u_x(y)$; then

$$u : M \times M \to M$$

is a smooth map having properties similar to (c1) – (c4). In particular, (c3) is generalized by

$$u_x(w(y,z)) = w(u_x(y), u_x(z))$$

for all $u, w \in S^1$, $x, y, z \in M$. In other words, we can attach to any point $x \in M$ an action of the circle group S^1 by automorphisms of the whole structure and fixing x. Such a situation is considered in greater generality by O. Loos in [Lo72].

Proposition VI.1.4. *If $(M, \nabla, \mathcal{J})$ is a twisted complex symmetric space, then for all p, $\mathcal{J}_p$ has a unique vector field extension (cf.* Def. I.A.1*) to an affine vector field J^p of ∇.*

Proof. This is the differentiated version of Prop. VI.1.3 (cf. Lemma I.A.2). ■

2. Ruled spaces

The preceding two propositions carry over to twisted polarized symmetric spaces because we only needed that $\mathcal{J}_p$ is a derivation of R_p, and this property holds by Lemma III.3.4 also for twisted polarized symmetric spaces. However, an axiomatic characterization of these spaces in the spirit of Def. VI.1.1 is less straightforward than in the twisted complex case. The reason for this is, of course, that the operator $e^{t\mathcal{J}_p}$ can no longer be interpreted as multiplication by a real or complex scalar.

Definition VI.2.1. A *ruled space* is a manifold M together with a family of smooth maps

$$\mu_r : M \times M \to M, \quad (x,y) \mapsto \mu_r(x,y) =: \widetilde{r}_x(y),$$

parametrized by $r \in \mathbb{R}^*$, such that for all $x,y,z \in M$ and $r,t \in \mathbb{R}^*$,
(p1) $r_x(x) = x$
(p2) $\widetilde{r}_x \circ \widetilde{t}_x = \widetilde{(rt)}_x$, $\widetilde{1}_x = \mathrm{id}_M$,
(p3) $\widetilde{r}_x(\mu_t(y,z)) = \mu_t(\widetilde{r}_x(y), \widetilde{r}_x(z))$
(p4) The tangent map $T_x(\widetilde{r}_x)$ is diagonalizable with two eigenvalues, r and r^{-1}. ■

Theorem VI.2.2. *The (germs of) ruled spaces are precisely the (germs of) twisted polarized symmetric spaces.*

Proof. We use the same arguments as in the proof of Th. VI.1.2: If $(M, (\mu_r)_{r\in\mathbb{R}^*})$ is a ruled space, then it follows immediately from (p1) – (p4) that $\mu := \mu_{-1}$ defines on M the structure of a symmetric space in the sense of Def. I.4.4. From (p3) with $r = -1$ it follows that the whole structure is invariant under $G(M)$. Therefore the polarization defined by $\mathcal{J}_p := \frac{d}{dr}|_{r=1}(T_p(\widetilde{r}_p))$ is $G(M)$-invariant. It is twisted: $\mathcal{J}_p$ is a derivation of R_p because all $\widetilde{r}_p$ are, by (p3), automorphisms.

Conversely, assume $(M, \nabla, \mathcal{J})$ is a twisted polarized symmetric space. Since, for all $x \in M$, $\mathcal{J}_x$ is a derivation of R_x, $e^{t\mathcal{J}_x}$ is an automorphism and therefore extends to unique affine automorphism of M, denoted by $\widetilde{r}_x$ if $r = e^t$. We let $\mu_r(x,y) := \widetilde{r}_x(y)$ for positive r. If r is negative, then we let $\widetilde{r}_x(y) := \mu(x, \mu_{-r}(x,y))$, where μ is the multiplication map of the symmetric space M. The properties (p1) – (p4) are now easily verified, cf. the proof of Th. VI.1.2. ■

Let us denote by $\mathfrak{X}(M) = \mathfrak{X}(M)^+ \oplus \mathfrak{X}(M)^-$ the eigenspace decomposition of the operator $\mathcal{J} : \mathfrak{X}(M) \to \mathfrak{X}(M)$, and write $X = X^+ + X^-$ for the corresponding decomposition of a vector field X. By a *integral submanifold* for $\mathfrak{X}(M)^+$ we mean a submanifold $N \subset M$ such that $T_xN \subset (T_xM)^+$ for all $x \in N$. We say that N is *maximal* if the stronger condition $T_xN = (T_xM)^+$ holds, and similarly for $\mathfrak{X}(M)^-$.

Proposition VI.2.3. *Let $\mathcal{J}$ be an invariant polarization on the germ (M,o) of a symmetric space.*

(i) *There exist (germs of) maximal integral submanifolds M^+ and M^- for $\mathfrak{X}(M)^+$ resp. $\mathfrak{X}(M)^-$ intersecting transversally at o. The spaces M^+ and M^- are (germs of) subsymmetric spaces of M and define a product chart around o.*

(ii) *Every almost para-holomorphic (local) diffeomorphism $g : M \to M$ is actually para-holomorphic, i.e. it is a direct product of (local) diffeomorphisms $g^+ : M^+ \to M^+$, $g^- : M^- \to M^-$:*

$$g((x,a)) = (g^+(x), g^-(a)) \quad (x \in M^+, a \in M^-).$$

Proof. (i) In Appendix A (Prop. VI.A.3) it is proved that $\mathcal{J}$ is *integrable* in the sense that the invariance algebra $\mathfrak{g}(\mathcal{J})$ of $\mathcal{J}$ decomposes into a direct sum of ideals:

$$\mathfrak{g}(\mathcal{J}) = \mathfrak{g}(\mathcal{J})^+ \oplus \mathfrak{g}(\mathcal{J})^-; \quad \mathfrak{g}(\mathcal{J})^\pm \subset \mathfrak{X}(M)^\pm.$$

Thus the projections $\mathrm{pr}_\pm : \mathfrak{g}(\mathcal{J}) \to \mathfrak{g}(J)^\pm$, $X \mapsto X^\pm$ are Lie algebra homomorphisms. Since $\mathcal{J}$ is invariant under $\mathfrak{g}$, we have $\mathfrak{g} \subset \mathfrak{g}(\mathcal{J})$. Let $\mathfrak{g}^\pm := \mathrm{pr}_\pm(\mathfrak{g})$ and

$$M^\pm := (\text{germ of}) \exp(\mathfrak{g}^\pm).o.$$

Then the $M^{\pm}$ are maximal integral submanifolds of $\mathfrak{X}(M)^{\pm}$: in fact, the tangent vectors $v \in (T_p M^{\pm})$ are all of the form $v = X_p$ with $X \in \mathfrak{g}^{\pm}$, and therefore $T_p(M^{\pm}) = (T_p M)^{\pm}$. It follows that M^+ and M^- intersect transversally at o because $T_o M = T_o M^+ \oplus T_o M^-$. Next we are going to show that

$$M^+ \times M^- \to M, \quad (\exp(X).o, \exp(Y).o) \mapsto \exp(X+Y).o$$

($X \in \mathfrak{g}^+, Y \in \mathfrak{g}^-$) defines a product chart of M: it is well-defined since X and Y commute, and it is smooth by standard arguments. It is a local diffeomorphism since (under the natural identification $T_o(M^+ \times M^-) \cong T_o(M^+) \times T_o(M^-)$) its differential at the origin is just the identity.

(ii) Let us write, in local product coordinates $V = V^+ \times V^-$, $g((x_1,x_2)) = (g_1(x_1,x_2), g_2(x_1,x_2))$ with locally defined smooth maps $g_1 : V \to V^+$, $g_2 : V \to V^-$. Then the condition $g_* \mathcal{J} = \mathcal{J}$ reads in these coordinates $\partial_{\mathcal{J} X} g = \mathcal{J} \partial_X g$. For $X \in \mathfrak{X}^+$ (i.e. $\mathcal{J} X = X$) this implies that $\mathcal{J} \partial_X g$ is a map from V to V^+, and thus $\partial_X g_2 = 0$. Similarly, $\partial_Y g_1 = 0$ for all $Y \in \mathfrak{X}^-$. This implies that $g = g^+ \times g^-$ with $g^+(x) = g_1(x,o)$, $g^-(a) = g_2(o,a)$. (One could also use other coordinates in which $\mathcal{J}$ is given by the matrix $\left(\begin{smallmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)$; if we write $g = (u,v)$ in these coordinates, the condition $\partial_{\mathcal{J} X} g = \mathcal{J} \partial_X g$ takes the form of the "para Cauchy-Riemann equations"

$$\partial_x u = \partial_y v, \quad \partial_y u = \partial_x v,$$

where x and y are related to the coordinates (x', y') used above via $x' = x + y$, $y' = x - y$.) ■

Now we are ready to decompose the (germ of) the multiplication map

$$\mu_r : M \times M \cong M^+ \times M^- \times M^+ \times M^- \to M \cong M^+ \times M^-$$

in the product chart and to obtain an *integrated version of the Jordan pair concept*:

Theorem VI.2.4.

(i) *Let* $(M, (\mu_r)_{r \in \mathbb{R}^*})$ *be a (germ of a) ruled space and let for* $(x,a) \in M \cong M^+ \times M^-$,

$$r^+_{x,a} := (\widetilde{r}_{(x,a)})^+ : M^+ \to M^+, \quad r^-_{a,x} := (\widetilde{r^{-1}}_{(x,a)})^- : M^- \to M^-,$$

and

$$\mu^{\pm}_r : M^{\pm} \times M^{\mp} \times M^{\pm} \to M^{\pm}, \quad (x,y,z) \mapsto r^{\pm}_{x,y}(z).$$

Then the following holds: for all $a,b,c \in M^+$, $x,y,z \in M^-$, $r,s \in \mathbb{R}^*$,

(p1') $\mu^+_r(x,a,x) = x$, $\mu^-_r(a,x,a) = a$,

(p2') $r^+_{x,a} s^+_{x,a} = (rs)^+_{x,a}$, $1^+_{x,a} = \mathrm{id}_{M^+}$,
$r^-_{a,x} s^-_{a,x} = (rs)^-_{a,x}$, $1^-_{a,x} = \mathrm{id}_{M^-}$,

(p3') $r^+_{x,a}(\mu^+_s(y,b,z)) = \mu^+_s(r^+_{x,a}(y), (r^{-1})^-_{a,x}(b), r^+_{x,a}(z))$,
$r^-_{a,x}(\mu^-_s(b,y,c)) = \mu^-_s(r^-_{a,x}(b), (r^{-1})^-_{x,a}(y), r^-_{a,x}(c))$,

(p4') $T_x r^+_{x,a} = r\,\mathrm{id}_{T_x M^+}$, $T_a r^-_{a,x} = r\,\mathrm{id}_{T_a M^+}$.

(ii) *Conversely, if* $M^{\pm}$ *are manifolds and* $\mu^{\pm}_r : M^{\pm} \times M^{\mp} \times M^{\pm} \to M^{\pm}$ *(*$r \in \mathbb{R}^*$*) maps having the properties* (p1') – (p4')*, then*

$$\mu_r : M \times M \to M, \quad \mu_r((x,a),(y,b)) := (\mu^+_r(x,a,y), \mu^-_{r^{-1}}(a,x,b))$$

defines on $M = M^+ \times M^-$ *a (germ of a) ruled space.*

Proof. The properties (p1') – (p4') are equivalent to (p1) – (p4), written out by decomposing elements of M into M^+- and M^--parts. ■

Note that the map

$$r_{x,a} \times r_{a,x} : M \to M$$

has differential $r\,\mathrm{id}_{T_{(x,a)}M}$ at $(x,a) \in M$ and can therefore be interpreted as a sort of "mulitplication by r on M w.r.t. the point (x,a)"; however, unlike $\widetilde{r}_{(x,a)} : M \times M$ (which has differential $r\,\mathrm{id}_{T_x M^+} \times r^{-1}\,\mathrm{id}_{T_a M^-}$ at (x,a)) it is in general *not* an automorphism of the whole structure – the only exception are the values $r = 1$ and $r = -1$; as already remarked, the value $r = -1$ belongs to the symmetric space structure of M.

3. Integrated version of Jordan triple systems

As we have seen in Ch. III, general Jordan triple systems are para-real forms of twisted polarised Jordan triple systems (=Jordan pairs). Thus we have to describe the structure obtained by adding a para-conjugation τ to the structure from the preceding section. Let $M = M^+ \times M^-$ be a polarized space with base point $o = (o^+, o^-)$. A para-conjugation is an automorphism τ of M of order 2 such that $\tau_* \mathcal{J} = -\mathcal{J}$. It induces an involution $\tau_*(g) := \tau g \tau$ on $G(M)$, and we have the formulae

$$(\tau_* g)^+ = g^-, \quad (\tau_* g)^- = g^+.$$

Assume further that $\tau(o) = o$. Then $\tau : M^+ \to M^-$ is a local diffeomorphism which we use to identify M^+ with M^-, thus considering τ as given by the formula

$$\tau((x,a)) = (a,x)$$

and the τ-fixed subspace as the (germ of the) diagonal in M:

$$M^\tau = \{(x,x) \in M^+ \times M^- \,|\, x \in M^+\}.$$

Under τ, the maps $\mu_r : M^\pm \times M^\mp \times M^\pm \to M^\pm$ correspond to a single map

$$\pi_r : M^+ \times M^+ \times M^+ \to M^+, \quad (x,y,z) \mapsto \pi_r(x,y,z) =: r_{x,y}(z),$$

and (p1') – (p4') translate to

(p1") $\pi_r(x,a,x) = x$,

(p2") $r_{x,y} \circ s_{x,y} = (rs)_{x,y}$, $1_{x,y} = \mathrm{id}_M$,

(p3") $r_{x,y}(\pi_s(a,b,c)) = \pi_s(r_{x,y}(a), r^{-1}_{y,x}(b), r_{x,y}(c))$,

(p4") $T_x(r_{x,y}) = r\,\mathrm{id}_{T_x M^+}$.

Taking $r = -1$ and restricting to the diagonal, we get the symmetric space structure of the τ-fixed space M^τ; i.e.

$$\mu((x,x),(y,y)) = (\pi_{-1}(x,x,y), \pi_{-1}(x,x,y)).$$

Since M^+, M^- and M^τ are diffeomorphic, we may also transfer this structure to M^+. Summing up, we have

Theorem VI.3.1.

(i) *There is a bijection between (real finite-dimensional) Jordan triple systems and germs (M^+, o) of manifolds with a family of germs of diffeomorphisms $\pi_r : M \times M \times M \to M$ satisfying* (p1") – (p4").

(ii) *The integrated version of the Jordan-Lie functor is the map associating to $(\pi_r)_{r \in \mathbb{R}^*}$ the multiplication map $\mu(x,y) := \pi_{-1}(x,x,y)$.* ∎

It is in fact possible to give an explicit formula for the maps π_r in terms of the corresponding JTS T (see Section X.3.2). It turns out that this formula is rational and thus allows to globalize the germs of structures described in this chapter.

Appendix A: Integrability of almost complex structures

Definition VI.A.1. Recall that an almost complex structure (resp. polarization) is a tensor field $\mathcal{J}$ of type (1,1) such that $\mathcal{J}_p^2 = -\mathbf{1}_p$ (resp. $\mathcal{J}_p^2 = \mathbf{1}_p$) for all $p \in M$. It is called *integrable* if its *torsion tensor*

$$N(X,Y) := [\mathcal{J}X, \mathcal{J}Y] + \mathcal{J}^2[X,Y] - \mathcal{J}[X, \mathcal{J}Y] - \mathcal{J}[\mathcal{J}X, Y]$$

vanishes. ∎

Proposition VI.A.2. *Let $\mathcal{J}$ be a G-invariant almost complex structure or polarization on a symmetric space $M = G/H$. Then $\mathcal{J}$ is integrable.*

Proof. We use the canonical connection ∇ of M defined in Prop. I.2.1. Since $[X,Y] = \nabla_X Y - \nabla_Y X$ for all $X, Y \in \mathfrak{X}(M)$, we get

$$\begin{aligned} N(X,Y) &= \nabla_{\mathcal{J}X}\mathcal{J}Y - \nabla_{\mathcal{J}Y}\mathcal{J}X + \mathcal{J}^2(\nabla_X Y - \nabla_Y X) \\ &\quad - \mathcal{J}(\nabla_X \mathcal{J}Y - \nabla_{\mathcal{J}Y} X - \nabla_{\mathcal{J}X} Y + \nabla_Y \mathcal{J}X). \end{aligned}$$

As remarked in Section I.2, $G \cdot \mathcal{J} = \mathcal{J}$ implies $\nabla \mathcal{J} = 0$; this means that $\nabla_Z \mathcal{J} S = \mathcal{J} \nabla_Z S$ for all $Z, S \in \mathfrak{X}(M)$. Using this property, all terms in the above expression of $N(X,Y)$ cancel out. ∎

We consider the invariance group $G(\mathcal{J})$ of $\mathcal{J}$; this is the group of diffeomorphisms g of M such that for all $Y \in \mathfrak{X}(M)$, $g_*(\mathcal{J}Y) = \mathcal{J} g_* Y$ holds. We call $G(\mathcal{J})$ also the *group of almost (para-) holomorphic transformations of* $(M, \mathcal{J})$. The corresponding infinitesimal object is the Lie algebra

$$\mathfrak{g}(\mathcal{J}) := \{X \in \mathfrak{X}(M) \mid \forall Y \in \mathfrak{X}(M) : [X, \mathcal{J}Y] = \mathcal{J}[X,Y]\}. \tag{A.1}$$

We remark that for all $X, Y \in \mathfrak{g}(\mathcal{J})$,

$$N(X,Y) = [\mathcal{J}X, \mathcal{J}Y] - \mathcal{J}^2[X,Y],$$

and therefore, if $\mathcal{J}$ is integrable, then we have for all $X, Y \in \mathfrak{g}(\mathcal{J})$,

$$[\mathcal{J}X, \mathcal{J}Y] = \mathcal{J}^2[X,Y]. \tag{A.2}$$

Proposition VI.A.3. *Let $\mathcal{J}$ be an integrable almost complex structure or a polarization on a manifold M and assume that $\mathfrak{g}(\mathcal{J})$ contains for any point $p \in M$ a local basis of $\mathfrak{X}(M)$ around p. Then $\mathfrak{g}(\mathcal{J})$ is stable under the map $X \mapsto \mathcal{J}X$. In particular,*

(i) *if $\mathcal{J}^2 = -\mathrm{id}_{\mathfrak{X}(M)}$, then $\mathfrak{g}(\mathcal{J})$ is a complex Lie algebra with complex structure $\mathcal{J}$;*

(ii) *if $\mathcal{J}^2 = \mathrm{id}_{\mathfrak{X}(M)}$, then $\mathfrak{g}(\mathcal{J})$ is the direct sum of its ideals*

$$\mathfrak{g}(\mathcal{J})^{\pm} := \{X \in \mathfrak{g}(\mathcal{J}) \mid X = \pm \mathcal{J}X\}.$$

Proof. Let $X \in \mathfrak{g}(\mathcal{J})$ and $Y \in \mathfrak{X}(M)$. On a neighbourhood of a point $p \in M$ we can write $Y = \sum_{i=1}^m f_i Y_i$ with smooth functions f_i and $Y_i \in \mathfrak{g}(\mathcal{J})$. Without loss of generality we may assume that $m = 1$ and write $Y = f\widetilde{Y}$. Then using Equation (A.2), locally,

$$\begin{aligned} [\mathcal{J}X, \mathcal{J}Y] &= [\mathcal{J}X, f\,\mathcal{J}\widetilde{Y}] = \mathrm{d}f(\mathcal{J}X) \cdot \mathcal{J}\widetilde{Y} + f[\mathcal{J}X, \mathcal{J}\widetilde{Y}] \\ &= \mathcal{J}(\mathrm{d}f(\mathcal{J}X) \cdot \widetilde{Y} + f[\mathcal{J}X, \widetilde{Y}]) \\ &= \mathcal{J}[\mathcal{J}X, Y]. \end{aligned}$$

Using a partition of unity, we get the same relation for general $Y \in \mathfrak{X}(M)$, whence $\mathcal{J}X \in \mathfrak{g}(\mathcal{J})$, proving the claim. Now (i) is an immediate consequence, and in order to prove (ii), it remains only to show that $\mathfrak{g}(\mathcal{J})^{\pm}$ are indeed ideals of $\mathfrak{g}(\mathcal{J})$: let $X \in \mathfrak{g}(\mathcal{J})^+$, $Y \in \mathfrak{g}(\mathcal{J})^-$; then $[X,Y] = [\mathcal{J}X, Y] = \mathcal{J}[X,Y] = [X, \mathcal{J}Y] = -[X,Y]$, and therefore $[\mathfrak{g}(\mathcal{J})^+, \mathfrak{g}(\mathcal{J})^-] = 0$. ∎

Proposition VI.A.4. *Let $\mathcal{J}$ be an integrable almost para-complex structure on M satisfying the assumptions of the preceding proposition. Assume τ is a para-conjugation, i.e. a diffeomorphism with $\tau_*(\mathcal{J}) = -\mathcal{J}$ and $\tau^2 = \mathrm{id}_M$. Then the maps*

$$p_\pm : \mathfrak{g}(\mathcal{J}) \to \mathfrak{g}(\mathcal{J})^{\tau_*}, \quad X \mapsto \frac{1}{2}(X + \tau_* X \pm \mathcal{J}(X - \tau_* X))$$

are Lie-algebra homomorphisms.

Proof. We can write $p_\pm = r_\pm \circ \mathrm{pr}_\pm$, where

$$\mathrm{pr}_\pm : \mathfrak{g}(\mathcal{J}) \to \mathfrak{g}(\mathcal{J})^+, \quad X \mapsto \frac{1}{2}(X \pm \mathcal{J} X)$$

are the projections onto the ideals introduced in the preceding proposition, and

$$r_\pm : \mathfrak{g}(\mathcal{J})^\pm \to \mathfrak{g}(\mathcal{J})^{\tau_*}, \quad X \mapsto X + \tau_* X.$$

We prove that the maps $r_\pm$ are Lie algebra homomorphisms: if $X \in \mathfrak{g}(\mathcal{J})^\pm$, then $\tau_* X \in \mathfrak{g}(\mathcal{J})^\mp$ and therefore $[X, \tau_* X] = 0$. Thus for $X, Y \in \mathfrak{g}(\mathcal{J})^\pm$,

$$[\tau_* X + X, \tau_* Y + Y] = [\tau_* X, \tau_* Y] + [X, Y] = \tau_*[X, Y] + [X, Y]. \qquad \blacksquare$$

Notes for Chapter VI.

With some suitable modifications, the properties (p1') – (p4') carry over to the case of more general base fields, and thus one can base on them a more general global theory of Jordan structures than presented here (work in progress).

Appendix A. It is well-known that invariant almost complex structures on symmetric spaces are integrable (cf. [KoNo69]); for the corresponding fact on para-complex structures cf. [KanKo85].

Chapter VII: The conformal Lie algebra

There is a by now classical construction associating to a Jordan triple system or a Jordan pair a 3-graded Lie algebra of quadratic polynomial vector fields; it has been found independently by Kantor, Koecher and Tits and is therefore often referred to as the *Kantor-Koecher-Tits construction.* This chapter is devoted to a geometric version of this constructon. Essentially, this means that we proceed in the same way as in the preceding chapter, but on the level of Lie algebras instead of working with local diffeomorphisms.

As explained in the introduction to Chapter VI, the infinitesimal version of the dilatation maps $r_x : M \to M$ is the *Euler vector field* $E^x := \frac{d}{dr}|_{r=1} r_x$, and the distribution $(E^x)_{x \in M}$ of Euler vector fields is the infinitesimal version of the multiplication map $M \times M \to M$, $(x,y) \mapsto r_x(y)$. The *(inner) conformal Lie algebra* or *Kantor-Koecher-Tits algebra* is then the Lie algebra generated by all Euler vector fields; it contains the Lie algebra of the transvection group $G(M)$ of the symmetric space M as a proper subalgebra. A fixed Euler vector field induces a 3-grading on the conformal Lie algebra, which naturally leads to a realization of the algebra as an algebra of polynomial vector fields on a vector space V. This vector space V can be seen as a natural chart (which we call *Jordan coordinates*) of the space M around a given point. The Jordan coordinates are the generalization of the well-known *Harish-Chandra imbedding* of a Hermitian symmetric space. In case of the unit disc it is the natural realization in the vector space $V = \mathbb{C}$.

It is clear that, as in the preceding chapter, we use the *integrability* in a crucial way. Most of the results of this chapter could have been derived from what we have done there; but in order to treat the Kantor-Koecher-Tits construction as a topic in its own right we have kept the exposition rather independent of Chapter VI by using only Lie-algebraic methods.

1. Euler operators and conformal Lie algebra

1.1. The twisted complex and twisted polarized case. Assume that $M = G/H$ is a symmetric space with an invariant (1,1)-tensor field $\mathcal{J}$ such that $\mathcal{J}^2 = -\mathbf{1}$ or $\mathcal{J}^2 = \mathbf{1}$. Let us assume that $G = G(M)$ and denote by $\mathfrak{g} = \mathfrak{g}(M)$ its Lie algebra. By invariance of $\mathcal{J}$, the Lie algebra $\mathfrak{g}$ belongs to the invariance algebra $\mathfrak{g}(\mathcal{J})$ of $\mathcal{J}$. If a base point $o \in M$ is chosen, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the usual decomposition w.r.t. this base point.

Lemma VII.1.1. *The vector space*

$$\mathfrak{g}^b := \mathfrak{g}(M) + \mathcal{J}\,\mathfrak{g}(M)$$

is a Lie-subalgebra of $\mathfrak{g}(\mathcal{J})$. *Denote by*

$$\mathfrak{co}(M,\mathcal{J}) := \{X \in \mathfrak{g}(\mathcal{J}) | \, [X, \mathfrak{g}^b] \subset \mathfrak{g}^b\}$$

its normalizer in $\mathfrak{g}(\mathcal{J})$. *If* $\mathcal{J}^2 = -\mathbf{1}$, *then both algebras are complex Lie algebras, with multiplication by* i *represented by* $\mathcal{J}$, *and if* $\mathcal{J}^2 = \mathbf{1}$, *then both split into a direct sum of ideals which are the eigenspaces of* $\mathcal{J}$.

Proof. This is an immediate consequence of the integrability of $\mathcal{J}$, cf. Appendix VI.A, in particular Prop. VI.A.3. ■

If $\mathcal{J}$ is straight, then it is easily seen that $\mathfrak{g}$ is already stable under $\mathcal{J}$, and the preceding lemma is uninteresting. Therefore let us assume from now on that $\mathcal{J}$ is twisted.

Definition VII.1.2. Let T be a twisted complex or polarized JTS and $(M,\mathcal{J})$ be the associated (germ of a) twisted complex resp. polarized symmetric space. The Lie algebra

$$\mathfrak{co}(T) := \mathfrak{co}(M,\mathcal{J})$$

defined in the preceding lemma is called the *conformal Lie algebra* of T, and the Lie algebra $\mathfrak{g}^b$ is called the *inner conformal Lie algebra* of T. ■

Lemma VII.1.3. *If* $(M,\mathcal{J})$ *is twisted complex or twisted para-complex and* J^p *denotes, for* $p \in M$, *the vector field extension of* $\mathcal{J}_p$ *from* Prop. VI.1.4, *then the vector field*

$$E^p := \mathcal{J}^{-1} J^p \in \mathfrak{g}^b$$

is a vector field extension of the operator $\mathbf{1}_p = \mathrm{id}_{T_pM}$.

Proof. We remark first that $(E^p)_p = \mathcal{J}_p^{-1}(J^p)_p = 0$. Next, since $J^p \in \mathfrak{g}(\mathcal{J})$ by construction and since $\mathfrak{g}(\mathcal{J})$ is stable under $\mathcal{J}$ (Prop. V.A.3), $E^p = \mathcal{J}^{-1} J^p$ belongs to $\mathfrak{g}(\mathcal{J})$. Fix $p \in M$ and choose a vector field extension $\mathbf{v}$ of $v \in T_pM$ such that $\mathbf{v} \in \mathfrak{g}(\mathcal{J})$, e.g. take $\mathbf{v} = l_p(v)$ (cf. Section I.2.1). Now

$$[\mathbf{v}, E^p]_p = [\mathbf{v}, \mathcal{J}^{-1} J^p]_p = \mathcal{J}_p^{-1}[\mathbf{v}, J^p]_p = \mathcal{J}_p^{-1} \mathcal{J}^p v = v;$$

thus E^p is a vector field extension of $\mathbf{1}_p$. ■

In the following proposition we use the notation

$$\widehat{\mathfrak{q}} := \mathrm{ad}(E^o)\mathfrak{q} = \{[E^o, X] \mid X \in \mathfrak{q}\} \subset \mathfrak{g}^b \tag{1.1}$$

and

$$\mathfrak{q}^b := \mathfrak{q} + \widehat{\mathfrak{q}} \subset \mathfrak{g}^b. \tag{1.2}$$

Proposition VII.1.4. *In the situation of the preceding lemma, the following holds:*

(E1) $[[\widehat{\mathfrak{q}},\widehat{\mathfrak{q}}],\widehat{\mathfrak{q}}] \subset \widehat{\mathfrak{q}}$ *and* $[[\mathfrak{q}^b,\mathfrak{q}^b],\mathfrak{q}^b] \subset \mathfrak{q}^b$ *(i.e,* $\widehat{\mathfrak{q}}$ *and* $\mathfrak{q}^b$ *are Lie triple-systems).*

(E2) $[[\mathfrak{q},\widehat{\mathfrak{q}}],\mathfrak{q}] \subset \widehat{\mathfrak{q}}$.

(E3) *For all* $X \in \mathfrak{q}^b$, $\mathrm{ad}(E^o)^2 X = X$.

(E4) *For all* $X, Y \in \mathfrak{q}^b$, $\mathrm{ad}(E^o)[X,Y] = 0$.

Proof. Let us assume first that $\mathcal{J}$ is twisted complex. Since $\mathcal{J}$ defines on $\mathfrak{g}(\mathcal{J})$ the structure of a complex Lie algebra, the inclusion

$$\iota : \mathfrak{g} \to \mathfrak{g}(\mathcal{J})$$

has a unique $\mathbb{C}$-linear extension to a homomorphism

$$\iota_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g} \to \mathfrak{g}(\mathcal{J}), \quad (X + iY) \mapsto X + \mathcal{J} Y.$$

The vector field J^o is an affine vector field vanishing at the origin, i.e. $J^o \in \mathfrak{h}$. Thus $\iota_{\mathbb{C}}(i^{-1}J^o) = \mathcal{J}^{-1}\,J^o = E^o$, and since $\iota_{\mathbb{C}}$ is a homomorphism, the diagram

$$\begin{array}{ccc} \mathfrak{g}_{\mathbb{C}} & \xrightarrow{\iota_{\mathbb{C}}} & \mathfrak{g}(\mathcal{J}) \\ -\operatorname{ad}(iJ^o)\downarrow & & \downarrow \operatorname{ad}(E^o) \\ \mathfrak{g}_{\mathbb{C}} & \xrightarrow{\iota_{\mathbb{C}}} & \mathfrak{g}(\mathcal{J}) \end{array}$$

commutes. Using that $[J^o,\mathfrak{q}] = \mathfrak{q}$, this implies $\widehat{\mathfrak{q}} = \mathcal{J}[J^o,\mathfrak{q}] = \mathcal{J}\,\mathfrak{q} = \iota_{\mathbb{C}}(i\mathfrak{q})$ and $\mathfrak{q}^b = \iota_{\mathbb{C}}(\mathfrak{q}_{\mathbb{C}})$. Since $i\mathfrak{q}$ and $\mathfrak{q}_{\mathbb{C}}$ are Lie triple systems, so are their images under $\iota_{\mathbb{C}}$, proving (E1). The relation $[[\mathfrak{q},i\mathfrak{q}],\mathfrak{q}] = i[[\mathfrak{q},\mathfrak{q}],\mathfrak{q}] \subset i\mathfrak{q}$ yields (E2).

By assumption, $\operatorname{ad}(J^o)$ is an invariant twisted complex structure on $\mathfrak{q}$; therefore $\operatorname{ad}(J^o) : \mathfrak{q}_{\mathbb{C}} \to \mathfrak{q}_{\mathbb{C}}$, being its $\mathbb{C}$-linear continuation, is an invariant twisted complex structure on $\mathfrak{q}_{\mathbb{C}}$. Since $(i\operatorname{ad}(J^o))^2 = i^2(\operatorname{ad}(J^o))^2$ is the identity on $\mathfrak{q}_{\mathbb{C}}$, it follows that $\operatorname{ad}(E^o)^2$ is the identity on $\mathfrak{q}^b$, whence (E3). Since, for all $X \in \mathfrak{h}_{\mathbb{C}}$, $[iJ^o,X] = i[J^o,X] = 0$, it follows that $[E^o,Y] = 0$ for all $Y \in \iota_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}})$, in particular for all $[U,V]$, $U,V \in \mathfrak{q}^b = \iota_{\mathbb{C}}(\mathfrak{q}_{\mathbb{C}})$; whence (E4).

The twisted polarized case can be treated in a similar way, replacing $\mathfrak{g}_{\mathbb{C}}$ by a direct sum $\mathfrak{g} \oplus \mathfrak{g}$. ■

Note that (E3) and (E4) imply that $[\operatorname{ad}(E^o)X,Y] + [X,\operatorname{ad}(E^o)Y] = \operatorname{ad}(E^o)[X,Y] = 0$, i.e.

$$\text{(E5)} \qquad \forall X,Y,Z \in \mathfrak{q}^b : \quad [[\operatorname{ad}(E^o)X,Y],Z] = -[[X,\operatorname{ad}(E^o)Y],Z]$$

holds. Using the terminology introduced in Def. III.3.2, (E3), (E4) and (E5) say that $\operatorname{ad}(E^o)$ is a twisted invariant polarization on the LTS $\mathfrak{q}^b$, and (E1) and (E2) say that $\mathfrak{q}$ has the essential properties of a para-real form of this polarized LTS. In fact, if the homomorphism $\iota_{\mathbb{C}}$ from the preceding proof is injective, then $\mathfrak{q}^b$ is isomorphic to the polarized LTS $\mathfrak{q}_{\mathbb{C}}$; we return to this point below (Section 2.4).

Definition VII.1.5. An *Euler operator on a symmetric space* $M = G/H$ is given by a vector field extension $E := E^o$ of $\mathbf{1}_o := \mathrm{id}_{T_oM}$ having properties (E1) – (E4). Property (E4) allows to define a vector field extension E^p of $\mathbf{1}_p$ for all $p \in M$ in a G-invariant way which is uniquely determined by E^o; then $(E^p)_{p\in M}$ will be called a *distribution of Euler operators on* M. ■

We will soon see that, in a suitable chart, an Euler operator on a symmetric space is realized by the usual Euler operator (see Example I.A.3). Note that according to Lemma I.A.2, the flow φ_t of E^p is a (local) extension of $e^t\mathbf{1}_p$; in the notation of the preceding chapter we have $\varphi_t(x) = r_{p,p}(x)$ with $r = e^t$.

1.2. The Euler operator of a symmetric space with twist. Euler operators can be restricted to (para-) real forms, and the restriction defines an Euler operator on the corresponding symmetric space with twist:

Proposition VII.1.6. *If τ is a (para-)conjugation of the twisted complex (resp. twisted polarized) space $(M,\mathcal{J})$ such that $\tau(o) = o$, then the Euler vector field E^o is invariant under τ_*, and it defines an Euler operator on the real form M^τ of M.*

Proof. We have $\tau_*(J^o) = -J^o$ since $\tau_*(J^o)$ is an affine vector field extending $\tau_*(\mathcal{J}_o) = -\mathcal{J}_o$. Using this, we get

$$\tau_*(E^o) = \tau_*(\mathcal{J}^{-1}\,J^o) = -\mathcal{J}^{-1}\,\tau_*(J^o) = E^o.$$

Thus E^o can be restricted to a (locally defined) vector field on M^τ, and $\mathfrak{q}^\tau + \operatorname{ad}(E^o)\mathfrak{q}^\tau$ is the real form $(\mathfrak{q}^b)^\tau$ of $\mathfrak{q}^b$; the properties (E1) – (E4) continue to hold for all $X,Y,Z \in (\mathfrak{q}^b)^\tau$. ■

Definition VII.1.7. Let T be a para-real form of a twisted polarized JTS $T_{ph\mathbb{C}}$ w.r.t. the para-conjugation τ. The *conformal Lie algebra* of T is the τ-fixed subalgebra

$$\mathfrak{co}(T) := (\mathfrak{co}(T_{ph\mathbb{C}}))^\tau$$

of the conformal Lie algebra of $T_{ph\mathbb{C}}$. The *inner conformal Lie algebra* of T is the subalgebra $(\mathfrak{g}^b)^\tau$ of $\mathfrak{co}(T)$. ■

The conformal Lie algebra is then also the normalizer of the inner conformal Lie algebra in $\mathfrak{X}(M)$ (cf. Lemma VII.1.1). The Euler operators associated to M^τ belong by definition to the conformal Lie algebra of M^τ, and since $\mathfrak{g}^b = \mathfrak{g} + \mathcal{J}\,\mathfrak{g}$, we have

$$(\mathfrak{g}^b)^\tau = \mathfrak{g}^\tau + [E^o, \mathfrak{g}^\tau].$$

Next we describe how the JTS T corresponding to the symmetric space with twist M^τ can be recovered from the Euler operator:

Proposition VII.1.8. *Let $(E^p)_{p\in M}$ be a distribution of Euler operators on a symmetric space $M = G/H$ and recall from* Section I.2.1 *the bijection $l_p : T_pM \to \mathfrak{q}^p$, inverse of the evaluation map $\mathfrak{q}^p \to T_pM$.*

(i) *The formula*

$$T(X,Y,Z)_p := [[l_p(X_p), \frac{l_p(Y_p) + \mathrm{ad}(E^p)l_p(Y_p)}{2}], l_p(Z_p)]_p$$

defines a Jordan-extension of the curvature tensor R of M.

(ii) *If M is a symmetric space with twist and $(E^p)_{p\in M}$ is the associated distribution of Euler operators, then T is equal to the structure tensor of M.*

Proof. (i) From the definition of T it is clear that T is a tensor field (i.e. it is function-linear in all three arguments) and that it is G-invariant. Thus we have to show that T_o is a Jordan-extension of R_o. If we identify T_oM with $\mathfrak{q}$ and let $X, Y, Z \in \mathfrak{q}$, then, using that E^o is an extension of $\mathbf{1}_o$,

$$T_o(X,Y,Z) = \frac{1}{2}([[X,Y],Z] - \mathrm{ad}(E^o)[[X, \mathrm{ad}(E^o)Y], Z]). \tag{1.3}$$

As we have remarked after the proof of Prop. VII.1.4, $\mathrm{ad}(E^o)$ is an *invariant twisted polarization* on the LTS $\mathfrak{q}^b$, cf. Def. III.3.2. Therefore the formula

$$\widetilde{T}(X,Y,Z) := \frac{1}{2}([[X,Y],Z] - \mathrm{ad}(E^o)[[X, \mathrm{ad}(E^o)Y], Z])$$

defines a Jordan-extension of the LTS $\mathfrak{q}^b$ (Prop. III.3.6). Because of (E1), (E2) and (E3), $\mathfrak{q}$ is stable under $\widetilde{T}$, and the restriction of $\widetilde{T}$ to a triple product T_o on $\mathfrak{q}$ (given by Eqn. (3.1)) is a Jordan-extension of $\mathfrak{q}$.

(ii) Let us assume first that M is a twisted complex symmetric space and (E^p) its associated field of Euler operators. Then the structure tensor of M, evaluated at the base point and multiplied by a factor 2, is for $X, Y, Z \in \mathfrak{q}$ given by

$$\begin{aligned} -(R(X,Y)Z - \mathcal{J}\,R(X, \mathcal{J}^{-1}\,Y)Z)_o &= ([[X,Y],Z] - \mathrm{ad}(J^o)[[X, \mathrm{ad}(J^o)^{-1}Y], Z])_o \\ &= ([[X,Y],Z] - \mathcal{J}\,\mathrm{ad}(E^o)[[X, \mathrm{ad}(E^o)^{-1}\,\mathcal{J}^{-1}\,Y], Z])_o \\ &= ([[X,Y],Z] - \mathrm{ad}(E^o)[[X, \mathrm{ad}(E^o)Y], Z])_o \\ &= ([[X,Y],Z]_o + [[X, \mathrm{ad}(E^o)Y], Z]_o) \end{aligned}$$

where we have used that $\mathcal{J}$ commutes with $\mathrm{ad}(H)$ for all $H \in \mathfrak{g}(\mathcal{J})$. It follows that the structure tensor is is precisely the tensor T_o defined in the claim. Passing to real forms, the claim follows since both Euler operator and structure tensor can be restricted to real forms. ■

2. The Kantor-Koecher-Tits construction

2.1. Jordan coordinates and the local Harish-Chandra imbedding. Let E^o be an Euler operator on the (germ of a) symmetric space $M = G/H$ with base point o.

Lemma VII.2.1. *The inner conformal Lie algebra $\mathfrak{g}^b$ is 3-graded. More precisely, $\mathfrak{g}^b$ is stable under the derivation $\mathrm{ad}(E^o)$ and decomposes as a direct sum*

$$\mathfrak{g}^b = \mathfrak{m}^{-1} \oplus \mathfrak{m}^0 \oplus \mathfrak{m}^1$$

of the eigenspaces $\mathfrak{m}^k$ for the eigenvalues $k = -1, 0, 1$ of $\mathrm{ad}(E^o)$, and $[\mathfrak{m}^k, \mathfrak{m}^l] \subset \mathfrak{m}^{k+l}$. In particular, the spaces $\mathfrak{m}^{\pm 1}$ are abelian and $[[\mathfrak{m}^1, \mathfrak{m}^{-1}], \mathfrak{m}^{\pm 1}] \subset \mathfrak{m}^{\pm 1}$. Moreover,

$$\mathfrak{q}^b = \mathfrak{m}^{-1} \oplus \mathfrak{m}^1, \quad [\mathfrak{q}^b, \mathfrak{q}^b] = \mathfrak{m}^0.$$

Proof. The algebra $\mathfrak{g}$ is generated by $\mathfrak{q}$, and therefore the algebra $\mathfrak{g}^b = \mathfrak{g} + [E^o, \mathfrak{g}]$ is generated by $\mathfrak{q}^b = \mathfrak{q} + [E^o, \mathfrak{q}]$. More precisely, $\mathfrak{g}^b = [\mathfrak{q}^b, \mathfrak{q}^b] + \mathfrak{q}^b$ since $\mathfrak{q}^b$ is a LTS. The space $\mathfrak{q}^b$ is stable under $\mathrm{ad}(E^o)$ whose square is, by (E3), the identity there; therefore $\mathfrak{q}^b \subset \mathfrak{m}^1 \oplus \mathfrak{m}^{-1}$. On the other hand, by (E4), $[\mathfrak{q}^b, \mathfrak{q}^b] \subset \mathfrak{m}^0$, and we actually have equalities. This establishes the decomposition of $\mathfrak{g}^b$ into eigenspaces.

Now, since $\mathrm{ad}(E^o)$ is a derivation, for all $X \in \mathfrak{m}^k$, $Y \in \mathfrak{m}^l$: $\mathrm{ad}(E^o)[X, Y] = [\mathrm{ad}(E^o)X, Y] + [X, \mathrm{ad}(E^o)Y] = [kX, Y] + [X, lY] = (k + l)[X, Y]$, proving the statement about the grading. ∎

We will use the notation $\mathfrak{m}^{\pm} := \mathfrak{m}^{\pm 1}$, $\mathfrak{h}^b := \mathfrak{m}^0$. Note that $X_o = [X, E^o]_o = 0$ for all $X \in \mathfrak{h}^b$, and also $X_o = 0$ for all $X \in \mathfrak{m}^+$ since $X_o = [E^o, X]_o = -[X, E^o]_o = -X_o$, using again that E^o is an extension of $\mathbf{1}_o$.

Lemma VII.2.2. *Every tangent vector $v \in T_oM$ has a unique vector field extension by an element of $\mathfrak{m}^-$; this element is given by the formula*

$$\mathbf{v} := \frac{1}{2}(l_o v - [E^o, l_o v]),$$

where l_o is the inverse of the bijective evaluation map $\mathfrak{q} \to T_oM$.

Proof. Since $\mathrm{ad}(E^o)^2 = \mathrm{id}$ on $\mathfrak{q}^b$, it follows that the projector onto $\mathfrak{m}^-$ is $P := \frac{1}{2}(\mathrm{id} - \mathrm{ad}(E^o))$. Its image is equal to $P(\mathfrak{q})$ because $\mathfrak{q}^b = \mathfrak{q} + \mathrm{ad}(E^o)\mathfrak{q}$. Thus $\mathfrak{m}^- = P(\mathfrak{q}) = \{\mathbf{v} | \, v \in T_oM\}$. We evaluate at the base point:

$$\mathbf{v}_o = \frac{1}{2}(l_o v + [l_o v, E^o])_o = \frac{1}{2}(v + \mathbf{1}_o(l_o v)_o) = v.$$

It follows that the evalution map $\mathfrak{m}^- \to T_oM$ is bijective. ∎

Definition VII.2.3. If E^o is an Euler operator on a symmetric space M, we define a chart, called *Jordan coordinates*, as follows: to a vector v lying in a suitable neighborhood U of 0 in $V := T_oM$ we assign the value of the integral curve $\varphi_t(o)$ of $\mathbf{v}$ at $t = 1$.

Having fixed the base point o, we will consider Jordan coordinates as an identification of an open neighborhood U of $o \in M$ with an open domain U in $V = T_oM$, i.e. we use the same notation v for $v \in U$ and for the corresponding point in M. Vector fields on U will be identified with smooth functions $U \to V$; thus the vector fields $\mathbf{v} \in \mathfrak{m}^-$ are identified with the constant vector fields (cf. Appendix I.A). ∎

The notion of Jordan coordinates can be seen as a *local Harish-Chandra imbedding*: in fact, integrating the vector fields $X \in \mathfrak{q}$ yields a local realization of M as an open domain in the vector space $V \cong \mathfrak{m}^-$. When M is a Hermitian symmetric space, this realization is known to be global and is precisely the well-known Harish-Chandra imbedding. More generally, when M is twisted complex or twisted polarized, then Jordan coordinates define a *complex* resp. *para-complex atlas:* since $\mathfrak{m}^- \subset \mathfrak{g}^b \subset \mathfrak{g}(\mathcal{J})$, it follows that $\mathbf{v} \cdot \mathcal{J} = 0$ for all $v \in V$, i.e. $\mathcal{J}$ is a translation invariant (=constant) tensor field on V. Thus the chart we have defined is indeed (para-) holomorphic, and almost (para-) holomorphic local diffeomorphisms of M are actually (para-) holomorphic. This can be seen as an elementary version of the theorem of Newlander and Nirenberg for twisted (para-) complex symmetric spaces.

2.2. Geometric version of the Kantor-Koecher-Tits construction.

Theorem VII.2.4. *Let E^o be an Euler operator on a symmetric space M and $\mathfrak{g}^b$ the associated inner conformal Lie algebra. Then $\mathfrak{g}^b$ is represented in Jordan coordinates as a Lie algebra of quadratic vector fields, i.e. by polynomial maps $V \to V$ of degree at most two, such that the grading*

$$\mathfrak{g}^b = \mathfrak{m}^- \oplus \mathfrak{h}^b \oplus \mathfrak{m}^+$$

(cf. Lemma VII.2.1*) coincides with the grading given by the degree. More precisely, if we define for $v \in V$ a polynomial $\mathbf{p}_v$ by*

$$\mathbf{p}_v(x) := \frac{1}{2} T(x, v, x),$$

where $T = T_o$ is the trilinear map on $V = T_oM$ defined in Prop. VII.1.8*, then*

(1) $\mathfrak{m}^- = \{\mathbf{v} | \, v \in V\}$,
(2) $\mathfrak{m}^+ = \{\mathbf{p}_v | \, v \in V\}$,
(3) $\mathfrak{q} = \{\mathbf{v} - \mathbf{p}_v | \, v \in V\}$,
(4) $\widehat{\mathfrak{q}} = \{\mathbf{v} + \mathbf{p}_v | \, v \in V\}$,
(5) $\mathfrak{h}^b = [\mathfrak{m}^+, \mathfrak{m}^-] = \mathrm{Span}\{T(w, v) | \, v, w \in V\}$*, where* $T(u, v)p := T(u, v, p)$,
(6) E^o *is identified with the usual Euler operator on* V*, i.e.* $E^o(x) = x$.

Proof. (1) is true by definition of the Jordan coordinates. In order to prove the other relations, recall formula (I.A.2) for the Lie bracket of two vector fields X, Y, considered as smooth functions on V. This formula implies that for any Y and $v \in V$,

$$\mathrm{ad}(\mathbf{v})^k Y(p) = \mathrm{D}^k Y(p) \cdot (\otimes^k v),$$

where $\mathrm{D}^k Y : V \to \mathrm{Hom}(S^k V, V)$ is the ordinary k-th total differential of $Y : V \to V$. Now, it follows from Lemma VII.2.1 that $[\mathfrak{m}^-, [\mathfrak{m}^-, [\mathfrak{m}^-, \mathfrak{g}^b]]] = 0$, and therefore, for all $Y \in \mathfrak{g}^b$ and p in the domain of the Jordan coordinates,

$$\mathrm{D}^3 Y(p) = 0.$$

Integration shows that Y is quadratic as claimed.

Let us prove (6): E^o is a linear vector field since $[\mathbf{v}, [\mathbf{v}, E^o]] = -[\mathbf{v}, \mathbf{v}] = 0$ and $(E^o)_o = 0$. Since $[\mathbf{v}, E^o] = \mathbf{v}_o = v$, $E^o = \mathrm{id}_V$.

Now, the solutions of the Euler differential equation

$$[E^o, X](p) = \mathrm{D}X(p) \cdot p - X(p) = k\, X(p)$$

for $k \in \mathbb{N}_0$ are precisely the homogeneous polynomials of degree $k+1$; therefore $\mathfrak{h}^b$ is represented by linear vector fields and $\mathfrak{m}^+$ by homogenous quadratic ones.

It remains to calculate the second differential of the vector field $l_o w \in \mathfrak{q}$ for $w \in V$. We use that for all $w \in \mathfrak{q}$, $l_o w = \mathbf{w} + \widetilde{\mathbf{w}}$ with $\widetilde{\mathbf{w}} := \frac{1}{2}(l_o w + [E^o, l_o w])$ is the decomposition of $l_o w$ in a constant term plus a homogeneous quadratic polynomial. Using the commutativity of $\mathfrak{m}^+$ and of $\mathfrak{m}^-$, we get $[\mathbf{v}, l_o w] = [\mathbf{v}, \widetilde{\mathbf{w}}] = [l_o v, \widetilde{\mathbf{w}}]$ and

$$\begin{aligned}
(\mathrm{D}^2(l_o w))(0) \cdot v \otimes v &= [\mathbf{v}, [\mathbf{v}, l_o w]]_o \\
&= [\mathbf{v}, [l_o v, \widetilde{\mathbf{w}}]]_o \\
&= [l_o v, [l_o v, \widetilde{\mathbf{w}}]]_o \\
&= T_o(v, w, v)
\end{aligned}$$

when comparing with the formula from Prop. VII.1.8. Since $\mathrm{D}^2\mathbf{w} = 0$, we have $\mathrm{D}^2(l_o w) = \mathrm{D}^2(\widetilde{\mathbf{w}})$; the vector field $\widetilde{\mathbf{w}}$ being homogeneous quadratic, it is given by the formula

$$\widetilde{\mathbf{w}}(x) = \frac{1}{2}D^2(l_o w)(0)(x,x) = -\frac{1}{2}T(x,w,x) = -\mathbf{p}_w(x).$$

This proves (2) and gives the formula

$$(l_o w)(x) = w - \frac{1}{2}T(x,w,x),$$

proving (3). Next, $[E^o, \mathbf{v} - \mathbf{p}_v] = -\mathbf{v} - \mathbf{p}_v$, proving (4).

We prove (5): Clearly, $[\mathfrak{q}^b, \mathfrak{q}^b] = [\mathfrak{m}^+, \mathfrak{m}^-]$, and we have explicitly

$$[\mathbf{p}_v, \mathbf{w}](p) = -(\mathrm{D}\mathbf{p}_v)(p)w = -T(w,v,p) = -T(w,v)p; \tag{2.1}$$

this implies the claim. ∎

Proposition VII.2.5. *There is a canonical bijection between Jordan-extensions of the curvature tensor and Euler operators on a symmetric space M.*

Proof. In Prop. VII.1.8 we have associated to an Euler operator E^o a Jordan-extension T of R. Conversely, given a Jordan-extension T of R, we let E^o be the Euler operator induced from the twisted complexification given by T.

These two constructions are indeed inverse to each other: Starting with a Jordan-extension T, we get again T from the associated Euler operator E^o (Prop. VII.1.8 (ii)). Starting with an Euler operator E^o, let T be the associated Jordan-extension. Then it is easily verified that in Jordan coordinates associated to E^o the Euler operator derived from T is again the usual Euler operator on V and thus coincides with E^o by part (6) of the preceding theorem. ∎

2.3. Local Borel imbedding.

Proposition VII.2.6. *The LTS $\widehat{\mathfrak{q}} = \mathrm{ad}(E^o)\mathfrak{q}$ is isomorphic to the dual of $\mathfrak{q}$ (i.e. its Lie triple bracket is isomorphic to the negative of the one in $\mathfrak{q}$). Consequently, the c-dual symmetric space of M has locally a canonical realization with base point o on the underlying manifold of M.*

Proof. From (E3) and (E4) we get for all $X, Y, Z \in \mathfrak{q}$,

$$[[\mathrm{ad}(E^o)X, \mathrm{ad}(E^o)Y], \mathrm{ad}(E^o)Z] = \mathrm{ad}(E^0)[[X, -\mathrm{ad}(E^o)^2 Y], Z] = -\mathrm{ad}(E^o)[[X,Y],Z];$$

therefore $\mathrm{ad}(E^o)$ is an isomorphism from $\mathfrak{q}$ onto the LTS obtained from $\widehat{\mathfrak{q}}$ by taking the negative of the usual triple Lie bracket. The second statement is obtained by integrating locally at o the vector fields $X \in \widehat{\mathfrak{q}}$. ∎

If M is a bounded symmetric domain, then it is well-known that M can be globally imbedded into its compact dual (the Borel imbedding). Thus the preceding proposition is indeed a local generalization of the Borel imbedding. The corresponding global statement will be more complicated: an imbedding of one space into its c-dual is globally only possible in the Riemannian case (see Ch. X).

2.4. Faithful Jordan triple systems.

Proposition VII.2.7. *Let E^o be an Euler operator on a symmetric space M corresponding to a Jordan-extension T. Then the following are equivalent:*

(1) $\mathfrak{q} \cap \widehat{\mathfrak{q}} = 0$

(2) *The map* $A : V \to \mathrm{Hom}(V \otimes V, V)$, $v \mapsto T(\cdot, v, \cdot)$ *is injective.*

If these properties hold, then the formula

$$\Theta(X+Y) := X - Y, \quad X \in \mathfrak{q},\, Y \in \widehat{\mathfrak{q}}$$

defines an involutive automorphism of the LTS $\mathfrak{q}^b$ such that $\Theta \circ \mathrm{ad}(E^o) = -\,\mathrm{ad}(E^o) \circ \Theta$. This automorphism extends to an involution of the inner conformal Lie algebra $\mathfrak{g}^b$ given in the notation of Th. VII.2.4 *by the formula*

$$\Theta(\mathbf{v} + T(a,b) + \mathbf{p}_w) = -\mathbf{w} - T(b,a) - \mathbf{p}_v.$$

Proof. The equivalence of (1) and (2) follows immediately from a comparison of formulas (3) and (4) of Th. VII.2.4.

Now we assume that $E^o = -\mathcal{J}\, J^o$ is the Euler operator of a twisted complex symmetric space, and as in the proof of Prop. VII.1.4 we consider the inclusion $\iota : \mathfrak{q} \to \mathfrak{g}(\mathcal{J})$ and its complexification

$$\iota_{\mathbb{C}} : \mathfrak{q}_{\mathbb{C}} = \mathfrak{q} \oplus i\mathfrak{q} \to \mathfrak{g}(\mathcal{J}), \quad X + iY \mapsto X + \mathcal{J}\, Y.$$

Its kernel is

$$\ker \iota_{\mathbb{C}} = \{X + i\,\mathcal{J}\, X \,|\, X \in \mathfrak{q}, \mathcal{J}\, X \in \mathfrak{q}\} = \{Z \in \mathfrak{q}_{\mathbb{C}} \,|\, i\,\mathcal{J}\, Z \in \mathfrak{q}_{\mathbb{C}}\}.$$

Thus

$$\ker \iota_{\mathbb{C}} = (\mathfrak{q} \cap \widehat{\mathfrak{q}})_{\mathbb{C}},$$

and we see that (1) holds iff $\iota_{\mathbb{C}}$ is injective. If this is the case, $\Theta|_{\mathfrak{q}_{\mathbb{C}}}$ is nothing but complex conjugation of $\mathfrak{q}_{\mathbb{C}}$ w.r.t. $\mathfrak{q}$, which has clearly the properties stated. Moreover, automorphisms of LTS extend always to automorphisms of the standard imbedding; thus Θ extends to an automorphism of $\mathfrak{g}^b$. It is one on $\mathfrak{q}$ and minus one on $\widehat{\mathfrak{q}}$, i.e. it is determined by the conditions

$$\Theta(\mathbf{v} - \mathbf{p}_v) = \mathbf{v} - \mathbf{p}_v, \quad \Theta(\mathbf{v} + \mathbf{p}_v) = -(\mathbf{v} + \mathbf{p}_v)$$

which together with $[\mathbf{p}_v, \mathbf{w}] = -T(w,v)$ imply the last formula of the claim.

Passing to real forms of twisted complex symmetric spaces, the claim now follows in the general case of a symmetric space with twist. ∎

Definition VII.2.8. A Jordan triple system T is called *faithful* if Condition (2) from the preceding proposition holds. ∎

Theorem VII.2.9. *Let (M,T) be a (germ of a) symmetric space with twist and $(M_{ph\mathbb{C}}, T_{ph\mathbb{C}})$ its twisted para-complexification. Denote by $\mathfrak{g}(M_{ph\mathbb{C}})$ the Lie algebra of $G(M_{ph\mathbb{C}})$, by $\mathfrak{D}er(M_{ph\mathbb{C}})$ the Lie algebra of the automorphism group of $M_{ph\mathbb{C}}$ and by τ the para-conjugation w.r.t. M, and let r_+ be as in* Prop. VI.A.4. *Then*

$$r_+ : \mathfrak{g}(M_{ph\mathbb{C}}) \to \mathfrak{g}^b$$

and

$$r_+ : \mathrm{Der}(M_{ph\mathbb{C}}) \to \mathfrak{co}(T)$$

are surjective homomorphisms of Lie algebras. They are injective if and only if T is faithful, and in this case $\Theta = r_+ \circ \tau_* \circ (r_+)^{-1}$ is the involution described in Prop. VII.2.7.

Proof. The proof is the para-complex version of the proofs of Prop. VII.1.4 and of Prop. VII.2.7. ∎

Faithfulness is a rather “weak” condition which is satisfied in all interesting cases:

Proposition VII.2.10. *A JTS T is faithful in the following two cases:*

(i) *T is non-degenerate.*

(ii) *T is associated to a unital Jordan algebra via the formula $T(x,y,z) = 2(x(yz) - y(xz) + (xy)z)$.*

Proof. (i) By definition, T is non-degenerate iff the map

$$^* : V \to V^*, \quad v \mapsto v^* := (x \mapsto \operatorname{tr} T(x,v))$$

is injective. But * is a composition of A and

$$\kappa : \operatorname{Hom}(V \otimes V, V) \to V^*, \quad B \mapsto (x \mapsto \operatorname{tr}(B(x,\cdot))).$$

Therefore A has to be injective, i.e. T is faithful.

(ii) If e is the unit element of the Jordan algebra V, we have the formula $A(v)(e \otimes e) = 2v$, forcing A to be injective. ■

Corollary VII.2.11. *In the category of semisimple symmetric spaces with twist the conformal Lie algebra depends functorially on the JTS T.*

Proof. JTS-homomorphisms induce homomorphisms of the corresponding twisted para-complexifications (Th. III.4.7). In the semisimple case, the standard imbedding $\mathfrak{g}(M_{ph\mathbb{C}})$ of the LTS $R_{ph\mathbb{C}}$ also depends functorially on the LTS (Th. V.1.9). But $\mathfrak{g}(M_{ph\mathbb{C}})$ is isomorphic to the inner conformal Lie algebra $\mathfrak{g}^b$ (Prop. VII.2.10 and Th.VII.2.9). Finally, in the semisimple case, the algebra $\mathfrak{g}(M_{ph\mathbb{C}})$ and therefore $\mathfrak{g}^b$ are semisimple (cf. Section V.3); thus all derivations of $\mathfrak{g}^b$ are inner and it follows that $\mathfrak{g}^b = \mathfrak{co}(T)$. Summing up, $\mathfrak{co}(T)$ depends functorially on T.■

3. General structure of the conformal Lie algebra

3.1. Conformal Lie algebra and structure algebra. We are going to show that the only distinction between the conformal Lie algebra and the inner conformal Lie algebra is in the linear part, leading to the important notion of the *structure algebra* of a JTS:

Theorem VII.3.1. *The conformal Lie algebra $\mathfrak{co}(T)$ is 3-graded; more precisely, it is the Lie algebra*

$$\mathfrak{co}(T) = \mathfrak{m}^- \oplus \mathfrak{str}(T) \oplus \mathfrak{m}^+$$

of quadratic vector fields, where $\mathfrak{m}^-$ and $\mathfrak{m}^+$ are as in Th. VII.2.4*, and the linear part $\mathfrak{str}(T)$ is given by*

$$\begin{aligned}\mathfrak{str}(T) &= \{X \in \mathfrak{gl}(V) \mid [X, \mathfrak{m}^+] \subset \mathfrak{m}^+\} \\ &= \{X \in \mathfrak{gl}(V) \mid \exists X^\sharp \in \mathfrak{gl}(V) : \forall u,v,w \in V : \\ &\qquad XT(u,v,w) = T(Xu,v,w) + T(u,X^\sharp v,w) + T(u,v,Xw)\}.\end{aligned}$$

The endomorphisms $T(u,v)$ $(u,v \in V)$ belong to $\mathfrak{str}(T)$, and one may choose $T(u,v)^\sharp = -T(v,u)$.

Proof. The Lie algebra $\mathfrak{co}(T)$ is the normalizer of $\mathfrak{g}^b$ in the Lie algebra of vector fields on M. Thus $[\mathfrak{m}^-, \mathfrak{co}(T)] \subset \mathfrak{g}^b$ and therefore $[\mathfrak{m}^-,[\mathfrak{m}^-,[\mathfrak{m}^-,[\mathfrak{m}^-,\mathfrak{co}(T)]]]] = 0$. As in the proof of Th. VII.2.4 we conclude that $\mathfrak{co}(T)$ is a Lie algebra of polynomial vector fields of degree at most 3.

Since $[E^o, \mathfrak{co}(T)] \subset \mathfrak{g}^b$, the Euler differential equation implies that $\mathfrak{co}(T)$ is actually a space of quadratic vector fields. It contains $\mathfrak{g}^b$; in particular, all constant vector fields belong to $\mathfrak{co}(T)$.

The homogeneous quadratic part of $\mathfrak{co}(T)$ is equal to $\mathfrak{m}^+$: in fact, if $X \in \mathfrak{co}(T)$ is homogeneous quadratic, then $X = [E^0, X] \in \mathfrak{g}^b$, and hence $X \in \mathfrak{m}^+$.

Let us determine the linear part: a linear vector field normalizes $\mathfrak{g}^b$ if and only if it normalizes the homogeneous quadratic part $\mathfrak{m}^+$ of $\mathfrak{g}^b$. This is expressed by the first equality for $\mathfrak{str}(T)$. More explicitly, it means that $[X, \mathbf{p}_v] = \mathbf{p}_w$ for some $w \in V$. The choice of w is in general not unique, but we can make it unique by requiring w to lie in a fixed complementary subspace V_1 of the kernel of $V \to \mathfrak{m}^+$, $v \mapsto \mathbf{p}_v$. We define $X^\sharp$ by $X^\sharp(v) := -w$, and then the relation $[X, \mathbf{p}_v] = \mathbf{p}_{-X^\sharp v} = \mathbf{p}_{[X^\sharp, \mathbf{v}]}$ holds for all $v \in V$. Written in terms of T, this means that for all $p \in V$,

$$XT(p,v,p) - T(Xp,v,p) - T(p,v,Xp) = T(p, X^\sharp v, p),$$

which leads to the second equality.

The operators $T(u,v)$ belong to $\mathfrak{g}^b$ and hence to $\mathfrak{co}(T)$, and from the defining identity (JT2) of a JTS it follows that one may choose $T(u,v)^\sharp = -T(v,u)$. ■

Definition VII.3.2. The *structure algebra* $\mathfrak{str}(T)$ of a JTS T is the linear part

$$\begin{aligned}\mathfrak{str}(T) = \{X \in \mathfrak{gl}(V) |\, \exists X^\sharp \in \mathfrak{gl}(V) :\ & \forall u,v,w \in V : \\ & XT(u,v,w) = T(Xu,v,w) + T(u,X^\sharp v,w) + T(u,v,Xw)\}\end{aligned}$$

of the conformal Lie algebra, and the *inner structure algebra* of T is the linear part

$$T(V \otimes V) = \operatorname{Span}(\{T(u,v) |\, u,v \in V\})$$

of the inner conformal Lie algebra. ■

If T is faithful, then the argument proving Prop. VII.2.7 shows that

$$\Theta(X) = X^\sharp$$

for all $X \in \mathfrak{str}(T)$.

Proposition VII.3.3. *If T is a non-degenerate JTS, then for all $X \in \mathfrak{str}(T)$ the relation $X^\sharp = -X^*$ holds, where * denotes the adjoint w.r.t. the trace form.*

Proof. We use the notation from the proof of Prop. VII.2.10 (i). By definition of $\sharp$, A has the equivariance property $X \cdot A(v) = A(X^\sharp v)$, and κ is natural, i.e $\kappa(X \cdot B) = -X^* \kappa(B)$. When we identify V and V^* via *, this proves the claim. ■

3.2. T-conformality and T-projectivity

Definition VII.3.4. A (possibly only locally defined) vector field X on V is called *T-conformal* if it is of class $\mathcal{C}^3$ and, for all $p \in V$ where X is defined, the first differential $\mathrm{D}X(p)$ belongs to $\mathfrak{str}(T)$. It is called *T-projective* if in addition, for all p where X is defined, the second differential $\mathrm{D}^2X(p) : V \otimes V \to V$ belongs to the space

$$W := A(V) = \{T(\cdot, v, \cdot) |\, v \in V\} \subset \operatorname{Hom}(V \otimes V, V).$$ ■

Proposition VII.3.5. *The Lie algebra $\mathfrak{co}(T)$ is a Lie algebra of T-conformal and T-projective vector fields.*

Proof. Both properties are trivial for the vector fields belonging to $\mathfrak{m}^-$ or $\mathfrak{str}(V)$. Let us assume that $X = \mathbf{p}_v \in \mathfrak{m}^+$. Then $\mathrm{D}X(p) = T(p,v) \in \mathfrak{str}(T)$ and $\mathrm{D}^2X(p) = T(\cdot, v, \cdot) \in W$. ■

We will prove later (Th. IX.1.2) that in the non-degenerate case the converse also holds.

3.3. Derivations and normalizer.

Theorem VII.3.6. *Every derivation of the conformal Lie algebra is an inner derivation, and the adjoint representation*

$$\mathrm{ad} : \mathfrak{co}(T) \to \mathfrak{D}er(\mathfrak{co}(T)), \quad X \mapsto \mathrm{ad}(X)$$

is an isomorphism onto.

Proof. Let us first show that ad is injective, i.e. the center of $\mathfrak{co}(T)$ is trivial. Note that, for any $X \in \mathfrak{X}(V)$,

$$[X, \mathbf{v}](p) = -\mathrm{D}X(p) \cdot v,$$
$$[X, E](p) = X(p) - \mathrm{D}X(p) \cdot p,$$

where $E = E^o$, and thus

$$X(p) = [X, E](p) - [X, \mathbf{p}](p). \tag{3.1}$$

Therefore, if X centralizes $\mathfrak{co}(T)$, then $X = 0$; in particular, the center of $\mathfrak{co}(T)$ is reduced to zero.

Let δ be a derivation of $\mathfrak{co}(T)$. We prove that δ is inner. W.l.o.g. we may assume that $\delta E \in \mathfrak{str}(T)$: In fact, writing $\delta E = S + Z$ with $S \in \mathfrak{str}(T)$ and $Z \in \mathfrak{m}^+ \oplus \mathfrak{m}^-$, we have $(\delta + \mathrm{ad}([E, Z]))E = S + Z + [[E, Z], E] = S + Z - Z = S \in \mathfrak{str}(T)$.

We claim that then $\delta E = 0$: In fact, for all $\mathbf{v} \in \mathfrak{m}^-$,

$$-\delta\mathbf{v} = \delta[E, \mathbf{v}] = [\delta E, \mathbf{v}] + [E, \delta\mathbf{v}].$$

We evaluate at 0 and use that $[E, Z](0) = -[Z, E](0) = -Z(0)$ for all $Z \in \mathfrak{co}(T)$. We obtain $[\delta E, \mathbf{v}](0) = 0$. But $[\delta E, \mathbf{v}] \in [\mathfrak{str}(T), \mathfrak{m}^-] \subset \mathfrak{m}^-$, and therefore $[\delta E, \mathbf{v}] = 0$ and $\delta E = 0$.

This implies $[\delta, \mathrm{ad}(E)] = \mathrm{ad}(\delta E) = 0$; i.e. δ preserves the eigenspaces of $\mathrm{ad}(E)$. Now, any derivation δ preserving the grading is uniquely determined by its restriction to $\mathfrak{m}^-$: δX for $X \in \mathfrak{str}(T)$ is uniquely determined by the condition $\delta[X, \mathbf{v}] = [\delta X, \mathbf{v}] + [X, \delta\mathbf{v}]$, and similarly $\delta[\mathbf{v}, \mathbf{p}_w] = [\delta\mathbf{v}, \mathbf{p}_w] + [\mathbf{v}, \delta\mathbf{p}_w]$. The same arguments hold for the Lie algebra $\mathrm{Pol}(V, V)$ of all polynomial vector fields on V: in fact, $\mathrm{ad}(H)$ with $H \in \mathfrak{gl}(V)$ given by $H(v) = -(\delta\mathbf{v})(0)$ is the unique extension of $\delta|_{\mathfrak{m}^-}$ to a derivation of $\mathrm{Pol}(V)$ preserving the grading. By unicity, we must have $\delta = \mathrm{ad}(H)|_{\mathfrak{co}(T)}$. This means that $\mathfrak{co}(T)$ is stable under $\mathrm{ad}(H)$, and moreover $[H, \mathfrak{m}^+] \subset \mathfrak{m}^+$ since $\mathrm{ad}(H)$ respects the grading. But according to Th. VII.3.1, this means precisely that $H \in \mathfrak{str}(T)$, und thus $\delta = \mathrm{ad}(H)$ is an inner derivation. ∎

Note that, in view of Eqn. (3.1), we have the explicit formula

$$\delta = \mathrm{ad}(X), \quad X(p) = (\delta E)(p) - (\delta\mathbf{p})(p)$$

for an arbitrary derivation δ of $\mathfrak{co}(T)$.

Corollary VII.3.7. *The Lie algebra $\mathfrak{co}(T)$ is equal to its normalizer in $\mathfrak{X}(V)$:*

$$\{X \in \mathfrak{X}(V) |\, [X, \mathfrak{co}(T)] \subset \mathfrak{co}(T)\} = \mathfrak{co}(T).$$

Proof. If $\mathrm{ad}(X)|_{\mathfrak{co}(T)}$ is a derivation, there exists by the preceding theorem $X' \in \mathfrak{co}(T)$ such that $\mathrm{ad}(X - X')|_{\mathfrak{co}(T)} = 0$; i.e. $X - X'$ centralizes $\mathfrak{co}(T)$. In the preceding proof we have seen that then $X - X' = 0$. ∎

3.4. Conformal Lie algebra and complexification functors.

Proposition VII.3.8. *The complexification functors from* Th.III.4.9 *induce the following "complexifications" of conformal Lie algebras:*

$$\begin{aligned}\mathfrak{co}(T_{\mathbb{C}}) &\cong \mathfrak{co}(T)_{\mathbb{C}}\\ \mathfrak{co}(T_{h\mathbb{C}}) &\cong \mathfrak{co}(T)_{\mathbb{C}}\\ \mathfrak{co}(T_{ph\mathbb{C}}) &\cong \mathfrak{co}(T)\times\mathfrak{co}(T).\end{aligned}$$

Proof. The three isomorphisms are easily established using the definition of the complexification functors for T (Prop. III.4.6) and the explicit realization of the (inner) conformal Lie algebra in Th. VII.2.4: $\mathfrak{co}(T_{\mathbb{C}})$ is given by the continuation of elements of $\mathfrak{co}(T)$ to holomorphic quadratic polynomials onto $V_{\mathbb{C}}$ and is thus canonically isomorphic to $\mathfrak{co}(T)_{\mathbb{C}}$. Next, from Th. VII.2.4 it follows that $\mathfrak{co}(T_{\mathbb{C}})$ and $\mathfrak{co}(T_{h\mathbb{C}})$ are the same algebras of vector fields and that $\mathfrak{co}(T_{ph\mathbb{C}})$ and $\mathfrak{co}(T_d)$ are the same algebras of vector fields. But the latter clearly is isomorphic to $\mathfrak{co}(T)\times\mathfrak{co}(T)$. ■

Notes for Chapter VII.

VII.2. The *Kantor-Koecher-Tits construction* associates to a Jordan algebra the polynomial Lie algebra described in Th. VII.2.4; see [Koe68]. Later K. Meyberg defined Jordan triple systems and extended the Kantor-Koecher-Tits construction for these ([Mey70], cf. [Sa80, Prop. I.7.1]). The original proofs are more computational, using the axiomatic properties of a JTS in order to verify that the given space of polynomials is indeed closed under the Lie bracket. Corollary VII.2.11 is due to E. Neher (cf. [Sa80, footnote on p.40]).

VII.3. Theorem VII.3.6 is due to Koecher ([Koe69b, Satz 3.1]).

Chapter VIII: Conformal group and conformal completion

Having defined the conformal Lie algebra $\mathfrak{co}(T)$, the next step in our program is obviously the definition of a suitable group $\mathrm{Co}(T)$ belonging to this Lie algebra which will be called the *conformal group*. This is not an abstract group, but a group coming in a particular realization, namely as a group of birational maps of a vector space, corresponding to the realization of the conformal Lie algebra as a Lie algebra of polynomial vector fields. Following Koecher ([Koe69a,b]) one may define the conformal group to be the group of birational maps of V which preserve the conformal Lie algebra, and this definition even makes sense over quite general base fields. Since we are working over the base field of real numbers, we can avoid the assumption of rationality and consider just locally defined smooth maps which preserve the conformal Lie algebra. It follows automatically that they are birational. Thus we start with a *pseudogroup of diffeomorphisms* (which is a useful concept in differential geometry, cf. [Ko72]) and end up with a group. We show that this group is a Lie group whose Lie algebra is the conformal Lie algebra (however, keeping in mind the sign-convention from Appendix I.A.1 and being pedantic, we should say that $\mathfrak{co}(T)$ is the Lie algebra of the opposite group $\mathrm{Co}(T)^{op}$).

The term *conformal group* is explained by the property that its elements are $\mathrm{Str}(T)$-*conformal* in the sense that their total differential (at every regular point) belongs to a certain linear group, namely to the *structure group* $\mathrm{Str}(T)$. In the case of a Jordan algebra belonging to a non-degenerate quadratic form b, the structure group is precisely the group of similarities of b (cf. Section II.3), and the condition just mentioned is equivalent to conformality in its usual sense. However, $\mathrm{Str}(T)$-conformality is not sufficient for a characterization of the conformal group in the general case as shown by the example of the JTS $M(1,n;\mathbb{R})$: in this case, $\mathrm{Co}(T) = \mathbb{P}\,\mathrm{Gl}(n+1,\mathbb{R})$ is the projective group and $\mathrm{Str}(T) = \mathrm{Gl}(n,\mathbb{R})$ is the whole general linear group; thus $\mathrm{Str}(T)$-conformality is an empty condition. We introduce a property called *T-projectivity* which is satisfied by all elements of $\mathrm{Co}(T)$; this is a condition on *second* differentials generalizing the description of the usual projective group by a basic theorem in differential geometry due to H. Weyl.

The birational maps forming the conformal group in general have singularities on the underlying vector space V. Put another way: the vector fields forming the conformal Lie algebra are in general not complete vector fields on V. One can remove the singularities und make the vector fields complete by adding "points at infinity" to the vector space V; the space V^c thus obtained is called the *conformal completion of* V (w.r.t a JTS T). It contains V as an open dense domain. For example, in the case corresponding to the Jordan algebra of a positive definite quadratic form, the conformal completion is the sphere S^n, obtained by adding a single point to $\mathbb{R}^n$. In the case where the conformal group is the ordinary projective group the conformal completion is the ordinary imbedding of $\mathbb{R}^n$ into the projective space $\mathbb{RP}^n$. These and other classical examples are described explicitly in Section 4. In all these cases the space V^c is actually compact and may also be called the *conformal compactification* of V.

1. Conformal group: general properties

1.1. Definition of the conformal group and of the structure group. We assume in the following that T is a JTS on a vector space V and $\mathfrak{co}(T)$ its conformal Lie algebra, considered as a Lie algebra of quadratic vector fields on V.

Theorem VIII.1.1. *Let $\varphi : U \to \varphi(U) \subset V$ be a local ($\mathcal{C}^1$-regular) diffeomorphism of V such that*

$$\varphi_*(\mathfrak{co}(T)) = \mathfrak{co}(T),$$

i.e. for all $X \in \mathfrak{co}(T)$, the vector field $\varphi_*(X)$ coincides on $\varphi(U)$ with an element of $\mathfrak{co}(T)$. Then φ has a unique extension to a birational map $\widetilde{\varphi}$ of V such that $\widetilde{\varphi}_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$. The birational maps thus obtained form a group.

Proof. Recall formula (I.A.5):

$$(\varphi_* X)(p) = (\mathrm{D}\varphi^{-1}(p))^{-1} \cdot X(\varphi^{-1}(p)).$$

We remark first that, since $(\varphi^{-1})_* = (\varphi_*)^{-1} = \varphi^*$, the conditions $\varphi_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$ and $\varphi^*(\mathfrak{co}(T)) = \mathfrak{co}(T)$ are equivalent.

Let $X = \mathbf{v}$ with $v \in V$ (constant vector field). By assumption, $\varphi^*\mathbf{v}$ coincides on U with an element of $\mathfrak{co}(T)$. Since $\mathfrak{co}(T)$ is a Lie algebra of (quadratic) polynomial vector fields (Th. VII.2.4), the map $U \to V$, $p \mapsto (\varphi^*\mathbf{v})(p) = (\mathrm{D}\varphi(p))^{-1}v$ is a (quadratic) polynomial for all $v \in V$, and thus

$$d_\varphi : U \to \mathrm{Gl}(V) \subset \mathrm{End}(V), \quad p \mapsto d_\varphi(p) := (\mathrm{D}\varphi(p))^{-1}$$

is (quadratic) polynomial. Now consider the Euler operator $E^o \in \mathfrak{co}(T)$. The same arguments as before show that

$$n_\varphi : U \to V, \quad p \mapsto (\varphi^* E^o)(p) = (\mathrm{D}\varphi(p))^{-1} \cdot \varphi(p)$$

is (quadratic) polynomial. Therefore

$$\varphi : U \to V, \quad p \mapsto \varphi(p) = d_\varphi(p)^{-1} \cdot n_\varphi(p)$$

is rational. We extend n_φ and d_φ to polynomials on V, and define

$$\widetilde{\varphi}(x) := d_\varphi(x)^{-1} \cdot n_\varphi(x)$$

for all x with $\det d_\varphi(x) \neq 0$ (since $\det d_\varphi$ is a non-zero polynomial, this is a non-empty Zariski-open set). Then $\widetilde{\varphi}$ is a birational map, its inverse given by $\widetilde{\varphi^{-1}}$, and for all $X \in \mathfrak{co}(T)$,

$$((\widetilde{\varphi})^* X)(p) = (\mathrm{D}\widetilde{\varphi}(p))^{-1} X(\widetilde{\varphi}(p)) = d_\varphi(p) X(\widetilde{\varphi}(p))$$

is rational in p, coinciding for $p \in U$ with the polynomial $Y := \varphi^* X \in \mathfrak{co}(T)$ and coincides thus everywhere with Y. In other words, $(\widetilde{\varphi})^*(\mathfrak{co}(T)) \subset \mathfrak{co}(T)$. The same being true for φ^{-1}, we have actually equality.

If φ and ψ with $\varphi_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$ and $\psi_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$ are given, then (although φ and ψ may be not composable) $\widetilde{\varphi}$ and $\widetilde{\psi}$ are always composable on a dense open set, and

$$(\widetilde{\varphi} \circ \widetilde{\psi})_*(\mathfrak{co}(T)) = \widetilde{\varphi}_*(\widetilde{\psi}_*(\mathfrak{co}(T))) = \mathfrak{co}(T).$$

This, and the remark made above that $(\widetilde{\varphi})^{-1} = \widetilde{\varphi^{-1}}$, prove the last claim. ∎

Definition VIII.1.2. The group of birational maps of V defined in the preceding theorem is denoted by $\mathrm{Co}(T)$ and is called the *conformal group of* T. For any $g \in \mathrm{Co}(T)$, the quadratic polynomial

$$d_g : V \to \mathrm{End}(V),$$

defined by the property

$$d_g(x)v = (g^*\mathbf{v})(x) \quad (x, v \in V),$$

is called its *denominator*, and the quadratic polynomial

$$n_g : V \to V,$$

defined by the property

$$n_g(x) = (g^*E)(x),$$

is called its *numerator* (cf. the preceding proof). ∎

According to preceding proof, if g is non-singular at x, we have the relations

$$d_g(x) = \mathrm{D}g(x)^{-1}; \quad g(x) = d_g(x)^{-1} \cdot n_g(x). \tag{1.1}$$

Theorem VIII.1.3. *The representation*

$$* : \mathrm{Co}(T) \to \mathrm{Aut}(\mathfrak{co}(T)), \quad g \mapsto g_*$$

is injective, and its image $\mathrm{Co}_*(T)$ *is an open subgroup of* $\mathrm{Aut}(\mathfrak{co}(T))$*; in particular,* $\mathrm{Co}(T)$ *has the structure of a Lie group with Lie algebra* $\mathfrak{co}(T)$*. The center of* $\mathrm{Co}(T)$ *is trivial.*

Proof. We have seen above that $g(p) = d_g(p)^{-1} n_g(p)$ with $d_g(p)v = (g^*\mathbf{v})(p)$ and $n_g(p) = (g^*E)(p)$. This defines an inverse of $*$ from $\mathrm{Co}_*(T)$ onto $\mathrm{Co}(T)$; thus $*$ is injective.

In order to prove that $\mathrm{Co}_*(T)$ is open in $\mathrm{Aut}(\mathfrak{co}(T))$ we are going to describe some special elements in $\mathrm{Co}(T)$. For $X \in \mathfrak{co}(T)$ let φ_t be the local flow of X. If $X \in \mathfrak{m}^-$ or $X \in \mathfrak{str}(T)$, then the flows are global on V; we write $\exp(X) := \varphi_1$. Explicitly, we have $\exp(\mathbf{v}) = t_v$ (translation by v, cf. Eqn. (I.A.6)), and for $X \in \mathfrak{str}(T)$, $\exp(X)$ is given by the usual exponential of a matrix.

Next we consider $X \in \mathfrak{m}^+$. Since X vanishes of order 2 at 0, there is a neighborhood U of 0 in V such that $\varphi_t(x)$ is defined for $x \in U$ and $t = 1$ (cf. Appendix A to this chapter). We let

$$\exp X : U \to V, \quad x \mapsto \varphi_1(x).$$

Since $X \in \mathfrak{co}(T)$, it follows that $(\varphi_t)_*$ preserves $\mathfrak{co}(T)$ in the sense of Th.VIII.1.1; in particular $\exp(X) \in \mathrm{Co}(T)$. For simplicity of notation, we identify $\exp(X)$ with its rational continuation guaranteed by Th.VIII.1.1. Then the relation

$$\forall Y \in \mathfrak{co}(T) : \quad (\exp X)_* Y = e^{-\,\mathrm{ad}(X)} Y$$

holds (the sign is explained by Eqn. (I.A.6)). Summing up, $e^{\mathrm{ad}(X)}$ belongs to $\mathrm{Co}_*(T)$ for all $X \in \mathfrak{m}^{\pm}$ and $X \in \mathfrak{str}(T)$; thus the identity component of the group $\mathrm{Int}(\mathfrak{co}(T))$ of inner automorphisms, generated by $e^{\mathrm{ad}(\mathfrak{co}(T))}$, belongs to $\mathrm{Co}_*(T)$. But this is equal to the identity component of $\mathrm{Aut}(\mathfrak{co}(T))$ because $\mathrm{ad}(\mathfrak{co}(T)) = \mathrm{Der}(\mathfrak{co}(T))$ (Th. VII.3.6). Thus $\mathrm{Co}_*(T)$ is open in $\mathrm{Aut}(\mathfrak{co}(T))$.

Finally, if g belongs to the center of $\mathrm{Co}(T)$, then it commutes with all elements of $e^{\mathrm{ad}(\mathfrak{co}(T))}$; therefore $g_* = \mathrm{id}_{\mathfrak{co}(T)}$ and $g = \mathrm{id}_V$ since $*$ is injective. ∎

It is possible to give an algebraic description of the image of $*$ (cf. the Notes). We content ourselves here with the corresponding result for the *linear* part of the conformal group.

Definition VIII.1.4. The linear part

$$\mathrm{Str}(T) := \mathrm{Co}(T) \cap \mathrm{Gl}(V)$$

of the conformal group is called the *structure group* of T. Its closure in $\mathrm{End}(V)$ is called the *structure monoid.* ■

Proposition VIII.1.5.

(i) *The structure group has the following descriptions:*

$$\begin{aligned}\mathrm{Str}(T) &= \{g \in \mathrm{Gl}(V) \mid g_*(\mathfrak{m}^+) = \mathfrak{m}^+\}\\ &= \{g \in \mathrm{Gl}(V) \mid \exists g^\sharp \in \mathrm{Gl}(V) : \forall u,v,w \in V :\ gT(u,v,w) = T(gu, g^\sharp v, gw)\}.\end{aligned}$$

It is a closed subgroup of $\mathrm{Gl}(V)$ *whose Lie algebra is* $\mathfrak{str}(T)$.

(ii) *The homomorphism*

$$* : \mathrm{Str}(T) \to \{\varphi \in \mathrm{Aut}(\mathfrak{co}(T)) \mid \varphi(E) = E\}, \quad g \mapsto g_*$$

is a bijection.

Proof. (i) An element $g \in \mathrm{Gl}(V)$ normalizes $\mathfrak{co}(T)$ if and only if it normalizes $\mathfrak{m}^+$ and $\mathfrak{str}(T)$, and this is the case if and only if it normalizes $\mathfrak{m}^+$ (because $\mathfrak{str}(T)$ is the normalizer of $\mathfrak{m}^+$ in $\mathfrak{gl}(V)$). This explains the first description. The second description is deduced as in the proof of Th. VII.3.1: the condition $g_*(\mathfrak{m}^+) = \mathfrak{m}^+$ means that, for every $v \in V$, there exists $w \in V$ such that $g_*(\mathbf{p}_v) = \mathbf{p}_w$. If we denote by V_0 the kernel of the linear map $\mathbf{p} : V \to \mathfrak{m}^+, \quad v \mapsto \mathbf{p}_v$, and require w to be in a fixed complementary subspace V_1 of V_0, then the choice of w becomes unique. Now we define $g^\sharp$ to be arbitrary invertible on V_0 and let $g^\sharp(v) := w$ for $v \in V_1$. Then clearly the condition

$$g_*(\mathbf{p}_v) = \mathbf{p}_{g^\sharp v}$$

holds for all $v \in V$; it is equivalent to the relation given in the second description of $\mathrm{Str}(T)$.

It is clear from these descriptions that $\mathrm{Str}(T)$ is a closed (even algebraic) subgroup of $\mathrm{Gl}(V)$ and that $\mathfrak{str}(T)$ is its Lie algebra.

(ii) The homomorphism is injective by the preceding theorem. In order to prove that it is surjective, we note first that the condition $\varphi(E) = E$ implies that φ preserves the eigenspaces of $\mathrm{ad}(E)$. In particular, $\varphi(\mathfrak{m}^-) = \mathfrak{m}^-$, i.e. for all $\mathbf{v} \in \mathfrak{m}^-$ there exists a unique $\mathbf{w} \in \mathfrak{m}^-$ such that $\varphi(\mathbf{v}) = \mathbf{w}$. We define the linear map g by $g(v) := w$ and claim that $\varphi(X) = g_*X$ for all $X \in \mathfrak{co}(T)$. For $X \in \mathfrak{m}^-$ this is true by definition of g. For $X \in \mathfrak{str}(T)$, the condition

$$\varphi[\mathbf{v}, X] = [\varphi(\mathbf{v}), \varphi(X)]$$

yields $g(X(v)) = (\varphi(X))(g(v))$ for all $v \in V$. Applied once more, this argument proves our claim also for $X \in \mathfrak{m}^+$. Therefore $\varphi = g_*$, and now the condition $g_*(\mathfrak{m}^+) = \mathfrak{m}^+$ implies as above that $g \in \mathrm{Str}(T)$. ■

Proposition VIII.1.6. *If T is faithful and Θ denotes the involution of $\mathfrak{g}^b$ described in* Prop. VII.2.7, *then for all $g \in \mathrm{Str}(T)$ the element $g^\sharp$ is unique, it belongs again to* $\mathrm{Str}(T)$, *and $\sharp : g \mapsto g^\sharp$ is an involutive automorphism of* $\mathrm{Str}(T)$, *determined by the condition*

$$(g^\sharp)_* = \Theta \circ g_* \circ \Theta.$$

Proof. Everything follows by comparing the equations $g_*(\mathbf{p}_v) = \mathbf{p}_{g^\sharp v}$ and

$$g_*\mathbf{p}_v = g_*(\Theta(-\mathbf{v})) = \Theta \circ (\Theta \circ g_* \circ \Theta)(-\mathbf{v}) = \mathbf{p}_{(\Theta \circ g_* \circ \Theta)(\mathbf{v})}$$

and noting that $g \mapsto g_*$ is injective. ■

If T is semisimple, then the involution $\sharp$ is given by

$$g^\sharp = (g^*)^{-1},$$

where the adjoint is taken w.r.t. the trace form; the proof is the same as of Prop. VII.3.3.

1.2. Conformality. We will make precise in which sense the conformal group is indeed "conformal". The question whether all conformal maps belong to the group $\mathrm{Co}(T)$ leads to the *Liouville theorem* to be discussed in the next chapter.

Definition VIII.1.7. Let V be a vector space and $G \subset \mathrm{Gl}(V)$ a closed subgroup. We say that a locally defined diffeomorphism g of V is G*-conformal*, if, for all x where g is defined,

$$\mathrm{D}g(x) \in G. \qquad \blacksquare$$

Theorem VIII.1.8. *All elements of* $\mathrm{Co}(T)$ *are* $\mathrm{Str}(T)$*-conformal.*

Proof. Given $x \in V$ with $\det d_g(x) \neq 0$, we let $y := g(x)$ and $g' := t_{-y} \circ g \circ t_x$. Then $g' \in \mathrm{Co}(T)$, $g'(0) = 0$ and $\mathrm{D}g'(0) = \mathrm{D}g(x)$. Replacing g by g', we may assume that $x = 0$ and $g(0) = 0$.

We claim that then $g_*(\mathfrak{m}^+) \subset \mathfrak{m}^+$. In order to prove this, note that $\mathfrak{m}^+$ is the subalgebra of elements of $\mathfrak{co}(T)$ vanishing of order 2 at the origin. Differentiating the expression $(g_*X)(p) = \mathrm{D}g(g^{-1}(x)) \cdot X(g^{-1}(p))$, we obtain

$$\begin{aligned}\mathrm{D}(g_*X)(p) = \big(\mathrm{D}^2 g(g^{-1}(p)) \cdot X(g^{-1}(p))\big) \circ \mathrm{D}g^{-1}(p) \\ + \mathrm{D}g(g^{-1}(p)) \circ \mathrm{D}X(g^{-1}(p)) \circ \mathrm{D}g^{-1}(p).\end{aligned} \tag{1.2}$$

It follows that, if $g(0) = 0$, $X(0) = 0$ and $\mathrm{D}X(0) = 0$, then $\mathrm{D}(g_*X)(0) = 0$. Since also $(g_*X)(0) = 0$, we have $g_*(\mathfrak{m}^+) \subset \mathfrak{m}^+$.

Next we want to show that for all $X \in \mathfrak{m}^+$, $g_*X = (\mathrm{D}g(0))_*X$. Since both sides of this equality are homogeneous quadratic vector fields, this is equivalent to the statement that

$$\mathrm{D}^2(g_*X)(0) = \mathrm{D}^2((\mathrm{D}g(0))_*X)(0).$$

In order to prove this, we differentiate the expression (1.2) once again and evaluate at 0. Then, under our assumptions on g and X, only one term remains, namely

$$\mathrm{D}^2(g_*X)(0) = \mathrm{D}g(0) \circ \mathrm{D}^2 X(0) \circ (\mathrm{D}g(0)^{-1} \otimes \mathrm{D}g(0)^{-1}).$$

If we replace in this expression g by $\mathrm{D}g(0)$, the right-hand side does not change, and therefore $g_*X = (\mathrm{D}g(0))_*X$.

We have shown that $\mathfrak{m}^+ = g_*(\mathfrak{m}^+) = (\mathrm{D}g(0))_*(\mathfrak{m}^+)$. According to Prop. VIII.1.5 (i), $\mathrm{D}g(0)$ belongs thus to the structure group. $\blacksquare$

Corollary VIII.1.9. *For all* $g \in \mathrm{Co}(T)$*, the image of the polynomial* d_g *is contained in the structure monoid.*

Proof. This is an immediate consequence of the definition of the structure monoid as the closure of the structure group in $\mathrm{End}(V)$. $\blacksquare$

1.3. Projectivity. Conformality was defined by a condition on the *first* differential. Similarly, projectivity will be defined by an additional condition on the *second* differential. The question whether all transformations which are projective in this sense belong to $\mathrm{Co}(T)$ leads to the *fundamental theorem* (Ch. IX).

Definition VIII.1.10. A locally defined diffeomorphism g of V is called T*-projective* if it is $\mathrm{Str}(T)$-conformal and in addition for all x where g is defined,

$$(\mathrm{D}g(x))^{-1} \circ \mathrm{D}^2 g(x) \in W,$$

where $W = \{T(\cdot, v, \cdot) | \, v \in V\} \subset \mathrm{Hom}(V \otimes V, V)$ is the subspace introduced in Def. VII.3.4. ■

We give a more conceptual definition of this condition: it is equivalent to requiring that

$$(g^* \nabla - \nabla)_x \in W, \tag{1.3}$$

where ∇ is the canonical flat connection of V (cf. Eqn. (I.B.4)). In other words, $g^*\nabla - \nabla$ is a section of the subbundle $\mathbf{W}$ of $\mathrm{Hom}(S^2(TV), V)$ with constant fiber $\mathbf{W}_p = W$. Conformality means just that g preserves $\mathbf{W}$. Therefore, if both g and h are T-projective, then

$$(gh)^* \nabla - \nabla = h^*(g^* \nabla - \nabla) + g^* \nabla - \nabla$$

is again a section of $\mathbf{W}$. This means that the composition of two T-projective maps, if it is defined, is again T-projective. In other words, locally defined T-projective diffeomorphisms form a *pseudogroup of diffeomorphisms* (cf. [Ko72] for the formal definition).

Theorem VIII.1.11. *All elements of* $\mathrm{Co}(T)$ *are* T*-projective.*

Proof. We know already that all elements of $\mathrm{Co}(T)$ are $\mathrm{Str}(T)$-conformal (Th. VIII.1.8). In order to prove T-projectivity, we reduce as in the proof of Th. VIII.1.8 to the case $x = 0$ and $g(x) = x$; composing with $\mathrm{D}g(0)^{-1}$, we may further reduce to the case $\mathrm{D}g(0) = \mathrm{id}_V$.

We specialize Eqn. (1.2) to the Euler operator $X = E$:

$$\mathrm{D}(g_* E)(p) = \big(\mathrm{D}^2 g(g^{-1}(p)) \cdot g^{-1}(p)\big) \circ \mathrm{D}g^{-1}(p) + \mathrm{id}_V .$$

Thus, under our assumptions on g, $\mathrm{D}(g_*E)(0) = \mathrm{id}_V$. Differentiating further and evaluating at 0, we find that

$$\mathrm{D}^2(g_*E)(0) = (\mathrm{D}^2 g)(0).$$

Since $g_*E \in \mathfrak{co}(T)$ by definition of $\mathrm{Co}(T)$, it follows from Prop. VII.3.5 that $(\mathrm{D}^2 g)(0) \in W$. ■

2. Conformal group: fine structure

2.1. Generators of the conformal group and generalized Harish-Chandra decomposition.

Proposition VIII.2.1. *An element* $g \in \mathrm{Co}(T)$ *is uniquely determined by its* 2*-jet (i.e. by the values* $g(p)$*,* $\mathrm{D}g(p)$ *and* $\mathrm{D}^2 g(p)$*) at one point* $p \in V$ *with* $\det d_g(p) \neq 0$*.*

Proof. Let $g_1, g_2 \in \mathrm{Co}(T)$ have the same 2-jet at a point $p \in V$. Then $g := g_1^{-1} g_2$ has the 2-jet of the identity at p. Replacing g by $t_{-g(p)} \circ g \circ t_p$, we may further assume that $p = 0$. We thus have to show that the only element $g \in \mathrm{Co}(T)$ with $g(0) = 0$, $\mathrm{D}g(0) = \mathrm{id}_V$ and $\mathrm{D}^2 g(0) = 0$ is the identity. The arguments proving Th. VII.1.11 show that, under these conditions on g, $g_* E = E$. Then, for all $v \in V$,

$$[E, g_* \mathbf{v}] = [g_* E, g_* \mathbf{v}] = g_*[E, \mathbf{v}] = -g_* \mathbf{v}.$$

However, the only solutions of the differential equation $[E, X] = -X$ are the constant vector fields, therefore $(g_* \mathbf{v})(p) = (\mathrm{D}g^{-1}(p))^{-1} v$ is constant; this means that $\mathrm{D}g^{-1} : V \to \mathrm{End}(V)$ is constant and thus g^{-1} is linear. Since $\mathrm{D}g(0) = \mathrm{id}_V$ by assumption, this implies that $g = \mathrm{id}_V$. ■

Proposition VIII.2.2. *Let* $\mathrm{Co}'(T) := \{g \in \mathrm{Co}(T) | \, d_g(0) \neq 0\}$. *Then the map*

$$\kappa : \mathrm{Co}'(T) \to V \times \mathrm{Str}(T) \times W, \quad g \mapsto (g(0), \mathrm{D}g(0), \mathrm{D}g(0)^{-1} \circ \mathrm{D}^2 g(0))$$

is bijective.

Proof. The map is well-defined since g is regular at 0 by definition of $\mathrm{Co}'(T)$, and $\mathrm{D}g(0)$ and $\mathrm{D}g(0)^{-1} D^2 g(0)$ lie in the correct spaces according to Th. VIII.1.8 and VIII.1.11. Since these data describe precisely the 2-jet of g, the preceding proposition implies now that κ is injective.

In order to prove that κ is surjective, we have to calculate the 2-jet of the element $\exp X$ with $X \in \mathfrak{m}^+$ (defined in the proof of Th. VIII.1.3). Note first that $(\exp X)(0) = 0$ and $(\mathrm{D} \exp X)(0) = \mathrm{id}_V$ (Lemma VIII.A.3). Therefore, as we have seen in the proof of Th. VIII.1.11,

$$\mathrm{D}^2(\exp X)(0) = \mathrm{D}^2((\exp X)_* E)(0),$$

and we can calculate

$$\begin{aligned}\mathrm{D}^2(\exp X)(0) &= \mathrm{D}^2(e^{-\operatorname{ad}(X)} E)(0) \\ &= \mathrm{D}^2(E - [X, E])(0) \\ &= -\mathrm{D}^2(X)(0)\end{aligned}$$

since $[X, E] = -X$ for all $X \in \mathfrak{m}^+$ und thus $\operatorname{ad}(X)^k E = 0$ for all $k > 1$.

Now, given $v, w \in V$, $h \in \mathrm{Str}(T)$, we let $g := t_v \circ h \circ \exp(\mathbf{p}_w) \in \mathrm{Co}(T)$ and prove that $\kappa(g) = (v, h, 2T(\cdot, w, \cdot))$: In fact, $g(0) = v$, $\mathrm{D}g(0) = h$ and $\mathrm{D}g(0)^{-1} \circ \mathrm{D}^2 g(0) = (\mathrm{D}^2 \exp(\mathbf{p}_w))(0) = \mathrm{D}^2(\mathbf{p}_w)(0) = 2T(\cdot, w, \cdot)$. ■

We collect the preceding results. By

$$\mathrm{Co}(T)_0 := \{g \in \mathrm{Co}(T) | \, d_g(0) \neq 0, g(0) = 0\}$$

we denote the "stabilizer of the base point", and by

$$\mathrm{Co}(T)_{00} := \{g \in \mathrm{Co}(T)_0 | \, \mathrm{D}g(0) = \mathrm{id}_V\}$$

the kernel of the first isotropy representation $g \mapsto \mathrm{D}g(0)$ of $\mathrm{Co}(T)_0$. These are subgroups of $\mathrm{Co}(T)$ lying in $\mathrm{Co}'(T)$.

Theorem VIII.2.3. $\mathrm{Co}(T)$ *is a Lie group with Lie algebra* $\mathfrak{co}(T)$. *It is generated by* $\exp(\mathfrak{m}^-)$, $\exp(\mathfrak{m}^+)$ *and* $\mathrm{Str}(T)$. *More precisely:*

(1) $\mathrm{Co}(T)_{00} = \exp(\mathfrak{m}^+)$.

(2) $\mathrm{Co}(T)_0 = \mathrm{Str}(T) \exp(\mathfrak{m}^+)$*; this is a semidirect product.*

(3) $\mathrm{Co}'(T) = \exp(\mathfrak{m}^-) \, \mathrm{Str}(T) \exp(\mathfrak{m}^+)$, *and this decomposition is unique, i.e. every element* g *of the open dense set* $\mathrm{Co}'(T) \subset \mathrm{Co}(T)$ *has a unique decomposition*

$$g = t_{g(0)} \, \mathrm{D}g(0) \, \exp(\mathbf{p}_w), \qquad t_{g(0)} \in \exp(\mathfrak{m}^-), \, \mathrm{D}g(0) \in \mathrm{Str}(V), \, \mathbf{p}_w \in \mathfrak{m}^+.$$

(4) $\mathrm{Co}(T) = \exp(\mathfrak{m}^-) \, \mathrm{Str}(T) \exp(\mathfrak{m}^+) \exp(\mathfrak{m}^-)$ *(the corresponding decomposition is not unique).*

(5) *If* T *is faithful, then* $\mathrm{Co}(T)$ *carries an involution* Θ *such that* $\Theta(\mathrm{Co}'(T))^{-1} = \mathrm{Co}'(T)$ *and for all* $v, w \in V$, $h \in \mathrm{Str}(T)$,

$$\Theta(t_v h \exp(\mathbf{p}_w))^{-1} = t_w (h^\sharp)^{-1} \exp(\mathbf{p}_v). \tag{2.1}$$

The decomposition from part (3) *can be written*

$$g = t_{g(0)} \, \mathrm{D}g(0) \, \Theta(t_{-\Theta(g)^{-1}(0)}), \tag{2.2}$$

and the relation

$$\mathrm{D}(\Theta(g^{-1}))(0) = \Theta(\mathrm{D}g(0))^{-1} \tag{2.3}$$

holds.

(6) *If* T *is associated to a unital Jordan algebra with Jordan inversion* j *via* $T(u,v,w) = 2(u(vw) - v(uw) + (uv)w)$, *then* $\mathrm{Co}(T)$ *is generated by the translations, elements of* $\mathrm{Str}(T)$ *and* j. *In this case,* T *is faithful and* Θ *is given by conjugation with* j.

Proof. Claims (1), (2) and (3) are a direct consequence of Prop. VIII.2.2.

(4) If $g \in \mathrm{Co}(T)$, we choose a point p with $d_g(p) \neq 0$; we let $g' := g \circ t_p$; then $d_{g'}(0) \neq 0$, and we can apply (3).

(5) The involutive automorphism Θ of $\mathfrak{co}(T)$ described in Prop. VII.2.7 induces by conjugation an involution of the group $\mathrm{Aut}(\mathfrak{co}(T))$ which is again denoted by Θ. We have to show that the subgroup $\mathrm{Co}_*(T) \subset \mathrm{Aut}(\mathfrak{co}(T))$ is stable under Θ. But this is easily checked on the generators of $\mathrm{Co}(T)$ exhibited in Part (4):

$$\Theta(t_v) = \Theta(\exp(\mathbf{v})) = \exp(\Theta(\mathbf{v})) = \exp(-\mathbf{p}_v)$$

and $\Theta(g) = g^\sharp$ for $g \in \mathrm{Str}(T)$ (cf. Prop. VIII.1.6). This also proves Equation (2.1) and that $\Theta(\mathrm{Co}'(T))^{-1} = \mathrm{Co}'(T)$.

If $g = vhn$ abbreviates the decomposition (3) of $g \in \mathrm{Co}'(T)$, then

$$\Theta(g)^{-1} = \Theta(n)^{-1}\Theta(h)^{-1}\Theta(v)^{-1}$$

is the corresponding decomposition of $\Theta(g)^{-1}$, and the uniqueness statement in (3) implies that $\Theta(n)^{-1} = t_{\Theta(g)^{-1}(0)}$ and $\Theta(h)^{-1} = \mathrm{D}(\Theta(g)^{-1})(0)$. This yields formulas (2.2) and (2.3).

(6) Recall (Prop. II.2.9) that j is a birational map of V with differential $\mathrm{D}j(x) = -Q(x)$. Thus, for $v \in V$, $jt_{tv}j$ is a one-parameter group of birational maps of V. It is generated by the vector field

$$X(p) = \frac{d}{dt}|_{t=0} j(tv + jp) = \mathrm{D}j(j(p)) \cdot v = -Q(x)v = -\mathbf{p}_v(p).$$

It follows that $\mathfrak{m}^+ = j_*(\mathfrak{m}^-)$; therefore $j_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$ and $j \in \mathrm{Co}(T)$. Comparing with Prop. VII.2.7, we see that $\Theta = j_*$. ∎

2.2. Quasi-inverse and Bergman operator. We are going to derive explicit formulae for the objects introduced in the preceding sections. For any JTS T on a vector space V we define:

Definition VIII.4.1.

(i) For $y \in V$ we let

$$\widetilde{t}_y := \exp(\mathbf{p}_y) \in \exp(\mathfrak{m}^+).$$

If $\widetilde{t}_y$ is defined at $x \in V$, then $x^y := \widetilde{t}_y(x)$ is called the *quasi-inverse of* x *with respect to* y.

(ii) The polynomial

$$B : V \times V \to \mathrm{End}(V), \quad (x,y) \mapsto B(x,y) := \mathrm{id}_V - T(x,y) + P(x)P(y),$$

where $P(z) = \frac{1}{2}T(z,\cdot,z)$, is called the *Bergman operator of the JTS* T. ∎

The term "quasi-inverse" is explained in [Lo77, Ch. 7]. If T is faithful, then $\widetilde{t}_y = \Theta(t_y)^{-1}$. In the general case, it would be more conceptual to define $\widetilde{t}_y$ for $y \in W$ with W as in Def. VII.3.4 and not for $y \in V$, and similarly to consider the Bergman operator as a map $B : V \times W \to \mathrm{End}(V)$.

Proposition VIII.2.5.

(i) *For all* $x, y \in V$,

$$d_{\exp(\mathbf{p}_y)}(x) = B(x,y).$$

In particular, $\exp(\mathbf{p}_y)$ *is regular at* x *if and only if* $B(x,y)$ *is invertible, and then*

$$B(x,y)^{-1} = (\mathrm{D}(\exp(\mathbf{p}_y)))(x) = (\mathrm{D}(\exp(\mathbf{p}_y \circ t_x))(0).$$

(ii) *For all* $x, y \in V$,

$$n_{\exp(\mathbf{p}_y)}(x) = x - P(x)y.$$

(iii) *If* $\det B(x,y) \neq 0$, *then* $B(x,y) \in \mathrm{Str}(T)$ *and*

$$\exp(\mathbf{p}_y)(x) = B(x,y)^{-1}(x - P(x)y).$$

Proof. (i) We are going to use the formula $[\mathbf{p}_a, \mathbf{b}] = -T(b,a)$ (Eqn. (VII.2.1)) and the equivariance property $[X, \mathbf{p}_a] = -\mathbf{p}_{X^\sharp a}$. According to the definition of the denominators, we have for all $v \in V$,

$$\begin{aligned}
(d_{\exp(\mathbf{p}_y)}(x)) \cdot v &= (\exp(\mathbf{p}_y)^* \mathbf{v})(x) = (e^{\mathrm{ad}(\mathbf{p}_y)}\mathbf{v})(x) \\
&= (\mathbf{v} + [\mathbf{p}_y, \mathbf{v}] + \frac{1}{2}[\mathbf{p}_y, [\mathbf{p}_y, \mathbf{v}]])(x) \\
&= v - T(v,y,x) + \frac{1}{2}[T(v,y), \mathbf{p}_y](x) \\
&= v - T(x,y)v + \frac{1}{2}\mathbf{p}_{T(y,v,y)}(x) \\
&= v - T(x,y)v + \frac{1}{4}T(x, T(y,v,y), x) = B(x,y)v.
\end{aligned}$$

The following claim now is a consequence of Eqn. (1.1).

(ii) We calculate the numerator

$$\begin{aligned}
n_{\exp(\mathbf{p}_y)}(x) &= (\exp(\mathbf{p}_y)^* E)(x) = (e^{\mathrm{ad}(\mathbf{p}_y)}E)(x) \\
&= x + [\mathbf{p}_y, E](x) \\
&= x - \mathbf{p}_y(x) = x - P(x)y.
\end{aligned}$$

(iii) According to Th. VIII.1.8, $d_{\exp(\mathbf{p}_y)}(x)$ belongs to the structure group whenever it is non-singular. Thus the first claim follows from Part (i). The second claim follows from (i) and (ii) because $\exp(\mathbf{p}_y)(x) = (d_{\exp(\mathbf{p}_y)}(x))^{-1} \cdot n_{\exp(\mathbf{p}_y)}(x)$. ∎

Proposition VIII.2.6.

(i) *If* T *is faithful and* $\det B(x,y) \neq 0$, *then*

$$B(x,y)^{-1} = \Theta(B(y,x)).$$

(ii) *If* T *comes from a unital Jordan algebra, we have for all invertible elements* x,

$$B(x,y) = P(x)P(j(x) - y).$$

Proof. (i) From part (i) of the preceding proposition together with Eqn. (2.3), letting $g = \exp(\mathbf{p}_x)t_y$, we get

$$\begin{aligned}
\Theta(B(y,x)) &= \Theta(\mathrm{D}(\exp(\mathbf{p}_x)t_y)(0))^{-1} = \mathrm{D}(\Theta(t_{-y}\exp(\mathbf{p}_{-x})))(0) \\
&= \mathrm{D}(\exp(\mathbf{p}_y t_x))(0) = B(x,y)^{-1}.
\end{aligned}$$

(ii) Using that $\Theta = j_*$ is conjugation by j (Th. VIII.2.3 (6)) and the formula $\mathrm{D}j(s) = -P(x)^{-1}$ together with the chain rule, we have

$$\begin{aligned} B(x,y)^{-1} &= (\mathrm{D}\exp(\mathbf{p}_y))(x) = (\mathrm{D}\Theta(-t_y))(x) \\ &= (\mathrm{D}(jt_{-y}j))(x) = \mathrm{D}j(j(x)-y)\circ \mathrm{D}j(x) \\ &= P(j(x)-y)^{-1}P(x)^{-1}. \end{aligned}$$

Taking inverses, we obtain the claim. ■

Proposition VIII.2.7. *Let T be faithful. Then, for all $x, y \in V$ with* $\det B(x,-y) \neq 0$*,*

$$\Theta(t_y)t_x = t_{\Theta(t_y)x}\; B(x,-y)^{-1}\; \Theta(t_{\Theta(t_x)(y)}).$$

Proof. Since $\det B(x,-y) \neq 0$, $\Theta(t_y)t_x$ belongs to $\mathrm{Co}'(T)$. Now the unique decomposition according to formula (2.2) along with $\mathrm{D}(\Theta(t_y)t_x)(0) = B(x,-y)^{-1}$ and $(-\mathrm{id})\,\Theta(t_z)\,(-\mathrm{id}) = \Theta(t_{-z})$ yields the claim. ■

The analogs of VIII.2.6 (i) and VIII.2.7 hold also for general JTS, and moreover one can show that any three homomorphisms of the groups $\exp(\mathfrak{m}^-)$, $\mathrm{Str}(T)$ and $\exp(\mathfrak{m}^+)$ into an arbitrary group Γ which are compatible with the relation just proved, have a unique common extension to a group homomorphism $\mathrm{Co}(T) \to \Gamma$ (cf. [Lo77, Th.8.11 (b)]). In this sense the preceding proposition completes the description of the conformal group by generators and relations.

Next we record the covariance properties of the Bergman-polynomial and of the quasi-inverse with respect to the conformal group. We assume in the following that T is faithful. For $g \in \mathrm{Co}(T)$ and $x \in V$ with $\det d_g(x) \neq 0$, we have the canonical decomposition of gt_x from Eqn. (2.2):

$$gt_x = t_{gx}\, p(g,x) \tag{2.5}$$

with

$$p(g,x) = \mathrm{D}g(x)\Theta(t_{-\Theta(gt_x)^{-1}(0)}) \in \mathrm{Co}(T)_0 = \mathrm{Str}(T)\exp(\mathfrak{m}^+). \tag{2.6}$$

The term $p(g,x)$ can be interpreted as a version of the second order tangent map of g at x, cf. Prop. VIII.2.2.

Proposition VIII.4.5. *For all $x, y \in V$ and $g \in \mathrm{Co}(T)$ for which the following expressions are defined, we have the equalities*

$$\Theta(t_{\Theta(g)y})^{-1}t_{gx} = \Theta\big(p(\Theta(g),y)\big)\;\; \Theta(t_y)^{-1}t_x\;\; p(g,x)^{-1}, \tag{2.7}$$

$$\Theta(t_{\Theta(g)y})^{-1}(gx) = \Theta\big(p(\Theta(g),y)\big)\,(\Theta(t_y)^{-1}(x)), \tag{2.8}$$

$$B(gx,\Theta(g)y) = \mathrm{D}g(x)\, B(x,y)\;\, \Theta(\mathrm{D}(\Theta(g))(y))^{-1}. \tag{2.9}$$

Proof. With $t_{gx} = gt_x p(g,x)^{-1}$ and $t_{\Theta(g)y} = \Theta(g)t_y p(\Theta(g),y)^{-1}$ we get

$$\begin{aligned} \Theta(t_{\Theta(g)y})^{-1}t_{gx} &= \Theta\big(\Theta(g)t_y p(\Theta(g),y)^{-1}\big)^{-1}\; gt_x\, p(g,x)^{-1} \\ &= \Theta\big(p(\Theta(g),y)\big)\,\Theta(t_y)^{-1}t_x\, p(g,x)^{-1}, \end{aligned}$$

proving (2.7). Now, (2.7) is an equation between conformal transformations, and taking the value of these transformations at the base point 0 and noting that $p(g,x)^{-1}(0) = 0$ yields (2.8). Taking the first total differential in (2.7) together with the relations $\mathrm{D}(\Theta(t_u)^{-1}t_v)(0) = B(v,u)^{-1}$ (Prop. VIII.2.5 (i)), $(\mathrm{D}p(g,x))(0) = \mathrm{D}g(x)$ and noting that $\Theta(p(\Theta(g),y))$ is an affine map of V and thus has constant differential yields (2.9). ■

Remark VIII.2.9. (1) Specialization of the covariance properties from the preceding proposition to elements of the form $g = B(u,v)$, $g = t_u$ or $g = \Theta(t_u)$ gives interesting algebraic identities – see Appendix B to this chapter.
(2) The covariance properties just proved can be understood as invariance properties of sections of certain bundles over the conformal completion (cf. next section). ■

3. The conformal completion and its dual

3.1. The conformal completion. We denote by $P := \mathrm{Str}(T)\exp(\mathfrak{m}^+)$ the stabilizer $\mathrm{Co}(T)_0$ and by $\mathfrak{p} := \mathfrak{str}(T) \oplus \mathfrak{m}^+$ its Lie algebra. Then P is closed in $\mathrm{Co}'(T)$ and in $\mathrm{Co}(T)$.

Definition VIII.3.1. The space $V^c := \mathrm{Co}(T)/P$ as well as the map

$$V \to V^c, \quad v \mapsto t_v P$$

are called the *conformal completion of* V (w.r.t T). If V^c is compact, then it is also called the *conformal compactification of* V. ■

Proposition VIII.3.2. *The conformal completion is an imbedding with open dense image.*

Proof. The map $V \to V^c$ is injective since $P \cap t_V = \{\mathrm{id}_V\}$. Its image is open dense since it is the image of the open dense set $\mathrm{Co}'(T) \subset \mathrm{Co}(T)$ under the surjective submersion $\pi : \mathrm{Co}(T) \to V^c$, $g \mapsto gP$. ■

We identify V with the corresponding open dense domain of V^c. It is clear that $\mathfrak{co}(T)$ is realized as an algebra of vector fields on V^c, and Thm. VIII.1.1 now says that $\mathrm{Co}(T)$ is precisely the group normalizing this algebra. It suffices to assume that diffeomorphisms normalizing $\mathfrak{co}(T)$ are only locally defined in order to conclude a global continuation onto V^c; in particular, all elements of $\mathfrak{co}(T)$ are complete vector fields on V^c, and in this sense V^c is "complete". It is known that, if T is non-degenerate, then P is a parabolic subgroup of the semisimple group $\mathrm{Co}(T)$ and hence V^c is compact (cf. Section X.6).

3.2. The dual space. Let us denote by $\mathrm{Aff}(V)$ the affine group of the vector space V and by

$$P' := \mathrm{Co}(T) \cap \mathrm{Aff}(V)$$

the group of affine conformal maps. The elements of P' are defined at 0, and from Th. VIII.2.3 we get $P' = t_V\,\mathrm{Str}(T)$.

Definition VIII.3.3. The space $W^c := \mathrm{Co}(T)/P'$ as well as the map

$$W \to W^c, \quad w \mapsto \exp(w)P'$$

(where we identify W with $\mathfrak{m}^+$ by identifying an element $X \in W \subset \mathrm{Hom}(S^2V, V)$ with the vector field $X(p) := X(p,p)$; cf. Def. VII.3.4) are called the *dual space of* V^c. ■

As above it is shown that $W \to W^c$ is an imbedding with open dense image. If T is faithful, then $P' = \Theta(P)$, and therefore $W^c \to V^c$, $gP' \mapsto \Theta(g)P$ is an isomorphism onto V^c with the action $g.x := (\Theta(g))(x)$.

The dual space W^c can be interpreted as the *space of conformal affinizations of* V^c: the inclusion $V \subset V^c$ provides a chart of an open dense domain of the conformal completion V^c. An element $g \in \mathrm{Co}(T)$ is an automorphism of this affine chart iff it preserves V together with its affine structure, i.e. iff it is an element of the group P'. Therefore the collection of all affine charts of the form "$V \subset V^c$" can be identified with $\mathrm{Co}(T)/P'$.

In the semisimple case, the interpretation of the dual space shows an even closer analogy with the case of classical projective geometry: the complement of the open dense set $V \subset V^c$ plays the role of a "hyperplane at infinity"; we denote it by

$$H_\infty := V^c \setminus V.$$

The group P' preserves H_∞; the converse is true if and only if T is semisimple (because an element of $\exp(\mathfrak{m}^+)$ preserves V if and only if it corresponds to an element in the radical of T, cf. [Lo75, 4.1]). Thus in the semisimple case W^c may also be interpreted as the "space of hyperplanes of type H_∞". The algebraic equations of these hyperplanes in the general case are as follows:

Proposition VIII.3.4. *For all $h \in \mathrm{Co}(T)$,*

$$h^{-1}.H_\infty = \{x \in V^c | \, x = g(0), g \in \mathrm{Co}(T), \det d_{hg}(0) = 0\};$$

in particular, $H_\infty = \{g(0) | \, g \in \mathrm{Co}(T), \det d_g(0) = 0\}$. In Jordan coordinates w.r.t. the origin, $h.H_\infty$ is described by the equation

$$V \cap h^{-1}.H_\infty = \{x \in V | \, \det d_h(x) = 0\}.$$

Proof. The point $g(0)$ belongs to V if and only if $\det d_g(0) \neq 0$: in fact, if $g(0) \in V$, then g is a local diffeomorphism of a neighborhood of 0 into V, and therefore $\det \mathrm{D}g(0) \neq 0$; conversely, if $\det d_g(0) \neq 0$, then the formula $g(x) = d_g(x)^{-1} n_g(x)$ defines locally on a neighborhood of 0 a diffeomorphism with values in V, and thus in particular $g(0)$ belongs to V. Therefore V is characterized by the condition $\det d_g(0) \neq 0$ and H_∞ by the condition $\det d_g(0) = 0$.

The description of $h^{-1}.H_\infty$ follows immediately, and if we choose $g = t_x$, we get the "affine" equation of this hyperplane. ■

In the important special case where T is associated to a Jordan algebra with Jordan inverse j, we get

$$V \cap j.H_\infty = \{x \in V | \, \det P(x) = 0\}. \tag{3.1}$$

The set $j.H_\infty$ is invariant under P since H_∞ is invariant under P'. Thus, if $g(0) =: a \in V$, then

$$V \cap g^{-1} j.H_\infty = \{x \in V | \, \det P(x-a) = 0\} \tag{3.2}$$

because w.l.o.g. we may choose $g = t_a$. In the case of a general JTS we have

$$V \cap \widetilde{t}_v^{-1}.H_\infty = \{x \in V | \, \det B(x,v) = 0\} \tag{3.3}$$

and

$$V \cap (\widetilde{t}_v t_w)^{-1}.H_\infty = \{x \in V | \, \det B(x+w,v) = 0\}. \tag{3.4}$$

Since $t_V \times \widetilde{t}_V \to W^c$, $(t_v, \widetilde{t}_w) \to t_v \widetilde{t}_w P'$ is surjective, the last equation is already the most general equation of a "hyperplane of type H_∞". The equations given above can be interpreted in terms of *vector bundles*. Recall from Appendix VIII.B the relation between representations induced from subgroups Q of a group L and sections of vector bundles over L/Q.

Lemma VIII.3.5. *The formula $F_0(g) := d_g(0)$ defines a t_V-invariant element of the representation $\mathrm{ind}_P^{\mathrm{Co}(T)}\pi_0$ induced from the representation*

$$\pi_0 : P \to \mathrm{Gl}(\mathrm{End}(V)), \quad p \mapsto (X \mapsto \mathrm{D}p(0) \circ X).$$

Proof. The transformation property under P follows from the equation

$$F_0(gp) = \mathrm{D}p(0)^{-1} d_g(0) = \pi_0(p)^{-1} F_0(g)$$

and t_V-invariance from $d_{t_v g}(0) = d_g(0)$. ∎

The lemma implies that $g \mapsto \det d_g(0)$ defines a t_V-invariant section of the bundle induced from $\det \circ \pi_0$, and H_∞ is precisely the set of zeroes of this section. As explained in Appendix B, elements of the representation induced from the first isotropy representation $p \mapsto \mathrm{D}p(0)$ correspond to vector fields; therefore elements of the representation induced from π_0 correspond to sections of the bundle $T(V^c) \otimes V^*$ (where V^* is considered as a trivial bundle over V^c), and elements induced from $\det \circ \pi$ correspond to sections of the bundle $\Lambda^n T(V^c)$, the dual of the canonical bundle $\Lambda^n T^*(V^c)$.

3.3. The global polarized space. We let the group $\mathrm{Co}(T)$ act on the direct product $V^c \times W^c$ by $g.(v, w) := (gv, gw)$. The origin $(0_V, 0_W)$ of $V^c \times W^c$ is often also denoted by $(0, 0)$.

Definition VIII.3.6. A point $(x, y) \in V^c \times W^c$ is called *finite* if x belongs to the range of the affine chart defined by y; i.e., if $y = h.0_W \in W^c$ with $h \in \mathrm{Co}(T)$, then x belongs to the open dense subset $U_y := h.V$ of V^c (which is independent of the choice of h). ∎

The condition $x \in h.V$ is equivalent to $g.x \in gh.V$ for all $g \in \mathrm{Co}(T)$, and therefore the set of finite points is a $\mathrm{Co}(T)$-invariant set in $V^c \times W^c$.

Theorem VIII.3.7. *The set of finite points in $V^c \times W^c$ is precisely the orbit*

$$X := \mathrm{Co}(T).(0, 0) \subset V^c \times W^c.$$

The orbit X is open dense in $V^c \times W^c$ and can be described as

$$X = \{(g.0, h.0) |\, g, h \in \mathrm{Co}(T), \det(d_{h^{-1}g}(0)) \neq 0\}.$$

Proof. Since $(0, 0)$ is finite, so is $g.(0, 0)$ for all $g \in \mathrm{Co}(T)$. Conversely, $(x, y) = (g.0, h.0)$ is a finite point iff so is $h^{-1}.(x, y) = (h^{-1}g.0, 0)$, and this is finite iff $h^{-1}g.0$ belongs to V, i.e. iff

$$\det(d_{h^{-1}g}(0)) \neq 0.$$

If this is the case, then there exists $p \in P'$ (e.g. $p = t_{h^{-1}g(0)}$) with $p.(0, 0) = (h^{-1}g.0, 0)$. It follows that $(x, y) = hp.(0, 0)$. (Moreover, we have the explicit equation $(g.0, h.0) = ht_{h^{-1}g(0)}.(0, 0)$; similarly one gets the equation

$$(g.0, h.0) = g\widetilde{t}_{g^{-1}h.0}.(0, 0) \tag{3.5}$$

which will have important consequences in Ch. X.)

Finally, the orbit is open dense in $V^c \times W^c$ since it is the image of the open dense subset

$$\{(g, h) \in \mathrm{Co}(T) \times \mathrm{Co}(T) |\, \det(d_{h^{-1}g}(0)) \neq 0\}$$

of $\mathrm{Co}(T) \times \mathrm{Co}(T)$ under the surjective submersion $\mathrm{Co}(T) \times \mathrm{Co}(T) \to V^c \times W^c$. ∎

From the theorem one reads off that a point $(x,y)=(t_x(0),\widetilde{t}_y(0))\in V\times W$ is finite if and only if

$$\det(d_{\widetilde{t}_{-y}t_x})(0)\neq 0.$$

If T is faithful, then we identify W^c and V^c via Θ, and X is described by

$$X\cap(V\times V)=\{(x,y)\in V\times V|\ \det B(x,y)\neq 0\}. \tag{3.6}$$

Theorem VIII.3.8. *As a homogenous space, the orbit* $X=\mathrm{Co}(T).(0,0)$ *is isomorphic to* $\mathrm{Co}(T)/\mathrm{Str}(T)$*. This is a twisted polarized symmetric space, and the integral submanifolds of the invariant polarization are given by the product structure of* $V^c\times W^c$*.*

Proof. The condition $g.(0,0)=(0,0)$ is equivalent to $g\in P\cap P'=\mathrm{Str}(T)$, and thus $X\cong\mathrm{Co}(T)/\mathrm{Str}(T)$ as a homogeneous space. It is a symmetric space since $\mathrm{Str}(T)$ is open in the fixed point group of the involution $(-\mathrm{id}_V)_*$ of $\mathrm{Co}(T)$ (to see this, it suffices to remark that the Lie algebra $\mathfrak{str}(T)$ is precisely the subalgebra of $\mathfrak{co}(T)$ fixed under $(-\mathrm{id}_V)_*$; this is follows from the polynomial realization of $\mathfrak{co}(T)$).

The space X carries an invariant polarization: by definition of the action of $\mathrm{Co}(T)$ on $V^c\times W^c$, the subsets of the form $\{v\}\times W^c$ resp. $V^c\times\{w\}$ are permuted under the action of $g\in\mathrm{Co}(T)$ and thus their intersections with X define integral submanifolds of an invariant polarization on X.

Finally, this polarization is twisted: in order to prove this, consider the element $r\,\mathrm{id}_V\in\mathrm{Co}(T)$ ($r\in\mathbb{R}^*$). It acts on W by scalar multiplication with the inverse of r since for all $w\in W\cong\mathfrak{m}^*$,

$$(r\,\mathrm{id}_V).\exp(w)P'=\exp((r\,\mathrm{id}_V)_*w)P'=\exp(r^{-1}w)P'.$$

Therefore the linear map $r\,\mathrm{id}_V\times r^{-1}\,\mathrm{id}_W$ is given by the action of an element of $\mathrm{Co}(T)$ and hence is an automorphism of the polarized symmetric space X. Taking the derivative $\frac{d}{dr}|_{r=1}$, we see that $J=\mathrm{id}_V\times(-\mathrm{id}_W)$ is a derivation of the curvature at the origin, and according to Lemma III.3.4, this means that J is twisted. ∎

3.4. The structure bundle. We may consider $V^c\times W^c$ as a bundle over V^c with typical fiber W^c. (It can be realized as the fiber bundle $\mathrm{Co}(T)\times_P W^c$.) Note that this bundle does not have a well-defined zero-section, but it has a distinguished subbundle: the fiber over $x\in V^c$ is the set of finite points having the form (x,y). This bundle can be realized as follows:

Proposition VIII.3.9. *The subbundle* $\mathbf{W}$ *of* $\mathrm{Hom}(S^2(TV),TV)$ *with constant fiber* $\mathbf{W}_p=W$ *(where* W *is as in* Def. VI.4.7*) has a unique extension to a* $\mathrm{Co}(T)$*-invariant subbundle (again denoted by* $\mathbf{W}$*) of the bundle* $\mathrm{Hom}(S^2(T(V^c)),T(V^c))$ *over* V^c*.*

Proof. We only have to check that $W=\mathbf{W}_0$ is invariant under the natural action of the stabilizer P. But this is clear since, for all $p\in P$, $\mathrm{D}p(0)$ belongs to $\mathrm{Str}(T)$ which is precisely the linear group preserving W. ∎

The arguments of the preceding proof show that, more generally, a locally defined diffeomorphism preserves the structure bundle if and only if it is $\mathrm{Str}(T)$-conformal.

Definition VII.3.10. The subbundle $\mathbf{W}$ of $\mathrm{Hom}(S^2T(V^c),T(V^c))$ is called the *structure bundle associated to* T*.* ∎

4. Conformal completion of the classical spaces

4.1. Conformal completion of $M(n,\mathbb{F})$. We consider the JTS defined on $V = M(n,\mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) by the formula

$$T(X,Y,Z) = XYZ + ZYX;$$

it is associated to the usual Jordan algebra structure on V having as Jordan inverse $j(X) = X^{-1}$ the usual matrix inverse (cf. Section II.3).

Proposition VIII.4.1. *The conformal completion of $V = M(n,\mathbb{F})$ is equivalent to the imbedding*

$$\Gamma : M(n,\mathbb{F}) \to \mathrm{Gr}_{n,2n}(\mathbb{F}), \quad X \mapsto \Gamma_X = \{(v, Xv) |\, v \in \mathbb{F}^n\}$$

into the Grassmannian of n-dimensional subspaces of $\mathbb{F}^{2n}$ associating to a linear map its graph. The identity component of the conformal group of V is

$$\mathrm{Co}(T)_o = \mathbb{P}\,\mathrm{Gl}(2n,\mathbb{F})_o$$

acting in the natural way on $\mathrm{Gr}_{n,2n}(\mathbb{F})$.

Proof. Note that $\Gamma(V) \subset \mathrm{Gr}_{n,2n}(\mathbb{F})$ is open dense, and since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\Gamma_A = \Gamma_{(aA+b)(cA+d)^{-1}}$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Gl}(2n,\mathbb{F})$ and $A \in M(n,\mathbb{F})$, the action of $\mathrm{Gl}(2n,\mathbb{F})$ in the coordinates given by $V \cong \Gamma(V)$ is given by the usual formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.X = (aX+b)(cX+d)^{-1}. \tag{4.1}$$

The effective group of this action is $\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{F})$. Note that

$$\begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}.X = X^{-1} = j(X), \tag{4.2}$$

$$\begin{pmatrix} \mathbf{1}_n & Z \\ 0 & \mathbf{1}_n \end{pmatrix}.X = X + Z = t_Z(X), \tag{4.3}$$

$$\begin{pmatrix} \mathbf{1}_n & 0 \\ Z & \mathbf{1}_n \end{pmatrix}.X = X(ZX+\mathbf{1})^{-1} = (Z + X^{-1})^{-1} = (jt_Zj)(X), \tag{4.4}$$

$$\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}.X = gXh^{-1}. \tag{4.5}$$

Eqn. (4.3) describes the translation group, (4.5) describes the (identity component of) the structure group (cf. Section II.3) and (4.2) describes the Jordan inverse. According to Th. VIII.2.3 (6) these elements generate $\mathrm{Co}(T)_o$. On the other hand, it is easily checked that the corresponding matrices generate $\mathrm{Gl}(2n,\mathbb{R})$; therefore the rational actions of $\mathrm{Co}(T)_o$ and of $\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{F})_o$ on V are the same. Since $V \cong \Gamma(V) \subset \mathrm{Gr}_{n,2n}(\mathbb{F})$ is open dense and $\mathrm{Gr}_{2n,n}(\mathbb{F})$ is homogeneous under $\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{F})$, it follows that $\mathrm{Gr}_{n,2n}(\mathbb{F})$ is the conformal completion of V. ■

4.2. Conformal completion of spaces of symmetric and skew-symmetric matrices. Given an non-degenerate ε-sesquilinear form $(x|y) = x^t A\varepsilon(y)$ on $\mathbb{F}^{2n}$, we denote by

$$\mathrm{Lag}(A,\varepsilon,\mathbb{F}) := \{W \in \mathrm{Gr}_{n,2n}(\mathbb{F}) |\, (W|W) = 0\}$$

the set of Lagrangian subspaces w.r.t. this form. This is the fixed point set of the map $\mathrm{Gr}_{n,2n}(\mathbb{F}) \to \mathrm{Gr}_{n,2n}(\mathbb{F})$, $W \mapsto W^\perp$ assigning to a subspace its orthocomplement.

Proposition VIII.4.2. *Let A be a Hermitian resp. a skew-Hermitian non-degenerate matrix. Then restriction of the imbedding from* Prop. VII.5.1 *to subspaces of $M(n,\mathbb{F})$ yields the following conformal completions and conformal groups:*

V	V^c	$\mathrm{Co}(T)_o$
$\mathrm{Sym}(A,\mathbb{F})$	$\mathrm{Lag}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right),\mathbb{F})_0$	$\mathbb{P}\mathrm{O}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right),\mathbb{F})_o$
$\mathrm{Asym}(A,\mathbb{F})$	$\mathrm{Lag}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right),\mathbb{F})_0$	$\mathbb{P}\mathrm{O}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right),\mathbb{F})_o$
$\mathrm{Herm}(A,\varepsilon,\mathbb{F})$	$\mathrm{Lag}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right),\varepsilon,\mathbb{F})_0$	$\mathbb{P}\mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right),\varepsilon,\mathbb{F})_o$
$\mathrm{Aherm}(A,\varepsilon,\mathbb{F})$	$\mathrm{Lag}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right),\varepsilon,\mathbb{F})_0$	$\mathbb{P}\mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right),\varepsilon,\mathbb{F})_o$

Proof. Let $X^* = A^{-1}X^tA$ be the adjoint of $X \in M(n,\mathbb{F})$ w.r.t. the form given by A. Then, for all $u, v \in \mathbb{F}^n$,

$$0 = (Xu|v) - (u|X^*v) = -b_1((u,Xu),(v,X^*v)) \tag{4.6}$$

with the form

$$b_1((x_1,x_2),(y_1,y_2)) = (x_1|y_2) - (x_2|y_1) \tag{4.7}$$

on $\mathbb{F}^{2n}$ which is given by the matrix $\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right)$. Thus $b_1(\Gamma_X, \Gamma_{X^*}) = 0$, and by reasons of dimension we have

$$\Gamma_{X^*} = (\Gamma_X)^\perp \tag{4.8}$$

w.r.t. the form b_1. Therefore $X = X^*$ is equivalent to $\Gamma_X \in \mathrm{Lag}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right),\varepsilon,\mathbb{F})$.

The group $\mathbb{P}\mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right),\varepsilon,\mathbb{F})$ acts on $V = \mathrm{Herm}(A,\varepsilon,\mathbb{F})$ by birational maps in the same way as $\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{F})$ acts on $M(n,\mathbb{F})$. The identity component of $\mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right),\varepsilon,\mathbb{F})$ is generated by the matrices $\left(\begin{smallmatrix} \mathbf{1}_n & X \\ 0 & \mathbf{1}_n \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} g & 0 \\ 0 & (g^*)^{-1} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mathbf{1}_n & 0 \\ X & \mathbf{1}_n \end{smallmatrix}\right)$ with $X \in V$ and $g \in \mathrm{Gl}(n,\mathbb{F})$ which yield, respectively, the actions of the translation group t_V, the structure group and the group $\Theta(t_V)$. Thus the rational actions of $\mathrm{Co}(T)_o$ and $\mathbb{P}\mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right),\varepsilon,\mathbb{F})$ on V are the same.

By Witt's theorem (cf. [Bou59, 4, no.3, Cor 2]), the group $\mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right),\varepsilon,\mathbb{F})$ acts transitively on the connected components of the variety of Lagrangian subspaces. (Note that the whole variety of Lagrangian subspaces is in general not connected – example: $\mathbb{F} = \mathbb{R}$, $A = \mathbf{1}_{2n+1}$ – and hence is not a conformal completion of V.) As in the proof of the preceding proposition, it follows that the connected component containing Γ_0 is a conformal completion of $V = \mathrm{Herm}(A,\varepsilon,\mathbb{F})$. For the other spaces the proof is carried out similarly. ∎

The preceding proof shows that the involutive automorphism "adjoint"

$$a : M(n,\mathbb{F}) \to M(n,\mathbb{F}), \quad X \mapsto a(X) := X^*$$

extends to the whole Grassmannian as the map "orthocomplement w.r.t. the form given by $\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right)$". A special case of the proposition is:

$$\mathrm{Sym}(n,\mathbb{R})^c = \mathrm{Lag}(J,\mathbb{R})_0, \qquad \mathrm{Co}(\mathrm{Sym}(n,\mathbb{R}))_o = \mathbb{P}\,\mathrm{Sp}(n,\mathbb{R}).$$

For $\mathrm{Asym}(n,\mathbb{R})$, we use the real Cayley transform R (cf. Eqn. (I.6.6)) in order to diagonalize the matrix $\left(\begin{smallmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)$. Thus

$$\mathrm{Asym}(n,\mathbb{R})^c = R \cdot \mathrm{Lag}(I_{n,n},\mathbb{R})_0, \qquad \mathrm{Co}(\mathrm{Asym}(n,\mathbb{R}))_o = \mathbb{P}(R\mathrm{O}(n,n)_o R^{-1}).$$

For $A = \mathbf{1}$ and $\mathbb{F} = \mathbb{C}, \mathbb{H}$, normal forms are obtained in the same way. The cases $A = F$ and $A = I_{p,q}$ are reduced to the case $A = \mathbf{1}$ by the following lemma:

Lemma VIII.4.3. *The following maps are isomorphisms of sub-JTS of the JTS $M(2n, \mathbb{F})$, resp. $M(p+q, \mathbb{F})$:*

$$\mathrm{Herm}(\mathbf{1}_{2m}, \mathbb{F}) \to \mathrm{Aherm}(J, \mathbb{F}), \quad X \mapsto JX,$$

$$\mathrm{Aherm}(\mathbf{1}_{2m}, \mathbb{F}) \to \mathrm{Herm}(J, \mathbb{F}), \quad X \mapsto JX,$$

$$\mathrm{Herm}(I_{p,q}, \mathbb{F}) \to \mathrm{Herm}(p+q, \mathbb{F}), \quad X \mapsto I_{p,q}X.$$

Proof. The relation $J^{-1}(JX)^t J = -JX^t$ implies that the first two maps are isomorphisms, and the relation $I_{p,q}(I_{p,q}X)^t I_{p,q} = I_{p,q}X^t$ implies that the third map is an isomorphism. ■

In particular, normal forms for the conformal completions and conformal groups of the Jordan algebras $\mathrm{Sym}(F, \mathbb{R})$ and $\mathrm{Sym}(F, \mathbb{C})$ can now be determined (see Table XII.3.1 for a complete list of normal forms). Finally we show that some varieties of Lagrangians can be identified with orthogonal resp. unitary groups:

Proposition VIII.4.4. *The graph-imbedding $X \mapsto \Gamma_X$ yields open imbeddings*

$$\mathrm{O}(A, \mathbb{F}) \to \mathrm{Lag}(\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \mathbb{F}), \quad \mathrm{U}(A, \varepsilon, \mathbb{F}) \to \mathrm{Lag}(\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \varepsilon, \mathbb{F}).$$

In particular, if we identify linear operators with their graphs, we have

$$\begin{aligned} \mathrm{Asym}(n, \mathbb{R})^c &= R \cdot \mathrm{O}(n)_o, \\ \mathrm{Herm}(n, \mathbb{C})^c \cong \mathrm{Aherm}(n, \mathbb{C})^c &= R \cdot \mathrm{U}(n), \\ \mathrm{Aherm}(n, \mathbb{H})^c &= R \cdot \mathrm{Sp}(n). \end{aligned}$$

Proof. As in the proof of Prop. VIII.4.2 we see that

$$\Gamma_{(g^*)^{-1}} = (\Gamma_g)^\perp,$$

this time the orthocomplement taken w.r.t. the form given by $\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$, and the first claim follows. For the proof of the second statement, note that varieties of Lagrangian subspaces, being closed in a Grassmannian, are always compact. Thus, if the orthogonal or unitary group imbedded as an open subset in the variety of Lagrangians is also compact, it follows that connected components agree if they have a point in common. Now the second claim follows since

$$R \cdot \mathrm{Aherm}(A, \varepsilon, \mathbb{F})^c = \mathrm{Lag}(\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}.\varepsilon, \mathbb{F})_{\Gamma_1} \supset \mathrm{U}(A, \varepsilon, \mathbb{F})_o,$$

and the last inclusion is an equality if $\mathrm{U}(A, \varepsilon, \mathbb{F})$ is compact. ■

4.3. Conformal completion of the spaces $M(p, q; \mathbb{F})$. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ we realize the space $V = M(p, q; \mathbb{F})$ as the -1-eigenspace

$$V = \{\begin{pmatrix} 0 & X \\ X^t & 0 \end{pmatrix} \mid X \in M(p, q; \mathbb{F})\} \subset \mathrm{Sym}(p+q; \mathbb{F})$$

of the involution $Z \mapsto I_{p,q} Z I_{p,q}$ of $\mathrm{Sym}(p+q; \mathbb{F})$ with the Jordan triple product induced from the ambient space. For $F = \mathbb{C}$ or $\mathbb{H}$ we have a similar realization inside $\mathrm{Herm}(p+q, \mathbb{F})$; the two realizations in the complex case are modifications of each other in the sense of Lemma III.4.5.

Proposition VIII.4.5. *The conformal completion of $M(p,q;\mathbb{F})$ is isomorphic to the graph-imbedding*

$$\Gamma : M(p,q;\mathbb{F}) \to \mathrm{Gr}_{p,p+q}(\mathbb{F}), \quad X \mapsto \Gamma_X,$$

and the identity component of its conformal group is isomorphic to the group $\mathbb{P}\,\mathrm{Gl}(p+q;\mathbb{F})$ acting in the usual way on the Grassmannian.

Proof. The claim is proved by the same arguments as Prop. VIII.4.1, with the only difference that for $p \neq q$ Eqn. (4.2) no longer makes sense. Eqns. (4.3) and (4.5) still describe the translation group, resp. the structure group, as is easily verified; one just has to check that Eqn. (4.4) still describes the group $\Theta(t_V)$. But this is clear from our realization of V as a subspace of $\mathrm{Herm}(p+q,\mathbb{F})$: the condition

$$\frac{d}{dt}|_{t=0}\Theta(t_{tZ})X = -XZX$$

holds on the ambient space $M(p+q,\mathbb{F})$ and therefore also on the subspaces in question. Now the proof can be finished as in the case $M(n,\mathbb{F})$. ∎

Of particular interest is the special case

$$\mathbb{F}^n = M(1,n;\mathbb{F}) \to \mathbb{F}\mathbb{P}^n, \quad X \mapsto \begin{pmatrix} 1 \\ X \end{pmatrix}$$

which is the usual imbedding of affine n-space into the projective n-space. In case $\mathbb{F} = \mathbb{R}$ the structure group is the whole linear group $\mathrm{Gl}(n,\mathbb{R})$, and therefore we can conclude that $\mathrm{Co}(M(1,n;\mathbb{R})) = \mathbb{P}\,\mathrm{Gl}(n+1,\mathbb{R}))$; this group is connected for n even and has two connected components for n odd.

4.4. Projective quadrics. We use notation introduced in Section IV.1.5.

Proposition VIII.4.6.

(i) *The identity component of the conformal group of the symmetric space with twist S^n is the group $\mathbb{P}\,\mathrm{SO}(n+1,1)$ acting in the natural way on the projective quadric $\{[x] \in \mathbb{P}(\mathbb{R}^{n+2}) |\ \sum_{j=1}^{n+1} x_j^2 - x_{n+2}^2 = 0\} \cong S^n$. This space is the conformal completion of the JTS $\mathbb{R}^n$, $T(x,y,z) = (x|z)y - (y|x)z - (y|z)x$.*

(ii) *The identity component of the conformal group of the symmetric space with twist $(S^n)_{h\mathbb{C}} = \mathrm{SO}(n+2)/(\mathrm{SO}(n) \times \mathrm{SO}(2))$ is the group $\mathbb{P}\,\mathrm{SO}(n+2,\mathbb{C})$ acting in the natural way on the projective quadric $\{[z] \in \mathbb{P}(\mathbb{C}^{n+2}) |\ \sum_{j=1}^{n+2} z_j^2 = 0\} \cong (S^n)_{ph\mathbb{C}}$. This space is the conformal completion of the JTS $\mathbb{C}^n$, $T(x,y,z) = (x|z)y - (y|x)z - (y|z)x$.*

Proof. (ii) Multiplication by i w.r.t. the base point o in the space $(S^n)_{h\mathbb{C}}$ is given by an element of the subgroup $\mathrm{SO}(2)$ of the stabilizer $\mathrm{SO}(n) \times \mathrm{SO}(2)$. The complexified group $\mathrm{SO}(2,\mathbb{C})$ acts holomorphically on the projective quadric in the natural way. According to our definition of the conformal Lie algebra in Ch. VI, the corresponding vector field (Euler operator) together with the vector fields induced from the $\mathrm{SO}(n+2)$-action generate the (inner) conformal Lie algebra. It follows that the action of $\mathrm{SO}(2,\mathbb{C})$ together with the action of $\mathrm{SO}(n+2)$ generate the identity component of the conformal group. But the subgroup of $\mathrm{SO}(n+2,\mathbb{C})$ generated by $\mathrm{SO}(n+2)$ and $\mathrm{SO}(2,\mathbb{C})$ is the whole group $\mathrm{SO}(n+2,\mathbb{C})$ which is therefore the identity component of the conformal group. In Section IV.1.5 we have seen that the corresponding JTS is $\mathbb{C}^n$ with the triple product $T(x,\overline{y},z)$. This is a modification of T and therefore has the same conformal group and conformal completion; the latter is given by the projective quadric since the group $\mathrm{SO}(n+2,\mathbb{C})$ acts transitively on it.

(i) This follows from part (ii) by passing to the real form S^n of $(S^n)_{h\mathbb{C}}$ (cf. Prop. IV.1.5). ∎

In the same way, passing to other real forms of the complex projective quadric, we see that the conformal completion of the JTS $\mathbb{R}^{p,q}$, which is the space $\mathbb{R}^n$ with the triple product

$$T(x,y,z) = xI_{p,q}z^t y - xI_{p,q}y^t z - zI_{p,q}y^t x,$$

is the projective completion of the quadric defined by the form with signature $(p+1,q)$ in $\mathbb{R}^{n+1}$, and the identity component of its conformal group is $\mathrm{SO}(p+1,q+1)$. The projective completion itself can be described as the space

$$\mathbb{P}(\mathrm{SO}(p+1)\times \mathrm{SO}(q+1))/\mathbb{P}(\mathrm{SO}(p)\times \mathrm{SO}(q)) = S^p \times S^q/((x,y)\sim(-x,-y)).$$

The Jordan algebra $V = \mathbb{R}^n$ $(p>0)$ with product $xy = b(x,e)y + b(y,e)x - b(x,y)e$, where b is the bilinear form given by $I_{p,q}$ and e is the first vector of the canonical basis, is the modification $T^{(\alpha)}$ of the JTS $\mathbb{R}^{p,q}$ by the automorphism α with matrix $I_{1,n-1}$. Therefore the projective quadric $S^p\times S^q/\sim$ is also the conformal completion of this Jordan algebra.

Appendix A: The flow of a vector field with a zero of order two

Proposition VIII.A.1. *If a vector field X on a vector space V vanishes of order 2 at the origin, then there exists a neighborhood U of 0 and a constant L, $0<L<1$, such that the flow $\varphi_t(x)$ of X is defined for all $x\in U$ and $|t|<1/L$.*

Proof. From $X(0)=0$ we deduce that $\varphi_t(0)=0$ for all $t\in R$. Since $\mathrm{D}X(0)=0$, there is a neighborhood U of 0 such that $X|_U$ satisfies a Lipschitz-condition with a Lipschitz-constant $L<1$. The proof of the existence theorem of Picard-Lindelöf (cf. [A74, p. 217]) shows that then $\varphi_t(x)$ is defined for all $x\in U$ and $|t|<1/L$. ■

Definition VIII.A.2. In the situation of the preceding proposition, we let

$$\exp X : U \to V, \quad x \mapsto \varphi_t(1).$$ ■

Lemma VIII.A.3. *If X and $\exp X$ are as above, then $(\mathrm{D}(\exp X))(0) = \mathrm{id}_V$.*

Proof. Let $\alpha(t) := (\mathrm{D}\varphi_t)(0)$. Then $\alpha(t)$ is defined at $t=1$, and we have $\alpha(0)=\mathrm{id}_V$, $\frac{d}{dt}|_{t=0}\alpha(t) = (\mathrm{D}\frac{d}{dt}|_{t=0}\varphi_t)(0) = (\mathrm{D}X)(0) = 0$, whence $\alpha(t)=0$ for all t where α is defined; in particular $\alpha(1)=0$. ■

Appendix B: Equivariant bundles over homogeneous spaces

In this appendix we describe a standard construction of fiber bundles over homogeneous spaces: our base space will be a smooth manifold $M=L/Q$ which is homogenous under the action of a Lie group L, and the typical fiber will be modelled on a smooth manifold U with a smooth action $Q \times U \to U$, $(q,u) \mapsto \pi(q)u$. We consider the following space of smooth functions from L to U:

$$\mathrm{ind}_Q^L \pi = \{F \in \mathcal{C}^\infty(L,U) | \, \forall q \in Q, g \in L : \, F(gq) = \pi(q)^{-1} F(g)\}. \tag{B.1}$$

The group L acts on this space by ordinary left translations: $g.F = F \circ g^{-1}$. In the important case where U is a vector space and π is a representation of Q, this construction is known as the *induced representation* (induced from Q to L).

Just as in the case of induced representations, we associate to the space (A.1) a fiber bundle

$$L \times_Q U \to M = L/Q \tag{B.2}$$

over M which is by definition the quotient space of $L \times U$ under the equivalence relation $(g,v) \sim (gq, \pi(q)^{-1}v)$ for $q \in Q$. We write $[g,v]$ for the equivalence class of (g,v). The group L acts by $l.[g,v] = [lg,v]$. Then the projection $[g,v] \mapsto gQ$ is L-equivariant, and the map $v \mapsto [g,v]$ sets up a (non-canonical) bijection of U with the fiber over gQ. The relation $[gq,v] = [g, \pi(q)v]$ allows to associate to a function $F \in \mathrm{ind}_Q^L \pi$ a well-defined section $s_F(gQ) := [g, F(g)]$, and standard arguments show that $F \mapsto s_F$ is an L-isomorphism from $\mathrm{ind}_Q^L \pi$ onto the space of smooth sections of $L \times_Q U$ (cf. e.g. [BtD85, p.144]). If π is a linear representation of Q, then we can define a vector bundle structure on $L \times_Q U$ by $[g,v] + [g,w] := [g, v+w]$, $r[g,v] := [g, rv]$.

Example VIII.B.1. (Tangent bundles) Consider the *first isotropy representation* $\pi = T_o$: we denote by $U = T_oM$ the tangent space at the base point $o = eQ$ of M, and

$$T_o : Q \to \mathrm{Gl}(T_oM), \quad q \mapsto T_o q. \tag{B.3}$$

Then $\mathrm{ind}_Q^L(T_o)$ is identified with the space $\mathfrak{X}(M)$ of vector fields on M (to $F \in \mathrm{ind}_Q^L(T_o)$ corresponds the vector field $X_{g.o} = T_o g \cdot F(g)$, and vice versa). Similarly, $\mathrm{ind}_Q^L(\otimes^k T_o \otimes \otimes^l T_o^*)$ is identified with the space of tensor fields of type (k,l), and similarly for symmetric and alternating powers. The second isotropy representation $q \mapsto T_o^{(2)} q$ (cf. Def. I.B.6) yields the second order tangent bundle whose sections are the second order differential operators, and so on. ■

Trivialization of bundles over vector spaces. As above, we consider a fiber bundle $L \times_Q U$ over $M = L/Q$ and assume in addition that L contains a vector group t_V with $t_V \cap Q = \{e\}$ and $\dim(t_V) = \dim M$. Then the orbit $t_V.o \subset M$ is open and isomorphic to V via the imbedding

$$V \to M, \quad v \mapsto t_v.o. \tag{B.4}$$

In this situation we associate to an element $F \in \mathrm{ind}_Q^L \pi$ a function

$$f := a(F) : V \to U, \quad v \mapsto F(t_v) \tag{B.5}$$

which we call the *affine picture of* F. The map $F \mapsto f = a(F)$ is called the *trivialization map.* For instance, the affine picture of a vector field is just a function $V \to V$, and so on. In the cases we will be interested in V is dense in M, and then the trivialization map is injective.

Next we want to describe the action of L in the affine picture. In analogy to Eqn. VIII (2.5) we define, for $g \in L$ and $x \in V$ such that $g.x \in V$, the element $p(g,x) \in Q$ by the condition

$$gt_x = t_{g.x} \, p(g,x). \tag{B.6}$$

Proposition VIII.B.2. *Let $F \in \mathrm{ind}_Q^L \pi$. Then for all $z \in V$ and $g \in L$ where the following expressions are defined, we have*

$$(a(F \circ g^{-1}))(z) = \pi(p(g^{-1}, z))^{-1} \cdot f(g^{-1}z).$$

Proof. By definition, $g^{-1}t_z = t_{g^{-1}z}p(g^{-1}, z)$. Using the transformation property of F from Eqn. (A.1), we get

$$a(F \circ g^{-1})(z) = F(g^{-1}t_z) = F(t_{g^{-1}z}\, p(g^{-1}, z)) = \pi(p(g^{-1}, z))^{-1} f(g^{-1}z). \qquad \blacksquare$$

Note that the right-hand side of the formula from the proposition does not define an action of L on "all" functions $V \to U$ because $g^{-1}z$ is not defined for all z.

Example VIII.B.3. If $\pi = T_o$ is the first isotropy representation, then we identify all tangent spaces of M over V with V and therefore have a natural identification

$$T_o(p(g, x)) = T_o(t_{-g.x} \circ g \circ t_x) = \mathrm{D}g(x).$$

Now Proposition VIII.B.2 is just the usual formula for the action of diffeomorphisms on vector spaces (I.A.5). More generally, if π factors through the linear isotropy representation, i.e. $\pi(p) = \pi'(\mathrm{D}p(0))$ with π' a representation of $T_o(Q)$, then the action from the proposition can be written

$$a(F \circ g^{-1})(z) = \pi'(\mathrm{D}g^{-1}(z))^{-1} \cdot f(g^{-1}z). \tag{B.7}$$

$\blacksquare$

Appendix C: Some identities for Jordan triple systems

Proposition VIII.2.8 implies some algebraic identities for Jordan triple systems. Let us mention some of them here in order to demonstrate that our methods yield non-trivial algebraic results; however, we are not going to use them in the sequel. For this reason we content ourselves to prove them only in the case of a faithful JTS; the method can be adapted to the case of a general JTS or Jordan pair. Using the notation $x^y = \Theta(t_y)^{-1}(x)$ (cf. Eqn. (2.4)), Equation (2.8) reads

$$(gx)^{\Theta(g)y} = \Theta(p(\Theta(g), y))x^y. \tag{C.1}$$

We will specialize this equation to elements of the form $g = t_u, \Theta(t_u), B(u,v)$ using that

$$\begin{aligned} p(t_u, x) &= \Theta(t_{\Theta(t_{x+u})^{-1}(0)})^{-1}, \\ p(\Theta(t_u), x) &= B(x, -u)^{-1}\Theta(t_{\Theta(t_x)(u)})^{-1}, \\ p(B(u,v), x) &= B(u,v). \end{aligned} \tag{C.2}$$

Now we let in (2.8) $g = B(u,v)$ and obtain

$$\Theta(t_{B(v,u)^{-1}y})^{-1}B(u,v)x = B(u,v)\Theta(t_y)^{-1}(x).$$

Replacing y by $B(v,u)y$ we get the *shifting formula*

$$B(u,v)(x^{B(v,u)y}) = (B(u,v)x)^y$$

from [Lo75] and [Lo77]. Next we let in (C.1) $g = \Theta(t_u)$, resp. $g = t_u$ and obtain in a similar way

$$x^{y+z} = (x^y)^z, \quad (x+z)^y = x^y + B(x,y)^{-1}(z^{(y^x)}).$$

These are the *addition formulae* from loc.cit. Letting in (2.9) $g = \Theta(t_u)$, resp. $g = t_u$ we obtain the identities

$$B(x,y)B(x^y,z) = B(x,y+z), \quad B(z,x^y)B(y,x) = B(y+z,x)$$

(identities JP33 and JP34 from loc. cit.). Letting in (2.9) $g = B(u,v)$ we obtain

$$B(B(u,v)x, B(v,u)^{-1}y) = B(u,v)B(x,y)B(u,v)^{-1}. \tag{C.3}$$

Putting it in a polynomial form and comparing homogeneous terms yields other identities. One of it is the identity

$$T(B(x,y)u, v, B(x,y)w) = B(x,y)T(u, B(y,x)v, w) \tag{C.4}$$

(cf. the identity JP26 in loc. cit.) which just reflects the fact that $B(x,y)$ belongs to the structure monoid. This is again a polynomial identity. Inserting the explicit formula for $B(x,y)$ and separating homogeneous terms of fixed degree in x and y, we get more identities. In particular, the term of degree 4 in x,y yields

$$T(P(x)P(y)u, v, P(x)P(y)w) = P(x)P(y)T(u, P(y)P(x)v, w), \tag{C.5}$$

which for $u = w$ is

$$P(P(x)P(y)u) = P(x)P(y)P(u)P(y)P(x). \tag{C.6}$$

Proposition VIII.C.1. *In any JTS the "fundamental formula"*

$$P(P(x)u) = P(x)P(u)P(x)$$

holds.

Proof. If there is an element y with $P(y) = \mathrm{id}_V$, then the claim follows from (C.6). Otherwise we cannot deduce it by straightforward specialization of the identities considered so far, and one has to use algebraic manipulations (cf. [Mey70, Satz 2.2]) in order to deduce the fundamental formula from (JT2). ■

Comparing with Chapter II, we remark that the fundamental formula has an obvious geometric significance only in the context of Jordan *algebras*, but not in the context of general Jordan triple systems. In fact, we will never use the fundamental formula in the general theory but only in Chapter XI where the link with Jordan *algebra* theory is discussed.

Notes for Chapter VIII.

VIII.1. and VIII.2. Our definition of the conformal group is a transposition of Koecher's approach ([Koe69a,b]) from an algebraic to a more analytic set-up. The use of only locally defined diffeomorphisms makes our approach less technical than Koecher's and is closer to problems in analysis related to pseudogroups of diffeomorphisms.

Theorem VIII.1.3 is also due to Koecher ([Koe69a, Satz II.2.2]), and in loc.cit. Satz I.2.1, Koecher gives a further description of the subgroup $\mathrm{Co}_*(T) \subset \mathrm{Aut}(\mathfrak{co}(T))$ as a Zariski-open subgroup which he calls the *group of essential automorphisms*; these are by definition the automorphisms τ of $\mathfrak{co}(T)$ for which the denominator d_τ defined by $(\tau\mathbf{v})(x) = d_\tau(x)v$ takes for some $x \in V$ values in $\mathrm{Gl}(V)$. Prop. VIII.1.4 is a weak version of this result.

The presentation of Theorems VIII.1.6, VIII.1.9 and VIII.2.3 given here is new; it is motivated by similar results in [Be96a]. In [Koe69a] the results corresponding to Th. VIII.2.3 are obtained in a more algebraic and more computational way, and Th. VIII.1.6 is then deduced as a corollary (cf. [Koe69a, p.370]).

VIII.3. An algebraic approach to the conformal group and conformal completion is due to O. Loos ([Lo78], [Lo79]; there the conformal group is called the *projective group*); see also [Lo77] where most of the results presented in this chapter can be found. The proofs given there rely on the algebraic theory of Jordan pairs ([Lo75]); in particular, the algebraic identities mentioned in Appendix C are needed.

VIII.4. The presentation given here is taken from on [Be96a]. See also [Lo78]. The realization of unitary groups via the "real Cayley transform" R (Prop. VII.4.4) is known from [Wey39] as "Cayley's rational parametrization of the orthogonal group".

Appendix C. A list of about 40 identities for Jordan pairs and Jordan triple systems can be found in [Lo75] and [Lo77]. As shows the example of the fundamental formula (which for triple systems is due to K. Meyberg [Mey70]), not all of these identities are obtained by a simple specialization of the formulas from Section 2.

Chapter IX: Liouville theorem and fundamental theorem

These two classical theorems characterize geometrically the conformal group, resp. the projective group: the former is precisely the group preserving the Euclidean metric of $\mathbb{R}^n$ up to a scalar function (if $n > 2$); the latter is precisely the group preserving straight lines in $\mathbb{R}^n$ (if $n > 1$). In both cases it is enough to assume that transformations having these geometric properties are only locally defined and have a rather weak regularity (we content ourselves with a $\mathcal{C}^4$-assumption) in order to conclude that they are actually given by birational maps of $\mathbb{R}^n$ and thus can be globally continued to almost everywhere defined transformations and even to everywhere defined maps if one passes to the compactification given by S^n, resp. by $\mathbb{RP}^n$. These results are generalized in Th. IX.1.1 for *non-degenerate* JTS not containing ideals isomorphic to $\mathbb{R}$ or $\mathbb{C}$ (these exceptions correspond to the restrictions $n > 2$, resp. $n > 1$ mentioned above). The structure of the statement and of its proof shows how in general the conditions of conformality and projectivity play together. In the classical Liouville-theorem, the condition of conformality is "leading" and the condition of projectivity is "hidden"; in the classical fundamental theorem, the latter is "leading" and the former is empty.

Specialization of our general theorem to various classical spaces takes interesting forms (Section 2): besides the classical results already mentioned, we obtain the determination of some *causal groups* and confirm a conjecture of I.E. Segal (Section 2.3), and we obtain new proofs of some results of Chow and Dieudonné (Section 2.4).

1. Liouville theorem and fundamental theorem

Let us recall from Definitions VII.3.4, VIII.1.7 and VIII.1.10 the notions of *conformality* and *projectivity*: given a JTS T on a vector space V, we introduce the following conditions on a vector field X, resp. on a locally defined diffeomorphism g. For all $p \in V$, where the expression is defined,

$$\begin{aligned} &(\mathrm{C1}) \quad \mathrm{D}X(p) \in \mathfrak{str}(T) \\ &(\mathrm{P1}) \quad \mathrm{D}^2X(p) \in W \\ &(\mathrm{C2}) \quad \mathrm{D}g(p) \in \mathrm{Str}(T) \\ &(\mathrm{P2}) \quad (\mathrm{D}g(p))^{-1} \circ \mathrm{D}^2g(p) \in W. \end{aligned} \tag{1.1}$$

Then conditions (C1) and (C2) define $\mathrm{Str}(T)$-conformality and (P1) and (P2) define T-projectivity.

Theorem IX.1.1. *Let T be a non-degenerate JTS containing no ideal isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

(i) (The Fundamental Theorem) *Any local T-projective diffeomorphism (of class $\mathcal{C}^4$) is birational, and its birational continuation belongs to* $\mathrm{Co}(T)$. *The group* $\mathrm{Co}(T)$ *is exactly the group of T-projective transformations.*

(ii) (The Liouville-Theorem) *If T is associated to a semisimple Jordan algebra via* $T(x,y,z) = 2(x(yz) - y(xz) + (xy)z)$, *then every local* $\mathrm{Str}(T)$*-conformal diffeomorphism is birational, and its birational continuation belongs to the group* $\mathrm{Co}(T)$, *i.e. it is a composition of the translations t_v, $v \in V$, of elements of* $\mathrm{Str}(V)$ *and of the Jordan inverse $j(x) = x^{-1}$. The group* $\mathrm{Co}(T)$ *is exactly the group of* $\mathrm{Str}(T)$*-conformal diffeomorphisms.*

Proof. We have already seen that elements of $\mathrm{Co}(T)$ satisfy the conditions (C2) and (P2) (Th. VIII.1.8 and Th. VIII.1.11). The proof of the converse occupies the remainder of this section. It is done in both cases by proving first an algebraic version on the level of vector fields, which we state next; the corresponding statement on the group-level is then deduced.

Theorem IX.1.2. *Let the assumptions be as in the preceding theorem.*

(i) (Infinitesimal version of the Fundamental Theorem) *Any locally defined T-projective vector field (of class $\mathcal{C}^3$) is polynomial and belongs to* $\mathfrak{co}(T)$. *The Lie algebra* $\mathfrak{co}(T)$ *is precisely the space of T-conformal vector fields.*

(ii) (Infinitesimal version of the Liouville Theorem) *If T is associated to a semisimple Jordan algebra, then* $\mathfrak{co}(T)$ *is exactly the space of* $\mathfrak{str}(T)$*-conformal vector fields. The Lie algebra* $\mathfrak{co}(T)$ *is precisely the space of* $\mathfrak{str}(T)$*-conformal vector fields.*

Proof. We have already seen that elements of $\mathfrak{co}(T)$ satisfy the conditions (C1) and (P1) of Eqn. (1.1) (Prop. VII.3.5). Conversely, let X be a vector field defined on some domain U and satisfying (C1) and (P1). We abbreviate $L := \mathrm{Str}(T) \subset \mathrm{Gl}(V)$ and $\mathfrak{l} := \mathfrak{str}(T) \subset \mathfrak{gl}(V)$. Then by (C1), $\mathrm{D}X$ is a map

$$\mathrm{D}X : U \to \mathfrak{l} \subset \mathrm{Hom}(V,V), \quad p \mapsto \mathrm{D}X(p);$$

therefore the second differential $\mathrm{D}^2X = \mathrm{D}(\mathrm{D}X)$ takes values in $\mathrm{Hom}(V,\mathfrak{l})$. On the other hand, $\mathrm{D}^2X(p)$ is a *symmetric* bilinear map $V \times V \to V$. Thus D^2X can be considered as a map

$$\begin{aligned}\mathrm{D}^2X : U \to \mathrm{Hom}_s(V,\mathfrak{l}) &:= \mathrm{Hom}(V,\mathfrak{l}) \cap \mathrm{Hom}(S^2V,V) \\ &= \{T : V \to \mathfrak{l} \,|\, \forall u,v \in V : T(u)v = T(v)u\}.\end{aligned}$$

Deriving further, we see that D^kX takes values in

$$\mathrm{Hom}_s(S^{k-1}V,\mathfrak{l}) := \mathrm{Hom}(S^kV,V) \cap \mathrm{Hom}(S^{k-1}V,\mathfrak{l}).$$

Note that this is an L- and an $\mathfrak{l}$-submodule of $\mathrm{Hom}(S^kV,V)$ (sometimes denoted by $\mathfrak{l}^{(k-1)}$, cf. [St83], [Ko72]). Similarly, condition (P1) implies that D^3X takes values in

$$\mathrm{Hom}_s(V,W) := \mathrm{Hom}(V,W) \cap \mathrm{Hom}(S^3V,V) = \{\alpha : S^3V \to V \,|\, \forall x \in V : \alpha(x,\cdot,\cdot) \in W\}.$$

Note that

$$W \subset \mathrm{Hom}_s(V,\mathfrak{l})$$

since $T(u,x) \in \mathfrak{str}(T) = \mathfrak{l}$ and $T(u,x)v = T(v,x)u$ for all $u,v,x \in V$. Thus condition (P1) is a refinement of condition (C1).

Lemma IX.1.3.

(i) *If* $\mathrm{Hom}_s(V,W) = 0$, *then the Lie algebra* $\mathfrak{co}(T)$ *is precisely the space of T-projective vector fields.*

(ii) *If* $\mathrm{Hom}_s(V,\mathfrak{g}) = W$ *and* $\mathrm{Hom}_s(S^2V,\mathfrak{g}) = 0$, *then the Lie algebra* $\mathfrak{co}(T)$ *is precisely the space of* $\mathfrak{l}$*-conformal vector fields.*

Proof. (i) If $\mathrm{Hom}_s(V,W) = 0$ and X is T-conformal, then $\mathrm{D}^3X = 0$ and we can integrate. We then find that X is in fact quadratic, given by

$$X(x) = X(0) + \mathrm{D}X(0)\cdot x + \frac{1}{2}\mathrm{D}^2X(0)\cdot(x,x).$$

Since $\mathrm{D}X(0) \in \mathfrak{str}(T)$ and $\mathrm{D}^2X(0) \in W$ by conformality, X is of the form given in Th. VII.3.1, i.e. $X \in \mathfrak{co}(T)$.

(ii) If $\mathrm{Hom}_s(S^2V,\mathfrak{l}) = 0$ and X is $\mathfrak{l}$-conformal, then $\mathrm{D}^3X = 0$, and we can conclude as above, using that $\mathrm{D}^2X(0) \in \mathrm{Hom}_s(V,\mathfrak{l}) = W$. ∎

Proposition IX.1.4. *If* T *is a non-degenerate JTS containing no ideal isomorphic to* $\mathbb{R}$ *or* $\mathbb{C}$, *then* $\mathrm{Hom}_s(V,W) = 0$.

Proof. According to Prop. VII.2.10 the map $A : V \to W$, $v \mapsto T(\cdot, v, \cdot)$ is bijective. Therefore

$$\mathrm{Hom}_s(V,W) \to \mathrm{End}(V), \quad \alpha \mapsto \beta := A^{-1}\circ\alpha$$

is injective, i.e., we define β by the relation $A(\beta) = \alpha$, that is $\alpha(u)\cdot v\otimes w = T(v,\beta u,w)$. By the assumption on α, this expression is totally symmetric in u,v,w. In particular, the relation

$$T(v,\beta u) = T(u,\beta v) \tag{1.2}$$

holds for all $u,v \in V$. When applying $\sharp$ to both sides, the relation $T(a,b)^\sharp = -T(b,a)$ (cf. Th. VII.3.1) yields $T(\beta u, v) = T(\beta v, u)$. Using the notaton $R(a,b) = T(a,b) - T(b,a)$, we get $R(v,\beta u) = R(u,\beta v)$, i.e.

$$R(v,\beta u) = -R(\beta v, u) \tag{1.3}$$

holds for all $u,v \in V$. We claim that this implies the relation

$$R(v,u)\circ\beta = -\beta\circ R(v,u) \tag{1.4}$$

for all $u,v \in V$. In order to prove this, we note that for all $x,y \in V$, $(y|\beta x) = \mathrm{tr}\,T(y,\beta x) = \mathrm{tr}\,T(\beta y, x) = (\beta y|x)$, i.e. β is self-adjoint. Using this and the relation

$$(R(x,y)u|v) = (R(u,v)x|y),$$

which holds for all $x,y,u,v \in V$ (cf. Lemma V.1.3), we get from (1.3)

$$\begin{aligned}(R(u,v)\beta x|y) &= (R(\beta x,y)u|v)\\ &= -(R(x,\beta y)u|v)\\ &= -(R(u,v)x|\beta y)\\ &= -(\beta R(u,v)x|y).\end{aligned}$$

Since the trace-form is non-degenerate, (1.4) follows by comparing the first and the last expression. From (1.4) we deduce that for all $a,b,u,v \in V$,

$$[R(a,b),R(u,v)]\circ\beta = \beta\circ[R(a,b),R(u,v)]. \tag{1.5}$$

Thus equations (1.4) and (1.5) together imply that for all $H \in [\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$,

$$H\circ\beta = \beta\circ H = 0. \tag{1.6}$$

It follows that $\beta(V) \subset V$ is a submodule on which $[\mathfrak{h}, \mathfrak{h}]$ acts trivially.

Let us assume now that T is simple and not isomorphic to the one-dimensional real or complex JTS. Then $R \neq 0$ and $[\mathfrak{h}, \mathfrak{h}] \neq 0$. Now, according to Th. V.4.4 and Cor. V.1.11, the following three cases can arise:

(a) V is an irreducible $[\mathfrak{h}, \mathfrak{h}]$-module of dimension bigger than one;

(b) V is the direct sum of two irreducible $[\mathfrak{h}, \mathfrak{h}]$-modules of dimension bigger than one;

(c) V is the direct sum of a trivial one-dimensional and an irreducible non-trivial $[\mathfrak{h}, \mathfrak{h}]$-module. This case arises precisely if T comes from a simple Jordan algebra (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with product $(x, y) \mapsto xy$ via the formula $T(u, v, w) = 2(u(vw) - v(uw) + (uv)w)$, and the trivial submodule is the space $\mathbb{K}e$ of multiples of the unit element e.

In the first two cases the only submodule on which $[\mathfrak{h}, \mathfrak{h}]$ acts trivially is zero, whence $\beta = 0$. In the third case β has to be a multiple of the projection onto the trivial one-dimensional module. Thus β is given by $\beta(x) = \lambda(x|e)e$ with $\lambda \in \mathbb{K}$, and

$$T(u, \beta v) = \lambda(v|e)T(u, e) = 2\lambda(v|e)L(u),$$

where $L(u)x = ux$. If $\lambda \neq 0$, then condition (1.2) with $v = e$ implies that $L(u) = (u|e)L(e) = (u|e)\,\mathrm{id}_V$ for all $u \in V$. Since $\mathfrak{h} = [L(V), L(V)]$, this implies that $\mathfrak{h} = 0$; this case corresponds to the simple Jordan algebra $V = \mathbb{K}$ and is excluded. Therefore $\lambda = 0$ and $\beta = 0$. (One could also reduce case (c) to (a) or (b) by replacing T by a suitable modification $T^{(\alpha)}$ (cf. Lemma III.4.5). The space W does not depend on the modification.)

Thus the proposition is proved in the case that T is simple. If T is a non-degenerate JTS on V, we decompose $V = \oplus_i V_i$ into simple ideals w.r.t. T; then $W = \oplus_i W_i$ where the W_i correspond to the V_i. From the definition of $\mathrm{Hom}_s(V, W)$ one easily gets that

$$\mathrm{Hom}_s(V, W) = \oplus_i \mathrm{Hom}_s(V_i, W_i),$$

and in this way the general statement is reduced to the case of a simple JTS. ■

The preceding proposition together with Lemma IX.1.3 (i) proves part (i) of Theorem IX.1.2.

Proposition IX.1.5. *If V is a semisimple Jordan algebra and $\mathfrak{l} = \mathfrak{str}(V)$ its structure algebra, then* $\mathrm{Hom}_s(V, \mathfrak{l}) = W$.

Proof. We proceed in three steps:

1. Recall the decomposition $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{q} = L(V)$ acts by self-adjoint operators and $\mathfrak{h} = \mathrm{Der}(V)$ acts by skew-symmetric operators w.r.t. the trace form $(x|y) = \mathrm{tr}\, L(xy)$. If $S \in \mathrm{Hom}_s(V, \mathfrak{l})$, we let $S(v) = S_1(v) + S_2(v)$ with $S_1(v) \in \mathfrak{h}$ and $S_2(v) \in \mathfrak{q}$. Then, for $u, v, w \in V$,

$$\begin{aligned}(S(u)v|w) &= (S(v)u|w) \\ &= (u|(S_2 - S_1)(v)w) \\ &= (u|S(v)w) - 2(u|S_1(v)w).\end{aligned}$$

Since the trace form is symmetric, we get

$$(S(u)v \mid w) - (S(v)w \mid u) = 2(S_1(v)u \mid w).$$

We take the sum of the three equations obtained by cyclic permutation of u, v, w and obtain

$$(w|S_1(u)v) + (v|S_1(w)u) + (u|S_1(v)w) = 0.$$

In particular, for $u = e$, we have

$$(w|S_1(e)v) + (v|S_1(w)e) - (S_1(v)e|w) = 0,$$

whence $(w|S_1(e)v) = 0$ since $\mathfrak{h} \cdot e = 0$. Since the trace form is non-degenerate, it follows that $S_1(e) = 0$. Thus $S(e) = S_2(e) \in \mathfrak{q}$ for all $S \in \mathrm{Hom}_s(V, \mathfrak{l})$.

2. Let $\widetilde{V}$ be the vector space V equipped with the L-action by $g \cdot v = (g^*)^{-1}(v)$ and the $\mathfrak{l}$-action by $X \cdot v = -X^*(v)$. We claim that the linear map

$$p : \mathrm{Hom}_s(V, \mathfrak{l}) \to \widetilde{V}, \quad S \mapsto S(e,e) := S(e)e$$

is $\mathfrak{l}$-equivariant. In fact, it is clearly equivariant w.r.t. $h \in \mathrm{Aut}(V)$ because $h(e) = e$, and therefore it is also equivariant w.r.t $\mathfrak{h} = \mathrm{Der}(V)$. Now let $X \in \mathfrak{q}$; then $X^* = X$, and by definition of the $\mathfrak{l}$-action in $\mathrm{Hom}(V, \mathrm{Hom}(V,V))$, we have for all $S \in \mathrm{Hom}_s(V, \mathfrak{l})$,

$$\begin{aligned} X \cdot p(S) - p(X \cdot S) &= -X(S(e,e)) - (X(S(e,e)) - S(Xe,e) - S(e,Xe)) \\ &= -2(X \circ S(e))(e) + 2(S(e) \circ X)(e) \\ &= 2[S(e), X](e) = 0 \end{aligned}$$

since, as seen above, $S(e) \in \mathfrak{q}$ and thus $[X, S(e)](e) \in [\mathfrak{q},\mathfrak{q}](e) = \mathfrak{h}(e) = 0$. Thus p commutes with the action of $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{q}$.

3. Since $p(A(v)) = T(e,v,e) = v$ for all $v \in V$, $p : \mathrm{Hom}_s(V, \mathfrak{l}) \to \widetilde{V}$ is surjective. We claim that p is also injective. Let $S \in \ker p$. Then, since p is equivariant, $g \cdot S \in \ker p$ for all $g \in L_o$. Thus

$$0 = (g \cdot S)(e \otimes e) = g(S(g^{-1}e \otimes g^{-1}e)),$$

so that $S(v \otimes v) = 0$ for all v belonging to the open orbit $\Omega = L(e) \subset V$. But since Ω is open in V, the set $d(\Omega) = \{v \otimes v|\ v \in \Omega\}$ generates S^2V as a vector space, and hence $S = 0$. Thus p is injective, and since $p \circ A = \mathrm{id}_V$, it follows that p is a bijection which is inverse to A. Therefore $\mathrm{Hom}_s(V, \mathfrak{l}) = A(V) = W$. ∎

Note that, for $S \in \mathrm{Hom}_s(S^2V, \mathfrak{l})$ and $v \in V$ fixed, $S(v, \cdot)$ belongs to $\mathrm{Hom}_s(V, \mathfrak{l})$. Therefore the preceding two propositions together imply that, if V is a semisimple Jordan algebra having no ideal isomorphic to $\mathbb{R}$ or $\mathbb{C}$, then $\mathrm{Hom}_s(S^2V, \mathfrak{l}) \subset \mathrm{Hom}_s(V, W) = 0$. Now part (ii) of Theorem IX.1.2 follows from Lemma IX.1.3 (ii), and Theorem IX.1.2 is completely proved. In order to deduce Theorem IX.1.1 from Theorem IX.1.2, we need the following proposition.

Proposition IX.1.6. *Let g be a locally defined diffeomorphism of V and X a vector field on V.*

(i) *If g satisfies* (C2) *and X satisfies* (C1)*, then g_*X satisfies* (C1).

(ii) *If X satisfies* (C1) *and* (P1) *and g satisfies* (C2) *and* (P2)*, then g_*X satisfies* (P1).

Proof. (i) We denote by $\Gamma(\mathbf{W})$ the space of smooth sections over V of the structure bundle $\mathbf{W}$; these are just the smooth functions $V \to W$. Thus $\Gamma(\mathbf{W})$ is a subspace of the space of smooth functions $V \to \mathrm{Hom}(S^2V, V)$, and condition (C2) means that g preserves this subspace. Similarly, (C1) is equivalent to the condition

$$X \cdot \Gamma(\mathbf{W}) \subset \Gamma(\mathbf{W}).$$

Thus, if g satisfies (C2) and X satisfies (C1), then

$$(g_*X) \cdot \Gamma(\mathbf{W}) = g \cdot (X \cdot (g^{-1} \cdot \Gamma(\mathbf{W}))) \subset \Gamma(\mathbf{W}).$$

(ii) Let ∇ be the canonical flat connection of V. Recall that (P2) is equivalent to $g \cdot \nabla - \nabla \in \Gamma(\mathbf{W})$. Similarly, (P1) is equivalent to

$$X \cdot \nabla \in \Gamma(\mathbf{W}).$$

Using this and part (i), we get

$$(g_*X) \cdot \nabla = g \cdot (X \cdot (g^{-1} \cdot \nabla)) \in g \cdot (X \cdot (\nabla + \Gamma(\mathbf{W}))) \subset g \cdot \Gamma(\mathbf{W}) \subset \Gamma(\mathbf{W});$$

thus g_*X satisfies (P1), and (ii) is proved. ∎

Now we prove part (i) of Theorem IX.1.1. By Theorem IX.1.2 (i), $\mathfrak{co}(T)$ is precisely the space of vector fields (of class $\mathcal{C}^3$) satisfying (C1) and (P1). If g is of class $\mathcal{C}^4$ and satisfies (C2) and (P2), then part (ii) of the preceding proposition implies that g_* preserves $\mathfrak{co}(T)$. Thus by Th. VIII.1.1, g is actually birational and belongs to the conformal group $\mathrm{Co}(T)$ (Def. VIII.1.2). In a similar way, Th. IX.1.2 (ii) and part (i) of the preceding proposition imply Th. IX.1.1 (ii).■

Remark IX.1.7. The proof of Th. IX.1.1 shows that a "fundamental theorem" holds whenever $\mathrm{Hom}_s(V,W) = 0$. ■

2. Application to the classical spaces

2.1. The fundamental theorem of projective geometry (real case).

Lemma IX.2.1. *A locally defined diffeomorphism φ of $V = \mathbb{R}^n$ ($n > 1$) preserves segments of straight lines if and only if there exists a locally defined one-form λ such that for all x where φ is defined and all $v \in V$,*

$$(\mathrm{D}\varphi(x))^{-1} \circ \mathrm{D}^2\varphi(x) \cdot (v \otimes v) = \langle \lambda(x), v \rangle v.$$

Proof. By assumption, if $\alpha(t) = x + tv$, then $\beta(t) := \varphi(\alpha(t))$ with t such that the expression is defined, is a segment of a straight line. This is equivalent to saying that $\beta'(t) = \mathrm{D}\varphi(\alpha(t)) \cdot v$ and $\beta''(t) = \mathrm{D}^2\varphi(\alpha(t)) \cdot (v \otimes v)$ are linearly dependent for all t in the domain of definition, i.e. for $y = \alpha(t)$ there exists a scalar $\lambda = \lambda(y; v)$ such that $\mathrm{D}^2\varphi(y) \cdot (v \otimes v) = \lambda(y; v)\mathrm{D}\varphi(y) \cdot v$. This is equivalent to the condition stated in the lemma, because the left-hand side is quadratic in v and therefore $\lambda(y; v)$ must be linear in v. ■

For $u, l, w \in \mathbb{R}^n$ we let

$$T(u, l, w) := ul^t w + wl^t u; \tag{2.1}$$

this is the JTS denoted in Section VIII.4 by $M(1, n; \mathbb{R})$. Then the condition of the preceding lemma can be rephrased as

$$(\mathrm{D}\varphi(x))^{-1} \circ \mathrm{D}^2\varphi(x) \in W = \{T(\cdot, v, \cdot) |\, v \in V\}$$

for all x where φ is defined. This is nothing but the condition (P2) for the JTS T. Since the structure group of T is equal to $\mathrm{Gl}(n, \mathbb{R})$, condition (C2) is empty in this case. Thus the preceding lemma says that a locally defined diffeomorphism of $\mathbb{R}^n$ preserves straight lines if and only if it is T-projective. Of course, this is a special case of a general theorem due to H.Weyl saying that two torsionfree affine connections have the same geodesics (as unparametrized lines) if and only if their difference is a section of the bundle with typical fiber W (cf. [Ei37]).

Theorem IX.2.2. *A locally defined diffeomorphism φ of $\mathbb{RP}^n$ ($n > 1$) preserves segments of straight lines if and only if it coincides on its domain of definition with an element of the general projective group $\mathbb{P}\,\mathrm{Gl}(n + 1, \mathbb{R})$.*

Proof. We have seen that, in the canonical affinization $\mathbb{R}^n$ of $\mathbb{RP}^n$, the condition on φ of the theorem is equivalent to T-projectivity. According to Theorem IX.3.1 (i) this is equivalent to $\varphi \in \mathrm{Co}(T)$. In Section VIII.4.3 we have determined $\mathrm{Co}(T)$: it is precisely the general projective group. ■

As an easy exercise, the reader may prove the crucial condition $\mathrm{Hom}_s(V,W)=0$ (Prop. IX.1.5) in the special case of the JTS (2.1) by elementary linear algebra. In the complex and quaternionic case, the proof of Lemma IX.2.1 has to be modified a little: one has to prove that $\lambda(y;v)$ is in fact $\mathbb{F}$-linear in v, $\mathbb{F}=\mathbb{C}$ resp. $\mathbb{H}$. Moreover, one should note that in the complex case $\mathrm{Co}(T)$ is strictly bigger than $\mathbb{P}\,\mathrm{Gl}(n+1,\mathbb{C})$ (it contains also the complex conjugation).

2.2. Theorems of Liouville and Lie. If $V=\mathbb{R}^n$ is equipped with the quadratic form $b(x,x)=x^t I_{p,q}x$, then a locally defined diffeomorphism φ is called *b-conformal* if φ preserves b up to a scalar function, i.e. for all x where φ is defined,

$$\mathrm{D}\varphi(x)\in L:=\{tg|\,g\in \mathrm{O}(p,q),t\in\mathbb{R}^*\}, \tag{2.2}$$

i.e. φ is L-conformal. The following theorem is due to Sophus Lie, who generalized the result of Liouville corresponding to the case $p=3$, $q=0$.

Theorem IX.2.3. *If φ is a locally defined b-conformal ($\mathcal{C}^4$-regular) diffeomorphism of $\mathbb{R}^n$ ($n>2$), then φ can be written as a composition of translations, of elements of the group L and the inversion $J(x)=\frac{x}{b(x,x)}$.*

Proof. The group L is the structure group of the semisimple Jordan algebra $V=\mathbb{R}^n$ defined by the quadratic form b (Section II.3). According to Th. IX.1.1 (ii) (note that the cases $n=1$ and $n=2$ are excluded), φ is L-conformal if and only if it belongs to the group $\mathrm{Co}(T)$ generated by the elements mentioned in the claim (it is harmless to replace the Jordan inverse j by J since $J=jg$ with an element $g\in L$). ∎

Note that according to Section VIII.4.4, $\mathrm{Co}(T)\cong \mathrm{SO}(p+1,q+1)$. The groups $\mathrm{SO}(n+1,1)$ are known from work of E. Cartan as the "conformal groups of the n-sphere"; in fact, S^n is the conformal compactification of the Jordan algebra $\mathbb{R}^n$ with a positive definite quadratic form (cf. Section VIII.4.4).

2.3. Causal groups and Segal's conjecture. If $\Omega\subset\mathbb{R}^n$ is a regular (i.e. open, convex and not containing straight lines) cone in $\mathbb{R}^n$, then a local diffeomeorphism φ of $\mathbb{R}^n$ is called Ω*-causal* or just *causal*, if for all x where φ is defined,

$$\mathrm{D}\varphi(x)\cdot\Omega=\Omega. \tag{2.3}$$

In other words, φ is $G(\Omega)$-conformal, where

$$G(\Omega):=\{g\in \mathrm{Gl}(n,\mathbb{R})|\,g(\Omega)=\Omega\}$$

is the group of linear automorphisms of Ω. The locally defined causal diffeomorphisms form the Ω*-causal pseudogroup.* Historically, the most important example is the Lorentz-cone in $\mathbb{R}^4$ leading to the *causal (pseudo-)group of special relativity.* Note that the Lorentz-cone is the symmetric cone associated to the Euclidean Jordan algebra $V=\mathrm{Herm}(2,\mathbb{C})$ which is isomorphic to the algebra corresponding to $p=3,q=1$ from the preceding theorem.

Theorem IX.2.4. *If V is a Euclidean Jordan algebra and Ω the associated symmetric cone, then every locally defined ($\mathcal{C}^4$-regular) Ω-causal diffeomorphism of V can be written as a composition of translations, elements of $G(\Omega)$ and of $-j$, where j is the Jordan inverse in V. The birational maps thus obtained form an open subgroup $\mathrm{Cau}(T)$ of $\mathrm{Co}(T)$.*

Proof. Note first that the transformations named in the claim are indeed causal. This being trivial for translations and elements of $G(\Omega)$, it remains only to be shown for $-j$. But $\mathrm{D}(-j)(x)=Q(x)^{-1}$, and $Q(x)\in G(\Omega)$ since $Q(x)$ preserves the space V' from Prop. II.2.9 and $Q(x)e=x^2\in\Omega$.

In order to prove that conversely all causal transformations are of this form, we need the fact that $G(\Omega) \subset \mathrm{Str}(V)$ which we will not prove here (cf. [FK94, Prop. VIII.2.8]). Since clearly $\exp(\mathfrak{str}(V)) \subset G(\Omega)$, it follows that $G(\Omega)$ is open in $\mathrm{Str}(V)$ (if V is simple, it is a normal subgroup of index 2, cf. loc.cit.). Now Ω-causal transformations φ are also $\mathrm{Str}(T)$-conformal, and we can apply Th. IX.1.1 (ii) in order to conclude that $\varphi \in \mathrm{Co}(T)$. The decomposition of $\mathrm{Co}(T)$ given in Th. VII.2.3 shows that necessarily φ is of the form given in the claim. Note that $-j$ is causal at every point where it is regular; therefore the elements of the form given in the claim form indeed a subgroup of $\mathrm{Co}(T)$ which is open in $\mathrm{Co}(T)$ because $G(\Omega)$ is open in $\mathrm{Str}(T)$. ■

The theorem implies that the causal group acts transitively on V^c. For $x = g.0 \in V^c$ with g in the causal group we define a cone $\Omega_x := T_0 g \cdot \Omega$ in the tangent space $T_x(V^c)$; this is well-defined since the stabilizer of the causal group at the base point 0 preserves $\Omega = \Omega_0$. Thus we obtain a *field of cones* $(\Omega_x)_{x \in V^c}$ on V^c. A diffeomorphism g is called *causal* if $\Omega_{g.x} = T_x g \cdot \Omega_x$ for all x where g is defined. Theorem IX.2.4 implies that every locally defined causal diffeomorphism of V^c has an extension to a global causal diffeomorphism of V^c, given by an element of the causal group. In this sense V^c may be called the *causal completion of* V. Here is a complete list of simple Euclidean Jordan algebras, their causal completions and causal groups (cf. Table XII.3.1):

V	V^c	$\mathrm{Co}(T)_o$
$\mathrm{Herm}(n,\mathbb{C})$	$\mathrm{U}(n)$	$\mathbb{P}\mathrm{U}(n,n)_o$
$\mathrm{Sym}(n,\mathbb{R})$	$\mathrm{Lag}(F,\mathbb{R}) \cong \mathrm{U}(n)/\mathrm{O}(n)$	$\mathbb{P}\,\mathrm{Sp}(n,\mathbb{R})$
$\mathrm{Herm}(n,\mathbb{H})$	$\mathrm{U}(2n)/\mathrm{Sp}(n)$	$\mathbb{P}\,\mathrm{SO}^*(4n)$
$\mathbb{R}^{n,1}$	$S^n \times S^1$	$\mathrm{SO}(n+1,2)$
$\mathrm{Herm}(3,\mathbb{O})$	$\mathrm{Herm}(3,\mathbb{O})^c$	$E_{7(-25)}$

The determination of the causal group of $\mathrm{U}(n)$ (first line of the table) confirms a conjecture by I.E. Segal ([Se76, p.35]).

2.4. Theorems of Chow and Dieudonné. A *linear relation of order* k ($k \in \{0,1,\ldots,q-1\}$) in the Grassmannian $\mathrm{Gr}_{p,p+q}(\mathbb{F})$ is a set of the form

$$[F] := \{W \in \mathrm{Gr}_{p,p+q}(\mathbb{F}) \mid W \subset F\}, \tag{2.4}$$

where $F \subset \mathbb{F}^{p+q}$ is a $p+k$-dimensional subspace. This is the analog of subspaces of projective space. We say that a locally defined diffeomorphism *preserves linear relations* if the image of a (piece of) a linear relation is again a (piece of) a linear relation. It is clear that linear relations are preserved under the canonical action of the group $\mathbb{P}\,\mathrm{Gl}(p+q,\mathbb{F})$.

Lemma IX.2.5. *In the chart given by the graph-imbedding $V := M(p,q;\mathbb{F}) \to \mathrm{Gr}_{p,p+q}(\mathbb{F})$, $X \mapsto \Gamma_X$, linear relations are represented by affine subspaces of V.*

Proof. Let $[F]$ be a linear relation having non-empty intersection with V. By applying translations (which are given by elements of $\mathbb{P}\,\mathrm{Gl}(p+q,\mathbb{F})$, cf. Eqn. (VIII.4.3)), we may assume that $\mathbb{F}^p \oplus 0 = \Gamma_0 \in [F]$. In other words, $\mathbb{F}^p \oplus 0 \subset F$, and therefore $F = \mathbb{F}^p \oplus F_2$ with a k-dimensional subspace $F_2 \subset 0 \oplus \mathbb{F}^q$. Now the condition $\Gamma_X \subset F$ is equivalent to the condition that the image of the linear operator X belongs to F_2. Clearly the space of such operators is a vector subspace of $M(p,q;\mathbb{F})$. ■

In contrast to the situation in the usual projective space, one cannot join two arbitrary points by a linear relation, and if this is possible, the linear relation is not unique. In fact, there is a linear relation of order k containing both Γ_X and Γ_Y if and only if $\dim(\Gamma_X \cap \Gamma_Y) \geq k$, i.e. $\mathrm{rk}(X - Y) \leq k$. The following result has been proved in another way by W.L. Chow ([Ch49]).

Theorem IX.2.6. *A locally defined diffeomorphism of $V = M(n, \mathbb{F})$ $(n > 1)$ preserving linear relations of order k, $k = 1, \ldots, n$, is an element of the group generated by $\mathbb{P}\,\mathrm{Gl}(2n, \mathbb{F})$, by maps of the form $W \mapsto W^{\perp}$ (w.r.t. some non-degenerate form), and (in the case $\mathbb{F} = \mathbb{C}$) by $Z \mapsto \overline{Z}$. The group thus obtained is the conformal group $\mathrm{Co}(T)$ of the Jordan algebra V.*

Proof. Note first that the transformations named in the claim indeed preserve linear relations. In order to prove that conversely every map φ preserving linear relations is of this form, we may assume that the domain of definition of φ is contained in the chart V. Then, since φ preserves linear relations and linear relations are represented in the chart V by affine subspaces (cf. the preceding lemma), if follows that (for all x where φ is defined) $\mathrm{D}\varphi(x)$ preserves linear relations through the origin. In particular, $\mathrm{D}\varphi(x)$ preserves the set $\{X \in V \,|\, \det(X) = 0\}$ which is the union of linear relations of order $n-1$. Since the polynomial det is irreducible, we may conclude that $\det \circ \mathrm{D}\varphi(x)$ is a multiple of the polynomial det, which in turn implies that $\mathrm{D}\varphi(x) \in \mathrm{Str}(V)$ ([FK94, p.161]). Thus φ is $\mathrm{Str}(V)$-conformal and belongs, according to Thm. IX.1.1 (ii) to the group $\mathrm{Co}(V)$. It only remains to be shown that $\mathrm{Co}(V)$ is generated by the elements given in the claim. We know already that $\mathrm{Co}(V)_o = \mathbb{P}\,\mathrm{Gl}(2n, \mathbb{F})_o$ (Prop. VIII.4.1). The other connected components are given by the connected components of $\mathrm{Str}(V)$ which are known (cf. Table XII.1.3). ∎

Similarly, we can characterize the groups preserving linear relations on the varieties of Lagrangians corresponding to the algebras $\mathrm{Sym}(A, \mathbb{R})$ and $\mathrm{Herm}(A, \mathbb{F})$ as the conformal groups of these algebras. This proves certain results of Dieudonné ([D63]).

Notes for Chapter IX.

IX.1. It has been known for a long time in the theory of *G-structures* that statements such as the Liouville-theorem correspond to the vanishing of the modules $\mathrm{Hom}_s(S^kV, \mathfrak{l})$ (usually denoted by $\mathfrak{l}^{(k)}$) for a certain k, cf. [St83], [Ko72]. However, to our knowledge the first complete and "global" proof of this fact is the one given in [Be96a, Th.2.1.4 and Th.2.2.1]. It seems that at present all known examples of Liouville-theorems in this sense are related to Jordan theory (where $\mathfrak{l}^{(2)} = 0$).

The proof of similar results by Kaneyuki ([Kan89, Th.6.2]) and Gindikin and Kaneyuki ([GiKa96, Th.3.3]) uses the heavy machinery of *Cartan-connections* developed by N. Tanaka. For several reasons we wished to avoid this theory. From a conceptual viewpoint, just as usual affine spaces are the model spaces for the introduction of affine connections, spaces with a non-degenerate JTS are the model spaces for the introduction of Cartan-connections. But it is clear that affine linear algebra logically precedes affine differential geometry, and therefore "elementary" Jordan theory should also precede the differential geometry of Cartan-connections.

The modules $\mathrm{Hom}_s(S^kV, \mathfrak{l})$ and $\mathrm{Hom}_s(V, W)$ are related to certain *Spencer cohomology groups*, cf. [Gui65]: the assumption that V contains no ideal isomorphic to $\mathbb{R}$ or $\mathbb{C}$ means that the Spencer cohomology group $H^{2,1}$ vanishes; then $\mathrm{Hom}_s(V, \mathfrak{l})/W$ is the Spencer cohomology group $H^{1,1}$, and Prop. IX.1.5 says that this cohomology group vanishes. In the literature it is claimed that already E. Cartan proved that (in the language used here) $\mathrm{Hom}_s(S^2V, \mathfrak{l}) = 0$ whenever $\mathfrak{l}$ is the structure algebra of a simple JTS not isomorphic to $M(1, n; \mathbb{F})$ (cf. [Gui65, Section 5], [Tan85, Lemma 1.12]; cf. also [Go87, Prop.4.2]); however, we have not been able to recognize this result in [Cartan]. From this claim it follows (by the general result from [Be96a]) that a Liouville-theorem holds for all simple JTS not isomorphic to $M(1, n; \mathbb{F})$ (this is essentially the theorem by Kaneyuki and Gindikin quoted above). Since we do not have a proof of the property $\mathrm{Hom}_s(V, \mathfrak{l}) = W$ in the spirit of Jordan theory if T does not belong to a Jordan algebra, we have formulated our Liouville-theorem only for Jordan algebras.

IX.2. For proofs of the fundamental theorem of projective geometry by linear algebra cf. [Ar66] or [B87]. A simple proof in the real case going back to Möbius is described in [Wey23]. Liouville stated his theorem without proof in [Li1850]. In [Sch54, p. 312] the proof of Lie's theorem is qualified as "tiresome and uninteresting". A short and interesting proof can be found in [Wey23]. H.Weyl very clearly remarked the formal and geometric similarities between the fundamental theorem and the Liouville theorem.

The notion of *causality* has its origins in the theory of relativity (cf. [Se76]). It has become an important aspect of geometry and analysis on Lie groups and symmetric spaces (cf. Section XI.3). At present it is not yet clear which role causal structures defined by symmetric cones (as in our examples) play in this general context.

The original proofs of the theorems of Chow ([Ch48]) and Dieudonné ([D63]) are similar to the algebraic proofs of the fundamental theorem mentioned above. Chow characterizes the "projective group" also as the algebraic automorphism group of Grassmannians resp. varieties of Lagrangians. This has been generalized by O. Loos for all conformal compactifications of non-degenerate JTS ([Lo78], [Lo79]).

Chapter X: Algebraic structures of symmetric spaces with twist

In this chapter we pass from the *local* theory of symmetric spaces with twist to their *global* theory. Here we encounter a deep difference between Jordan- and Lie-theory: a Lie algebra or a Lie triple system determines the associated Lie group, resp. the associated symmetric space only locally, and only by additional topological requirements (such as simple connectedness) one can determine a global object. In contrast, without any additional requirements a Jordan triple system determines a canonically associated globally symmetric space with twist. The key property of this space is not a topological one, but an algebraic one: it is an *algebraic symmetric space* in the sense of Def. I.4.4, and its global algebraic structure is entirely determined by its Jordan triple system.

In order to understand this difference, recall that the classical *Campbell-Hausdorff formula* (cf. e.g. [HN91, Satz I.4.8]) expresses the local group multiplication of a Lie group in the exponential chart. It is an analytic formula, given by a power series in terms of the Lie bracket. One can derive from the classical Campbell-Hausdorff formula a *Campbell-Hausdorff formula for symmetric spaces* expressing the multiplication map μ of a symmetric space (cf. Section I.4) on a neighborhood of the origin in the exponential chart; it is a power series in terms of the associated Lie triple system. In Jordan theory there is a canonical chart which we have called *Jordan coordinates* (Chapter VII). It turns out that the multiplication map μ of the corresponding symmetric space with twist is *rational* in this chart, and it can be explicitly calculated. Thus we get a *rational analog of the Campbell-Hausdorff formula* (Th. X.3.2) – it is still a complicated formula, but it is simpler than the original Campbell-Hausdorff formula. Moreover, it allows to extend the germ of a symmetric space we started with in a unique way to a global space M. An open dense part $M \cap V$ of M is covered by the Jordan coordinates; we call it the *affine part of* M and describe it as a Zariski-open domain in V, complementary to the set of zeros of a polynomial related to the *Bergman operator* defined in Chapter VIII (Th. X.2.1). Our "Campbell-Hausdorff formula" is global in the sense that it describes the multiplication map on this open dense set; since in general neither M nor $M \cap V$ are topologically connected, this is something which is completely out of reach in Lie theory.

There are many other features which reflect in one way or another the fact that the space M is globally determined by its JTS: first, the exponential map itself is described in Jordan coordinates by a formula generalizing the well-known formula expressing the exponential map of the Poincaré disc by the hyperbolic tangent of the Euclidean distance (Th. X.4.1). Again, our result is global in the sense that it holds on an open dense set which is in general non-connected; therefore once more we replace the more traditional analytic arguments by more algebraic ones. We prove that the global spaces share with simply connected spaces the property that germs of homomorphisms always extend to global ones (Th. X.3.5) and look at homomorphisms of one-dimensional spaces into M: in analogy to Lie-theory we call them *one-parameter subspaces* (Section 5). They are geodesics, but not every geodesic is a one-parameter subspace: only the geodesics starting in the direction of (generalized) *tripotents* are one-parameter subspaces. The twisted para-complexification of a one-parameter subspace leads to a natural homomorphism of $\mathrm{Sl}(2,\mathbb{R})$ into the conformal group of M and to the associated *Peirce-decomposition*.

Finally, in Section 6 we describe explicitly the pseudo-Riemannian metric, the invariant measure and the Cartan-involution associated to *non-degenerate* spaces.

We assume throughout in this chapter that T is a faithful JTS (although parts of the theory could be developed for arbitrary JTS).

1. Open symmetric orbits in the conformal completion

1.1. The open symmetric orbit associated to a JTS. Let T be a faithful JTS with conformal group $\mathrm{Co}(T)$, conformal completion $V^c = \mathrm{Co}(T)/P$ and Θ the canonical involution of $\mathrm{Co}(T)$. We are interested in the orbit M_0 of the base point $0 = 0_V$ under the identity component $G = \mathrm{Co}(T)^\Theta_o$ of the fixed point group of Θ,

$$M_0 := G.0 \subset V^c.$$

Proposition X.1.1. *The orbit M_0 is open in V^c, and it is a symmetric space with twist given by T. In Jordan coordinates the symmetry s_0 is described by $-\mathrm{id}_V$. As a homogeneous symmetric space, M_0 is isomorphic to the zero-component of the space* $\mathrm{Co}(T)^\Theta/\,\mathrm{Str}(T)^\Theta$.

Proof. The Lie algebra of G is $\mathfrak{g} = \mathfrak{co}(T)^\Theta$. It is stable under the involutive automorphism $(-\mathrm{id}_V)_*$, and the corresponding decomposition is

$$\mathfrak{g} = \mathrm{Der}(T) \oplus \mathfrak{q},$$

where $\mathfrak{q}$ is as in Th. VII.2.4 (3). Therefore the evaluation map $T_0M_0 \cong \mathfrak{q} \to T_0(V^c) = V$, $X \mapsto X(0)$ is surjective, and M_0 is open in V^c. Since Θ and $-\mathrm{id}_V$ commute, the group G is stable under conjugation by $-\mathrm{id}_V$. The stabilizer H of the origin 0 in $\mathrm{Co}(T)^\Theta$ is a subgroup of P which is stable under conjugation by $-\mathrm{id}_V$. It follows that H is a linear group; since it is Θ-stable, it is a subgroup of $\mathrm{Str}(T)^\Theta$. On the other hand, $\mathrm{Str}(T)^\Theta$ stabilizes the origin; therefore M_0 is the zero-component of $\mathrm{Co}(T)^\Theta/\,\mathrm{Str}(T)^\Theta$. The symmetry s_0 is now calculated as $s_0(g.0) = ((-\mathrm{id})_*g)(0) = -g(0)$.

Since $\mathrm{Str}(T)^\Theta = \mathrm{Aut}(T)$, it follows that $T = T_0$ is $\mathrm{Str}(T)^\Theta$-invariant and thus extends to a G-invariant tensor field on M_0. Moreover, T_0 is by construction a Jordan-extension of R_0, and thus T is a Jordan-extension of R. ∎

Definition X.1.2. The orbit M_0 is called the *open symmetric orbit associated to T*. ∎

Remark. If T is not faithful, we may define $G \subset \mathrm{Co}(T)$ to be the analytic subgroup with Lie algebra $\mathfrak{g} = \mathrm{Der}(T) \oplus \mathfrak{q} \subset \mathfrak{co}(T)$, and $M_0 := G.0 = G/H$ still is an open symmetric orbit. However, since H is in general not connected, we cannot assure that $H \subset \mathrm{Aut}(T)$.

1.2. Modifications of the open symmetric orbit. Let α be an element of the structure group of T with $\Theta(\alpha) = \alpha^{-1}$, and denote by α_* the conjugation by α in $\mathrm{Co}(T)$. Then $\Theta\alpha_*$ is an involution of $\mathrm{Co}(T)$: in fact,

$$(\Theta\alpha_*)^2 = \Theta\alpha_*\Theta\alpha_* = \Theta\Theta(\Theta(\alpha))_*\alpha_* = \Theta\Theta\alpha_*^{-1}\alpha_* = \mathrm{id}_{\mathrm{Co}(T)}.$$

We denote by $G^{(\alpha)} := (\mathrm{Co}(T)^{\Theta\alpha_*})_o$ the identity component of the corresponding fixed point group.

Proposition X.1.3. *The orbit* $M_0^{(\alpha)} := G^{(\alpha)}.0$ *is open in* V^c*; it is the open symmetric orbit associated to the JTS*

$$T^{(\alpha)}(u,v,w) = T(u,\alpha v,w).$$

The spaces $M_0^{(\alpha)}$ *and* $M_0^{(-\alpha)}$ *are c-dual symmetric spaces.*

Proof. The Lie algebra of $G^{(\alpha)}$ is $\mathfrak{g}^{(\alpha)} = \mathfrak{co}(T)^{\Theta\alpha_*}$. It is stable under $(-\operatorname{id})_*$; the corresponding decomposition is

$$\mathfrak{g}^{(\alpha)} = \mathfrak{h}^{(\alpha)} \oplus \mathfrak{q}^{(\alpha)}$$

with $\mathfrak{h}^{(\alpha)} = \mathfrak{str}(T)^{\Theta\alpha_*}$. From the formula (cf. Prop. VII.2.7)

$$\Theta\alpha_*(\mathbf{v} + \mathbf{p}_w) = \Theta(\alpha\mathbf{v} + \mathbf{p}_{\alpha^\sharp w}) = -\alpha^{-\mathbf{1}}\mathbf{w} - \mathbf{p}_{\alpha v}.$$

it follows that

$$\mathfrak{q}^{(\alpha)} = \{\mathbf{v} - \mathbf{p}_{\alpha v} |\, v \in V\}.$$

Thus the evaluation map $\mathfrak{q}^{(\alpha)} \to T_o(V^c) = V$, $X \mapsto X(0)$ is surjective, and as in the proof of Prop. X.1.1 it follows that $M_0^{(\alpha)}$ is an open orbit in V^c having the structure of a symmetric space with geodesic symmetry $-\operatorname{id}_V$ at the origin.

It remains to be shown that $T^{(\alpha)}$ is a Jordan-extension of the Lie triple product $R^{(\alpha)}$ of the LTS $\mathfrak{q}^{(\alpha)}$. In order to prove this, we denote by R the curvature corresponding to $\mathfrak{q}^{(\alpha)}$ and calculate

$$\begin{aligned} R(u,v)w &= -[[\mathbf{u} - \mathbf{p}_{\alpha u}, \mathbf{v} - \mathbf{p}_{\alpha v}], \mathbf{w} - \mathbf{p}_{\alpha w}]_0 \\ &= -[(T(v,\alpha u) - T(u,\alpha v)), \mathbf{w} - \mathbf{p}_{\alpha w}]_0 \\ &= -T(u,\alpha v,w) + T(v,\alpha u,w) = T^\alpha(v,u,w) - T^\alpha(u,v,w), \end{aligned}$$

thus $T^{(\alpha)}$ is a Jordan-extension of the LTS $\mathfrak{q}^{(\alpha)}$.

Finally, since $T^{(-\alpha)} = -T^{(\alpha)}$ and $R_{-T} = -R_T$, the Lie triple product of $\mathfrak{q}^{(\alpha)}$ is isomorphic to the negative of the one in $\mathfrak{q}^{(-\alpha)}$, and this means precisely that the two are c-dual. ■

In particular, the proposition shows that both $M_0 = M_0^{(\operatorname{id}_V)}$ and its c-dual space $M_0^{(-\operatorname{id}_V)}$ have a global realization on V^c. This is the global version of the local Borel-imbedding (Prop. VII.2.6; there $\mathfrak{q}^{(-\operatorname{id}_V)}$ was denoted by $\widehat{\mathfrak{q}}$). The more general statement given here means that all para-real forms of the para-Hermitian complexification $M_{ph\mathbb{C}}$ have a global realization on V^c (cf. Section IV.2). We illustrate this situation by two important examples (more examples will be given in Section XI.5):

Example X.1.4. (The general linear group) We consider the JTS on $V = M(n,\mathbb{R})$ given by $T(X,Y,Z) = XYZ + ZYX$. The identity component of the conformal group is $\mathbb{P}\operatorname{Gl}(2n,\mathbb{R})_o$, and Θ is given by conjugating with the matrix $F = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}$ which corresponds to the Jordan inverse $j(X) = X^{-1}$ (cf. Section VIII.4.1). The automorphism $(-\operatorname{id})_*$ is given by conjugating with the matrix $I_{n,n}$. Introducing a base change by the real Cayley transform R (Eqn. (I.6.6)), these two matrices are exchanged, and we see that the identity component G of $\operatorname{Co}(T)^\Theta$ is isomorphic to $\mathbb{P}(\operatorname{Gl}(n,\mathbb{R}) \times \operatorname{Gl}(n,\mathbb{R}))_o$ and its subgroup $G^{(-\operatorname{id})_*}$ is isomorphic to $\mathbb{P}(\operatorname{Gl}(n,\mathbb{R}))_o$, diagonally imbedded. It follows that

$$M_o \cong \mathbb{P}(\operatorname{Gl}(n,\mathbb{R})_o \times \operatorname{Gl}(n,\mathbb{R})_o)/\mathbb{P}\operatorname{Gl}(n,\mathbb{R})_o \cong \operatorname{Gl}(n,\mathbb{R})_o$$

is the (identity component of) the general linear group.

The c-dual space is obtained by replacing Θ by $(I_{n,n})_*\Theta = (I_{n,n}F)_* = J_*$, i.e. by conjugating with the matrix J. The group fixed under this automorphism is $\operatorname{Gl}(n,\mathbb{C})$, and we get the c-dual space $M_0^{(-\operatorname{id})} \cong \operatorname{Gl}(n,\mathbb{C})/\operatorname{Gl}(n,\mathbb{R})$ of $M_0 = \operatorname{Gl}(n,\mathbb{R})_o$. There are several other modifications $M_0^{(\alpha)}$ of the space $M = \operatorname{Gl}(n,\mathbb{R})$ (cf. Thm. XI.5.2). For instance, the automorphism $\alpha(X) = -X^t$ of $M(n,\mathbb{R})$ leads to the Grassmannian $\operatorname{Gr}_{n,2n}(\mathbb{R})$ (which is equal to V^c) to be discussed in the next example. ■

Example X.1.5. (Grassmannians.) We consider the JTS given on $V = M(p,q;\mathbb{R})$ by $T(X,Y,Z) = -(XY^tZ + ZY^tX)$. The identity component of the conformal group is $\mathbb{P}\,\mathrm{Gl}(n,\mathbb{R})_o$ ($n = p+q$), and Θ is given by $\Theta(g) = (g^t)^{-1}$ since the differential of Θ sends $\left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 & 0 \\ -X^t & 0 \end{smallmatrix}\right)$ as required. Thus $G = \mathrm{O}(n)_o$. The automorphism $(-\operatorname{id})_*$ is given by conjugating with the matrix $I_{p,q}$. It follows that

$$M_0 \cong \mathrm{O}(p+q)/(\mathrm{O}(p) \times \mathrm{O}(q))$$

is isomorphic to the real Grassmannian $\mathrm{Gr}_{p,p+q}(\mathbb{R})$. The c-dual space $M^{(-\operatorname{id})}$ is obtained by replacing Θ by $g \mapsto I_{p,q}(g^t)^{-1}I_{p,q}$, and one obtains the space $\mathrm{O}(p,q)/(\mathrm{O}(p)\times\mathrm{O}(q))$. ■

2. Harish-Chandra realization

Theorem X.2.1. *Let T be a faithful JTS, Θ the associated involution of the conformal group and $G = \mathrm{Co}(T)_o^\Theta$.*

(i) *The set*

$$M = \{h.0 \mid h \in \mathrm{Co}(T), \det(d_{\Theta(h)^{-1}h}(0)) \neq 0\} \subset V^c$$

is the union of open G-orbits in V^c. More precisely: the connected component M_0 is the open symmetric orbit defined in the preceding section, and for arbitrary $x = h.0 \in M$ ($\det(d_{\Theta(h)^{-1}h.0}(0)) \neq 0$), the connected component M_x of x is a homogeneous symmetric space, isomorphic to the modification $M_0^{(\alpha)}$ of M_0 with

$$\alpha = \Theta(t_{\Theta(h)^{-1}h.0})h^{-1}\Theta(h)t_{\Theta(h)^{-1}h.0}.$$

The space M is a symmetric space with twist.

(ii) *In Jordan coordinates, M is given by*

$$M \cap V = \{x \in V \mid \det B(x,x) \neq 0\},$$

and if $x \in V$, then the connected component M_x is isomorphic to $M_0^{(B(x,x)^{-1})}$.

Proof. (i) The condition $\det(d_{\Theta(h)^{-1}h}(0)) \neq 0$ means that $\Theta(h)^{-1}h(0) \in V$ (Prop. VIII.3.4). This condition is satisfied for $h \in \mathrm{Co}(T)$ if and only if it is satisfied for hp with $p \in P$, and thus M is well-defined.

The set M is $\mathrm{Co}(T)^\Theta$-invariant: replacing h by gh with $\Theta(g) = g$ does not affect the condition defining M. It follows that every open G-orbit is contained in M because M, being defined by algebraic equations, is open dense in V^c.

Let us prove that conversely every point of M belongs to an open G-orbit: assume $x = h.0$ with $\det(d_{\Theta(h)^{-1}h}(0)) \neq 0$. Then the element

$$g := \Theta(h)t_{\Theta(h)^{-1}h.0} \in \mathrm{Co}(T)$$

is well-defined. It satisfies the conditions $g.0 = h.0 = x$, $\Theta(g).0 = h.0 = x$. We apply g^{-1} to the orbit $G.x$:

$$g^{-1}(G.x) = \mathrm{Co}(T)_o^{g_*^{-1}\circ\Theta\circ g_*}.0 = \mathrm{Co}(T)_o^{\Theta\circ(\Theta(g)^{-1})_*\circ g_*}.0.$$

Consider the element

$$\Theta(g)^{-1}g = \Theta(t_{\Theta(h)^{-1}h.0})h^{-1}\Theta(h)t_{\Theta(h)^{-1}h.0} = \alpha.$$

It satisfies the conditions $\alpha(0) = \Theta(g)^{-1}.x = 0$, i.e. $\alpha \in P$, and $\Theta(\alpha) = g^{-1}\Theta(g) = \alpha^{-1}$, whence $\alpha \in \Theta(P)$. Together this implies $\alpha \in \mathrm{Str}(T)$. Now the property $\Theta(\alpha) = \alpha^{-1}$ means that α satisfies the assumption of Prop. X.1.2. It follows that

$$g^{-1}(G.x) = \mathrm{Co}(T)_o^{\Theta\circ\alpha_*}.0$$

is the modified space $M_0^{(\alpha)}$ which is open in V^c by Prop. X.1.2. Moreover, all connected components of M are symmetric spaces with twist and thus M also is one.

(ii) According to part (i), the point $x = t_x.0$ ($x \in V$) belongs to M if and only if $\det(d_{\Theta(t_x)^{-1}t_x}(0)) \neq 0$. By Prop. VIII.2.5, this is equivalent to $\det B(x,x) \neq 0$, proving the description of M in Jordan coordinates. Finally, if $h = t_x$ and $g = t_x\Theta(t_{\Theta(t_x)^{-1}x})$, then according to Prop. VIII.2.7 we have $\alpha = \Theta(g)^{-1}g = B(x,x)^{-1}$; thus $G.x \cong M_0^{(B(x,x)^{-1})}$. ■

Definition X.2.2. The symmetric space with twist M defined in the preceding theorem is called the *global space associated to the faithful JTS T*, and the set $M \cap V$ is called the *affine part of M*. ■

The reader may have remarked that in Section VIII.3 we have proved a very similar result: recall from Th. VIII.3.7 the *global polarized space*

$$\begin{aligned} X &= \{(g.0, h.0) \in V^c \times W^c |\, g,h \in \mathrm{Co}(T), \det(d_{h^{-1}g}(0)) \neq 0\} \\ &\cong \mathrm{Co}(T)/\,\mathrm{Str}(T) \end{aligned}$$

and its interpretation as the *set of finite points*. We will see that M is a para-real form of X: since T is faithful, the groups P and P' are related via $P' = \Theta(P)$, and the map

$$V^c \to W^c, \quad g.0_V \mapsto \Theta(g).0_W$$

is an equivariant bijection.

Lemma X.2.3. *The map $\tau : V^c \times W^c \to V^c \times W^c$, $(g.0, h.0) \mapsto (\Theta(h).0, \Theta(g).0)$ defines by restriction a para-conjugation of X.*

Proof. From the definition of the action of $\mathrm{Co}(T)$ on $V^c \times V^c$ it follows that $\tau(g.(x,y)) = \Theta(g).\tau((x,y))$, i.e. τ is equivariant and thus an automorphism. It is clearly involutive and exchanges the eigenspaces of the twisted polarization. Therefore it is a para-conjugation. ■

From now on we will identify W^c with the space V^c, equipped with the action of $g \in \mathrm{Co}(T)$ by $g.w = (\Theta(g))(w)$. Then $\tau((v,w)) = (w,v)$ is just the automorphism exchanging the two factors, and its fixed point space is the diagonal. We have proved:

Proposition X.2.4. *The space X^τ and the global space M are isomorphic:*

$$X^\tau = \{(x,x) \in V^c \times W^c |\, x \in M\}$$

The space $X \cong \mathrm{Co}(T)/\,\mathrm{Str}(T)$ is a global para-complexification of the global space M. ■

This realization of M explains the definition of the element $h \in \mathrm{Co}(T)$ in the proof of Th. X.2.1: it is defined by the condition $(x,x) = h.(0,0)$, and in this way we have used that X is homogeneous under $\mathrm{Co}(T)$. In contrast, the space M is in general neither homogeneous under G nor under $\mathrm{Co}(T)^\Theta$. Therefore one can often prove algebraic relations for M by proving them for X, using transitivity of the conformal group there, and then restrict to M (see Section 3).

Description in terms of line bundles. Recall from Section VIII.3 that we have described the "hyperplanes at infinity" in terms of sections of line bundles. The same can be done for the complement of M, which in a sense is the analog of a projective quadric. (In the case where $V^c = \mathbb{RP}^n$ is the ordinary real projective space, this interpretation coincides indeed with the classical one – see Ex. X.2.8 below.)

Lemma X.2.5. *The formula* $S(g) := d_{\Theta(g)^{-1}g}(0)$ *defines a* $\mathrm{Co}(T)^\Theta$*-invariant element of the representation* $\mathrm{ind}_P^{\mathrm{Co}(T)}\pi$ *induced from*

$$\pi : P \to \mathrm{End}(V), \quad p \mapsto (X \mapsto \mathrm{D}p(0) \circ X \circ \Theta(\mathrm{D}p(0))^{-1}).$$

The affine picture of the corresponding section of the vector bundle $\mathrm{Co}(T) \times_P \mathrm{Hom}(W,V)$ *(where* $\mathrm{Hom}(W,V)$ *is the representation space of* π *) is given by*

$$S(t_x) = B(x,x).$$

Proof. Recall that $d_\varphi(x) = (\mathrm{D}\varphi(x))^{-1}$ at points x where φ is regular. Let $p = hn$ with $h \in \mathrm{Str}(T)$, $n \in \exp(\mathbf{m}^+)$ and assume that $S(g)$ is invertible. Then

$$\begin{aligned} S(gp) &= (\mathrm{D}(\Theta(p)^{-1}\Theta(g)^{-1}gp)(0))^{-1} \\ &= (\Theta(h)^{-1}\,\mathrm{D}(\Theta(g)^{-1}g)(0)\,h)^{-1} \\ &= h^{-1} \circ (\mathrm{D}(\Theta(g)^{-1}g)(0))^{-1} \circ \Theta(h) \\ &= \pi(p)^{-1} S(g). \end{aligned}$$

Since the set of points where S is invertible is open dense in $\mathrm{Co}(T)$, the equation just proved holds by continuity for arbitrary g. This proves that S is an element of $\mathrm{ind}_P^{\mathrm{Co}(T)}\pi$. Next, for all $h \in G = \mathrm{Co}(T)^\Theta$,

$$S(hg) = d_{\Theta(hg)^{-1}hg}(0) = d_{\Theta(g)^{-1}g}(0) = S(g),$$

proving the invariance of S. The formula for $S(t_x)$ follows from Prop. VIII.2.5 (i). ■

As explained in Appendix VIII.B, the vector bundle $\mathrm{Co}(T) \times_P \mathrm{Hom}(W,V)$ can be interpreted as the bundle $\mathrm{Hom}(\mathbf{W}, T(V^c))$ (where $\mathbf{W}$ is structure bundle, induced from the representation $p \mapsto \Theta(\mathrm{D}p(0))$. The section S of the bundle $\mathrm{Hom}(\mathbf{W}, TM_0)$ over M_0 will be called the *inverse structure tensor* because for all $p \in M_0$, the isomorphism $S_p : \mathbf{W}_p \to T_pM_0$ is inverse to the isomorphism $T_pM_0 \to \mathbf{W}_p$ given by the structure tensor.

Corollary X.2.6. *The global space* M *is the complement of the zero-set of the section* $s := \det \circ S$ *of the line bundle* $\Lambda^n\mathbf{W}^* \otimes \Lambda^n T(V^c)$. ■

In the semisimple case, $\mathbf{W} \cong T^*(V^c)$, and then s is a section of the bundle $\Lambda^n T(V^c) \otimes \Lambda^n T(V^c)$ (square of the dual of the canonical bundle; cf. Section 6). – Similarly to Prop. VIII.3.4, one can also describe the translates of M:

$$h^{-1}.M = \{x \in V^c |\, x = g.0, \det(d_{\Theta(hg)^{-1}hg}(0)) \neq 0\},$$

and the affine picture of this set is given by the equation $\det(d_{\Theta(t_x)^{-1}\Theta(h)^{-1}ht_x}(0)) \neq 0$. The most important special case of this formula arises when there exists a *Cayley transform* h leading to *tube realizations* of M; see Section XI.2. In general, however, there is no canonical element $h \in \mathrm{Co}(T)$ sending the origin 0 to the boundary of M.

Example X.2.7. (The general linear group.) We consider the JTS on $V = M(n,\mathbb{R})$ given by $T(X,Y,Z) = XYZ + ZYX$. Then $P(X)Y = XYX$, and

$$B(X,Y)Z = Z - (XYZ + ZYX) + XYZYX = (\mathbf{1} - XY)Z(\mathbf{1} - YX).$$

Note that $\det B(X,Y) = \det((\mathbf{1} - XY)(\mathbf{1} - YX))^n$ and $\det B(X,X) = \det((\mathbf{1} - X)(\mathbf{1} + X))^{2n}$. The global space M associated to T is given in Jordan coordinates by the condition

$$M \cap V = \{X \in M(n,\mathbb{R}) |\, \mathrm{Det}((\mathbf{1} - X)(\mathbf{1} + X)) \neq 0\},$$

i.e. by the operators X having neither 1 nor -1 as eigenvalue. Thus the graph $W = \Gamma_X$ is characterized by the condition that its intersection with the diagonal and with the antidiagonal in $\mathbb{R}^n \oplus \mathbb{R}^n$ is zero. Applying the real Cayley transform R (i.e. a rotation by 45 degree), the transformed graphs $R(W)$ are characterized by the fact that the intersections with the axes of the product $\mathbb{R}^n \oplus \mathbb{R}^n$ are zero. This in turn means precisely that $R(W)$ belongs to an *invertible* linear operator. Thus $R(M) \subset V^c$ is the general linear group, and $M \cong \mathrm{Gl}(n, \mathbb{R})$.

The twisted para-complexification $\mathrm{Co}(T)/\mathrm{Str}(T) \cong \mathrm{Gl}(2n, \mathbb{R})/(\mathrm{Gl}(n, \mathbb{R}) \times \mathrm{Gl}(n, \mathbb{R}))$ of $\mathrm{Gl}(n, \mathbb{R})$ is given by

$$M_{ph\mathbb{C}} \cap (V \times V) = \{(X, Y) \in V \times V |\ \det((\mathbf{1} - XY)(\mathbf{1} - YX)) \neq 0\}.$$

In the case $n = 1$ the space $M_{ph\mathbb{C}} = \mathrm{Gl}(2, \mathbb{R})/(\mathrm{Gl}(1, \mathbb{R}) \times \mathrm{Gl}(1, \mathbb{R}))$ is isomorphic to the one-sheeted hyperboloid $\mathrm{Sl}(2, \mathbb{R})/\{dia(t, t^{-1}) | t \neq 0\}$, which in Jordan coordinates $V \times V = \mathbb{R} \times \mathbb{R}$ is described by the condition $B(x, y) = (1 - xy)^2 \neq 0$, i.e. $y \neq \frac{1}{x}$: The condition $\det B(x, x) = (1 - x^2)^2 = 0$ defines two points on $\mathbb{R}$ and thus M_0 is an interval $M_0 =]-1, 1[$. It corresponds to the segment of the diagonal cut off by the hyperbola $B(x, y) = 0$. ■

Example X.2.8. (Grassmannians.) We consider the JTS on $V = M(p, q; \mathbb{R})$ given by $T(X, Y, Z) = -(XY^tZ + ZY^tX)$. Then

$$B(X, Y)Z = (\mathbf{1} + XY^t)Z(\mathbf{1} + Y^tX).$$

Note that $\det B(X, X) = \det(\mathbf{1} + XX^t)^{2n}$. Since XX^t has only real non-negative eigenvalues, we have $\det B(X, X) \neq 0$ for all X. Therefore $M \cap V = V$ for the global space M associated to T. In fact, this could have been deduced already from Example X.1.5, where we have seen that $M_0 = \mathrm{Gr}_{p,p+q}(\mathbb{R}) = V^c$.

The twisted para-complexification $\mathrm{Gl}(p + q, \mathbb{R})/(\mathrm{Gl}(p, \mathbb{R}) \times \mathrm{Gl}(q, \mathbb{R}))$ of M is described in Jordan coordinates by

$$M_{ph\mathbb{C}} \cap (V \times V) = \{(X, Y) \in V \times V |\ \det((\mathbf{1} + XY^t)(\mathbf{1} + Y^tX)) \neq 0\}.$$

In the case $p = q = 1$ the space $M_{ph\mathbb{C}} = \mathrm{Gl}(2, \mathbb{R})/(\mathrm{Gl}(1, \mathbb{R}) \times \mathrm{Gl}(1, \mathbb{R}))$ is isomorphic to the one-sheeted hyperboloid $\mathrm{Sl}(2, \mathbb{R})/\{dia(t, t^{-1}) | t \neq 0\}$, which in Jordan coordinates $V \times V = \mathbb{R} \times \mathbb{R}$ is described by the condition $B(x, y) = (1 + xy)^2 \neq 0$, i.e. $y \neq -\frac{1}{x}$. The situation is c-dual to the one of $\mathrm{Gl}(1, \mathbb{R})$ described in the preceding example. The equation $\det B(x, x) = (1 + x^2)^2 = 0$ has no real solution, and thus $M \cap \mathbb{R}$ is the whole real axis, corresponding to the fact that the diagonal has empty intersection with the hyperbola $B(x, y) = 0$.

For $(p, q) = (1, n + 1)$, M is the real projective space $\mathbb{RP}^n$. The condition $\det B(x, x) = 0$ is then equivalent to

$$x^t x = -1$$

which has no real solution. Similarly one can realize the quadric given by $x^t I_{p,q} x = -1$ as the boundary of a suitably modified space $M^{(\alpha)}$. Thus all non-degenerate projective quadrics appear as boundaries of global spaces $M^{(\alpha)} \subset \mathbb{RP}^n$. ■

Note that, according to Lemma IV.1.4, the spaces $M = \mathrm{Gl}(1, \mathbb{R})$ and $M = \mathbb{RP}^1$ are (up to isomorphism) the only one-dimensional symmetric spaces with twist given by a faithful JTS; in this sense the preceding examples already describe the one-dimensional situation completely.

3. Jordan theoretic analog of the Campbell-Hausdorff formula

Our aim in this section is to show that the global space M with its atlas given by Jordan coordinates is a *real algebraic symmetric space* in the sense of Def. I.4.4 – that is, Jordan coordinates are algebraic, and the *multiplication map* $\mu : M \times M \to M$, $(x,y) \mapsto s_x(y)$ is rational.

Proposition X.3.1. *Jordan coordinates provide an algebraic atlas of the global space* M*. The transition transformation from Jordan coordinates around* 0 *to Jordan coordinates around* $x \in M$ *for* $x = g.0$ *with* $g \in \mathrm{Co}(T)$ *is, up to an element of* $\mathrm{Str}(T)$*, given by*

$$\Theta(g)t_{\Theta(g)^{-1}g(0)}.$$

Proof. If $x \in M_0$, $x = g.0$ with $g \in G$, then it is clear that the transition from Jordan coordinates around 0 to Jordan coordinates around x is simply given by $g = \Theta(g)$. In the general case, recall from the proof of Th. X.2.1 that for $h = \Theta(g)t_{\Theta(g)^{-1}g(0)}$,

$$h^{-1}(M_x) = M_0^{(\alpha)}$$

with α as in Th. X.2.1. It is easily seen that the transition transformation from Jordan coordinates of M_0 to those of $M_0^{(\alpha)}$ is simply α. Therefore $h\alpha$ is the transition transformation from Jordan coordinates of M_0 to those of $M_x = G.x$. ■

Theorem X.3.2. *The global space* M *associated to a faithful JTS* T *with its atlas given by Jordan coordinates around each point is an algebraic symmetric space in the sense of* Def. I.4.4. *In Jordan coordinates, its quadratic map* Q *is given for* $x \in M \cap V$ *by*

$$\begin{aligned} Q(x) &= t_x\ \Theta(t_{2\Theta(t_x)^{-1}x})\ t_x \\ &= \Theta(t_x)\ t_{2\Theta(t_x)^{-1}x}\ \Theta(t_x) \\ &= t_{2\Theta(t_x)x}\ B\big(2\Theta(t_x)x, -x\big)^{-1}\ \Theta(t_{2\Theta(t_x)x}); \end{aligned}$$

its multiplication map is given by

$$\begin{aligned} \mu(x,y) &= (Q(x))(-y) \\ &= x + B\big(x-y, -2\Theta(t_x)^{-1}x\big)^{-1} \cdot (x - y + 2P(x-y)\Theta(t_x)^{-1}x), \end{aligned}$$

and the squaring by

$$x^2 = Q(x)0 = 2\Theta(t_x)x = 2(\mathrm{id} + \frac{1}{2}P(x))^{-1}x.$$

Proof. We prove the formulas for Q and μ first for the polarized space X and then restrict to M. Let $(x,y) = (g.0, h.0) \in V^c \times W^c$. If $(x,y) \in X$, then $\Theta(g)^{-1}h.0 \in V$, and

$$(x,y) = g.(0, \Theta(g)^{-1}h.0) = g.(0, t_{\Theta(g)^{-1}h.0}.0) = g\Theta(t_{\Theta(g)^{-1}h.0}).(0,0).$$

Since $\mathrm{Co}(T)$ acts by automorphisms on the symmetric space X, the geodesic symmetry w.r.t. the point (x,y) is given by

$$s_{(x,y)} = g\Theta(t_{\Theta(g)^{-1}h.0}) \circ s_{(0,0)} \circ (g\Theta(t_{\Theta(g)^{-1}h.0}))^{-1} \tag{3.1}$$

with $s_{(0,0)} = -\mathrm{id}_{V\times W}$. Written out explicitly as an action on $V^c \times V^c$, this formula reads

$$\begin{aligned} s_{(x,y)}(v,w) = ((g\Theta(t_{\Theta(g)^{-1}h.0}) \circ (-\mathrm{id}) \circ (g\Theta(t_{\Theta(g)^{-1}h.0}))^{-1})(v), \\ (\Theta(g)t_{\Theta(g)^{-1}h.0} \circ (-\mathrm{id}) \circ (\Theta(g)t_{\Theta(g)^{-1}h.0})^{-1})(w)). \end{aligned} \tag{3.2}$$

If $(x,y) \in X \cap (V \times W)$, then we let $g = t_x$, $h = t_y$ and obtain

$$s_{(x,y)}(v,w) = t_x\Theta(t_{\Theta(t_{-x}(y)}) \circ (-\mathrm{id}) \circ (t_x\Theta(t_{\Theta(t_{-x}(y)}))^{-1} . (v,w). \tag{3.3}$$

This proves that $\mu((x,y),(v,w))$ is given by a rational formula. The multiplication map of M is simply obtained by restricting the one of X to the diagonal (Prop. X.2.4). Hence for $x = g.0$ it is given by

$$\mu(x,y) = g\Theta(t_{\Theta(g)^{-1}g.0}) \circ (-\mathrm{id}) \circ (g\Theta(t_{\Theta(g)^{-1}g.0}))^{-1}.y, \tag{3.4}$$

and therefore

$$Q(x) = s_x s_0 = g\Theta(t_{\Theta(g)^{-1}g.0}) \circ (-\mathrm{id})_*(g\Theta(t_{\Theta(g)^{-1}g.0}))^{-1}, \tag{3.5}$$

where $(-\mathrm{id})_*$ is conjugation by $-\mathrm{id}$. If $x \in V$, then we may choose $g = t_x$ and obtain from (3.5), taking into account of relations $(-\mathrm{id}) \circ t_z \circ (-\mathrm{id}) = t_{-z}$, $(-\mathrm{id}) \circ \Theta(t_z) \circ (-\mathrm{id}) = \Theta(t_{-z})$,

$$Q(x) = t_x\, \Theta(t_{\Theta(t_x)^{-1}x})^2\, t_x$$

proving the first formula for $Q(x)$. In order to deduce the second formula, note that $Q(x) \in G \subset \mathrm{Co}(T)^\Theta$ for all x in some neighborhood of the origin. For such x, $Q(x) = \Theta(Q(x))$, and thus the second formula holds. Now both expressions, being rational, must coincide for all $x \in M \cap V$ (and it follows that $\Theta(Q(x)) = Q(x)$ for all $x \in M \cap V$). The third formula is deduced from the second one using Prop. VIII.2.7: the first term in the "Harish-Chandra decomposition" $Q(x) = t_{Q(x).0}(DQ(x))(0)\Theta(t_{Q(x).0})$ (Eqn. VIII.(2.2), observing that $\Theta(Q(x)) = Q(x)$) is given by

$$Q(x).0 = \Theta(t_x)2\Theta(t_x)^{-1}x = 2\Theta(t_{2x})\Theta(t_x)^{-1}x = 2\Theta(t_x)x. \tag{3.6}$$

Now apply Prop. VIII.2.7 to the second expression for $Q(x)$, and the third one follows. The formula for μ is obtained by writing out the first formula for $Q(x)(-y) = s_x s_0(-y) = s_x(y)$ explicitly, using the formula $\Theta(t_a)^{-1}b = B(b,a)^{-1}(b - P(b)a)$ for the quasi-inverse.

Finally, we prove the formulas for the squaring. The formula $x^2 = 2\Theta(t_x)x$ has been proved above (Eqn. (3.6)). Explicitly, it reads

$$x^2 = 2\Theta(t_x)x = 2(\mathrm{id} + T(x,x) + P(x)^2)^{-1}(\mathrm{id} + P(x))x. \tag{3.7}$$

In Appendix A to this chapter, Cor. X.A.3, it is shown that the operators

$$(\mathrm{id} + P(x))^2 : \langle x\rangle \to \langle x\rangle \quad \text{and} \qquad \mathrm{id} + T(x,x) + P(x)^2 : \langle x\rangle \to \langle x\rangle$$

are well-defined and coincide, where $\langle x\rangle$ is the linear span of all $P(x)^j x = \frac{1}{2^j}T(x,x)^j x$, $j \in \mathbb{N}$ (they do in general not coincide on all of V). Therefore, if they are invertible, also their inverses coincide. Applying this observation to (3.7) yields the formula $x^2 = 2(\mathrm{id} + P(x))^{-1}x$. (In the same way one sees that

$$\Theta(t_x)^{-1}x = (\mathrm{id} - P(x))^{-1}x,$$

making some of the formulas more explicit.) ■

Since the formulas from the preceding theorem are rational, we have not specified in each case the domain of definition. In particular, one notes that x^2 is finite (i.e. belongs to V) iff $\Theta(t_x)x \in V$, i.e. iff $\det B(x,-x) \neq 0$. Since $B(x,-y)$ is the Bergman operator belonging to $T^{(-\mathrm{id})}$, this means that x belongs to the ("Borel-imbedded") c-dual space of M.

Corollary X.3.3. *With Q and μ as in the preceding theorem, the rational identities*

$$Q(Q(x)y) = Q(x)Q(y)Q(x), \quad Q(x^n) = Q(x)^n, \quad \mu(x^n, x^m) = x^{2n-m}$$

hold on V.

Proof. These identities hold by Lemma I.5.7 for all $x, y \in M$ for which the expressions are defined, and by rationality of both sides, they hold as rational identities on all of V. ■

The identities from the preceding corollary turn out to be very complicated if one wishes to write them explicitly in terms of the JTS T and its Bergman-operator. In order to get a slight impression we give here some rough Taylor expansions of the rational mappings defined in the theorem. Using that $(1-A)^{-1} = \sum_{k=0}^{\infty} A^k$ for all linear operators A in a neighborhood of zero, we get the expansions

$$\begin{aligned}
B(x,y)^{-1} &= \sum_k (T(x,y) - P(x)P(y))^k \\
&\sim \mathrm{id} + T(x,y) + T(x,y)^2 - P(x)P(y) \\
B(x,x)^{-1} &\sim \mathrm{id} + T(x,x) + T(x,x)^2 \\
B(x,-x)^{-1} &\sim \mathrm{id} - T(x,x) + T(x,x)^2 \\
\Theta(t_y)^{-1}x &\sim (\mathrm{id} + T(x,y))(x - P(x)y) \\
&\sim x + P(x)y - P(x)y = x + \frac{1}{2}P(x)y \\
\mu(x,y) &\sim x + (\mathrm{id} + T(x-y, -2\Theta(t_x)^{-1}x))(x - y + 2P(x-y)x) \\
&\sim x + (\mathrm{id} - 2T(x-y, x))(x - y + 2P(x-y)x) \\
&\sim 2x - y - 2P(x-y)x \\
Q(x)y &\sim 2x + y - 2P(x+y)x \\
x^2 &= 2\Theta(t_x)x \sim 2x - 2P(x)x.
\end{aligned}$$

3.2. Integrated version of Jordan structures: global formulas. Not only the symmetric space structure of M, but also the integrated Jordan structure of M from Ch. VI is given by a global and rational formula. Recall that the flow of the Euler vector field $E^{(x,y)}$ associated to a point $(x,y) \in X$ defines maps

$$r_{x,y} : V^c \times W^c \to V^c \times W^c, \quad (v, w) \mapsto (r^+_{x,y}v, r^-_{y,x}w)$$

for $r \in \mathbb{R}^*$ (cf. Section VI.2). Since $E^{(0,0)}$ is the usual Euler operator, we have $r_{0,0} = r\,\mathrm{id}_V \times r\,\mathrm{id}_W$. For a general point $(x,y) = (g.0, h.0) \in X$, we deduce analogously to the proof of Eqn. (3.1):

$$r_{x,y} = g\Theta(t_{\Theta(g)^{-1}h.0}) \circ r_{0,0} \circ (g\Theta(t_{\Theta(g)^{-1}h.0}))^{-1}.$$

If $(x,y) \in X \cap (V \times W)$, then we get similarly to (3.3)

$$r_{x,y} = t_x\Theta(t_{\Theta(t_x)^{-1}y}) \circ r_{0,0} \circ (t_x\Theta(t_{\Theta(t_x)^{-1}y}))^{-1}.$$

Note that this formula is rational. If one wishes, one can put it in another form by using the relations $r\,\mathrm{id}_V \circ t_z \circ r^{-1}\,\mathrm{id}_V = t_{rz}$, $r\,\mathrm{id}_V \circ \Theta(t_z) \circ r^{-1}\,\mathrm{id}_V = \Theta(t_{r^{-1}z})$.

3.3. Functorial properties. The next theorem states that the category of (faithful) Jordan triple systems is equivalent to the category of the associated global spaces. We know already (Th. III.4.7) that a similar equivalence of categories holds on the level of germs. Thus the main step is to prove that local homomorphisms always extend globally.

Theorem X.3.4. *Let $(T,V),(T',V')$ be faithful Jordan triple systems and M,M' the associated global spaces with multiplication maps μ,μ'.*

(i) *A germ φ_0 of a homomorphism of the pointed symmetric spaces with twist $(M,0)$, $(M',0')$ has a unique extension to a global homomorphism $\widetilde{\varphi}: M \to M'$. It is represented in Jordan coordinates by a linear map.*

(ii) *If $\widetilde{\varphi}$ is as in* (i)*, then it has a unique extension to a continuous (in fact, algebraic) map $V^c \to (V')^c$.*

(iii) *If $\widetilde{\varphi}$ is as in* (i)*, then it has a unique extension to a global homomorphism $\widetilde{\varphi}_{ph\mathbb{C}} : M_{ph\mathbb{C}} \to M'_{ph\mathbb{C}}$ of the global twisted paracomplexifications.*

Proof. (i) Let the germ φ_0 be given by the JTS-homomorphism $\varphi : V \to V'$ (cf. Th. III.4.7).

We expand $\mu(x,y)$ in a Taylor series containing only multilinear terms defined by the JTS T. In fact, all we need for this is the expansion (already used after Cor. X.3.3 for the Taylor expansions)

$$B(u,v)^{-1} = \sum_{k=0}^{\infty}(T(u,v) - P(u)P(v))^k$$

which holds for (u,v) in some neighborhood of the origin in $V \times W$. By usual manipulations of absolutely convergent power series we then get the desired expansion for the rational map $\mu(x,y)$ from Th. X.3.2.

Since φ is linear and compatible with T,T', it follows that the equality

$$\mu'(\varphi x, \varphi y) = \varphi\mu(x,y)$$

holds for all x,y in some neighborhood of the origin. Since both sides depend rationally on x,y, they coincide whenever $x,y,\mu(x,y) \in M \cap V$. Thus, letting $N := M \cap V$, the map $\varphi|_N : N \to V'$ extends the germ φ_0. We are going to show that $\varphi|_N$ extends to a global continuous homomorphism $M \to M'$. (Since N is open dense in M, there is at most one such extension.)

To this end let $z \in M \setminus N$; choose Jordan coordinates around z. Since N is dense in M, the intersection of their range with N contains a point x. Again we choose Jordan coordinates around x. Then φ is defined at x and extends again by a linear map in these new coordinates. In particular, it has a continuous extension to an open neighborhood of z, and this extension is independent of the choice of x because there is at most one continuous extension. By continuity it now follows that the global map φ thus defined is a homomorphism of symmetric spaces.

(ii) By density of M in V^c, the atlas given by Jordan coordinates around points in M extends to a rational atlas of V^c. In each of these charts $\widetilde{\varphi}$ is linear and therefore extends to a global map $V^c \to (V')^c$.

(iii) This follows by applying part (i) to the twisted complexification $M_{ph\mathbb{C}}$ of M. ■

Now we take a closer look at the case that φ is an *inclusion* of Jordan triple systems.

Proposition X.3.5. *Let (V,T) be a sub-JTS of the faithful JTS (V',T').*

(i) *The subspace $M_T \subset M'$ generated (as a symmetric space) by all $x \in (M' \cap V)$ is closed in M'.*

(ii) *The inclusion $V \subset V'$ induces an isomorphism $\iota : M \to M_T$.*

Proof. (i) Because T is a sub-JTS of T', the rational formula for μ' from Th. X.3.2 (resp. its Taylor-expansion used in the preceding proof) show that there is a neighborhood U of the origin in M' such that $M_T \cap U = M' \cap V \cap U$. Since we can choose Jordan coordinates around any point of M_T, this holds for all points of M_T and therefore M_T is closed.

(ii) By the preceding theorem, the inclusion induces a homomorphism $\iota : M \to M'$. Its image has to be contained in M_T. It follows that $\iota : M \to M_T$ is a covering (in fact, using Prop. I.3.5. it is proved as for Lie groups that a LTS-isomorphism, if it extends to a global homomorphism of associated connected spaces, induces a covering). We have to show that in our situation the covering is trivial with fibers containing one point each. Now, let $x \in M' \cap V$, U a neighborhood of x in $M' \cap V$ such that $\iota^{-1}(U)$ is a disjoint union of open sets U_i each homeomorphic to U. Then each U_i intersects V, and it follows immediately that $U_i = U$. Since every connected component of M' intersects V', it follows now that the covering ι is injective.■

The spaces M_T belonging to the sub-JTS T take the role of the "analytic subgroups" belonging to a Lie-subalgebra in Lie theory. In contrast to the situation in Lie theory, they are always closed, as shows the proposition. The important case that φ is the inclusion of a one-dimensional JTS leads to the definition of *one-parameter subspaces* and will be discussed in more detail in Section 5.

Complexifications. Next we discuss the case in which φ is the inclusion of a real form in a complex JTS. Let T be a real JTS on a real vector space V, let $T_{\mathbb{C}}$ be its $\mathbb{C}$-trilinear extension to a complex JTS on $V_{\mathbb{C}}$ (Prop. III.4.6). Recall that $R_{T_{\mathbb{C}}}$ is a straight complexification of R_T and $R_{(T_{\mathbb{C}})^\tau}$ a twisted complexification of R_T, where τ is complex conjugation of $V_{\mathbb{C}}$ w.r.t. V.

Proposition X.3.6.

(i) *The inclusion* $V = (V_{\mathbb{C}})^\tau \subset V_{\mathbb{C}}$ *induces natural inclusions*

$$\mathrm{Co}(T) \subset \mathrm{Co}(T_{\mathbb{C}}), \qquad V^c \subset (V_{\mathbb{C}})^c.$$

The image of the first inclusion is open in $\mathrm{Co}(T_{\mathbb{C}})^{\tau_*}$*, and the image of the second inclusion is open in* $((V_{\mathbb{C}})^c)^\tau$.

(ii) *The global space* $M_{\mathbb{C}}$ *associated to* $T_{\mathbb{C}}$ *is a global straight complexification of* M*, and the global space* $M_{h\mathbb{C}}$ *associated to* $T_{h\mathbb{C}}$ *is a global twisted complexification of* M.

Proof. (i) Note that τ, being an automorphism of $T_{\mathbb{C}}$ (considered as a real JTS), belongs to $\mathrm{Co}(T_{\mathbb{C}})$. The map

$$\mathfrak{co}(T_{\mathbb{C}})^{\tau_*} \to \mathfrak{co}(T), \quad X \mapsto X|_V$$

is an isomorphism; the inverse is given by assigning to $X \in \mathfrak{co}(T)$ its $\mathbb{C}$-polynomial continuation $X_{\mathbb{C}}$ to $V_{\mathbb{C}}$. It follows that, if we denote for $g \in \mathrm{Co}(T)$ by $g_{\mathbb{C}}$ its $\mathbb{C}$-rational extension onto $V_{\mathbb{C}}$, we obtain an injective map

$$\mathrm{Co}(T) \to \mathrm{Co}(T_{\mathbb{C}}), \quad g \mapsto g_{\mathbb{C}}.$$

From the corresponding situation on the Lie algebra level it follows that the image is open in $\mathrm{Co}(T_{\mathbb{C}})^{\tau_*}$. (In fact, it is a normal subgroup of index 2 in this group, the corresponding quotient being $\{\mathrm{id}, \tau\}$.)

We identify $\mathrm{Co}(T)$ with the corresponding subgroup of $\mathrm{Co}(T_{\mathbb{C}})$. Then, since the rational actions on V and on $V_{\mathbb{C}}$ are given by the same formulas, the stabilizer of $0_V \in V^c$ and the stabilizer of $0_{V_{\mathbb{C}}} \in (V_{\mathbb{C}})^c$ of the corresponding actions of $\mathrm{Co}(T)$ are the same, and it follows that

$$V^c \to (V_{\mathbb{C}})^c, \quad gP \mapsto gP_{\mathbb{C}}$$

is an injection. The affine picture of this inclusion is just $V \subset V_{\mathbb{C}}$. By reasons of dimension, V^c is open in $((V_{\mathbb{C}})^c)^\tau$.

(ii) We have to prove that M is a union of connected components of $(M_{\mathbb{C}})^\tau$. This follows from the description of the global space in terms of sections of line bundles (Cor. X.2.6): the section $\widetilde{s}$ describing $M_{\mathbb{C}}$, restricted to V^c, is just the square of the section s describing M

(due to the relation $\det_{\mathbb{R}}(A) = \det(A|_V)^2$, if A is a real endomorphism of the complex vector space $V_{\mathbb{C}}$). The same argument works for $M_{\mathbb{C}}$ replaced by the space $M_{h\mathbb{C}}$ associated to the modification $T_{ph\mathbb{C}}$ of $T_{\mathbb{C}}$.

Note finally that $M_{h\mathbb{C}}$ is indeed a globally circled space: In Jordan coordinates, multiplication $\mathcal{J}_o$ by i in the tangent space $T_o(M_{h\mathbb{C}}) = V_{\mathbb{C}}$ as well as its affine extension j_o to $M_{h\mathbb{C}}$ are both given by ordinary multiplication by i. ■

Proposition X.3.7. *If T is a real JTS and α an element of the structure variety of T, then the complexification diagram of the homogeneous symmetric space $M_0^{(\alpha)}$ with twist $T^{(\alpha)}$ is given by*

$$\begin{array}{ccccc} & \nearrow & \mathrm{Co}(T_{\mathbb{C}})^{\Theta\alpha_*}/\,\mathrm{Str}(T_{\mathbb{C}})^{\Theta\alpha_*} & \searrow & \\ \mathrm{Co}(T)^{\Theta\alpha_*}/\,\mathrm{Str}(T_{\mathbb{C}})^{\Theta\alpha_*} & \to & \mathrm{Co}(T_{\mathbb{C}})^{\Theta\alpha_*\tau_*}/\,\mathrm{Str}(T)^{\Theta\alpha_*\tau_*} & \to & \mathrm{Co}(T_{\mathbb{C}})/\,\mathrm{Str}(T_{\mathbb{C}}) \\ & \searrow & \mathrm{Co}(T)/\,\mathrm{Str}(T) & \nearrow & \end{array}$$

where Θ is the involution of $\mathrm{Co}(T)$ *defined by T.*

Proof. Propositions X.1.1 and X.1.3 show that $M_0^{(\alpha)} \cong (\mathrm{Co}(T)^{\Theta\alpha_*}/\,\mathrm{Str}(T)^{\Theta\alpha_*})_0$. From Prop. X.2.4 we know that the twisted para-complexification is given by the bottom line. The top line is of the form $G_{\mathbb{C}}/H_{\mathbb{C}}$ if $M = G/H$ and therefore describes the straight complexification. Finally, comparison with Prop. X.1.3 shows that the middle term is the symmetric space belonging to the JTS

$$\mathfrak{q}_{h\mathbb{C}} = \{\mathbf{v} - \mathbf{p}_{\tau(\alpha v)} |\, v \in V_{\mathbb{C}}\}$$

which corresponds to the JTS $T_{h\mathbb{C}}^{(\alpha)}$, and therefore it describes the twisted complexification of $M^{(\alpha)}$. Finally, the last term is obtained by a straight complexification of the bottom term. ■

4. The exponential map

For any symmetric space M, the exponential map $\mathrm{Exp} : T_oM \to M$ w.r.t. the base point o is defined by $\mathrm{Exp}(v) = \exp(l_o v).o$ (cf. Prop. I.5.8). If M is the global space associated to a faithful JTS T, then $M \cap V$ is an open neighborhood of the base point $0 \in M$. As usual, we identify V with T_0M. We let

$$V_f := \{v \in V|\ \mathrm{Exp}(v) \in M \cap V\} = \{v \in V|\ \det \mathrm{D}(\exp(l_0 v)(0)) \neq 0\} \tag{4.1}$$

be the set of points whose image under the exponential map is “finite”. The restriction of the exponential map to this set,

$$\mathrm{Exp} : T_0M = V \supset V_f \to (M \cap V) \subset V, \tag{4.2}$$

is a smooth map $V \supset V_f \to V$ such that $\mathrm{Exp}(0) = 0$ and, moreover, $\mathrm{Exp}(-v) = -\mathrm{Exp}(v)$ since $-\mathrm{id}_V$ is the geodesic symmetry w.r.t. 0. Furthermore, it will commute with the stabilizer group $\mathrm{Aut}(T)$ because elements of this group are linear, i.e. are identified with their differential at the base point, and the general relation $\mathrm{Exp} \circ \mathrm{D}h(0) = h \circ \mathrm{Exp}$ holds for all elements h of the isotropy group H. In this section we give an explicit formula for Exp.

Trigonometric and hyperbolic functions on a JTS. Recall the notation $P(v) = \frac{1}{2}T(v,\cdot,v)$ and define the following power series

$$\sinh(v) := \sinh_T(v) := \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(P(v))^k v, \tag{4.3}$$

$$\cosh(v) := \cosh_T(v) := \sum_{k=0}^{\infty} \frac{1}{(2k)!}(P(v))^k, \tag{4.4}$$

(In the following we drop the index T if there is no risk of confusion.) The power series defining sinh and cosh converge absolutely on the whole vector space V, and since the analytic function $v \mapsto \mathrm{Det}(\cosh v)$ does not vanish at the origin, it is non-zero almost everywhere, and thus the function

$$\tanh(v) := \tanh_T(v) := \cosh(v)^{-1}\sinh(v) \tag{4.5}$$

is defined almost everywhere. The usual manipulations of power series yield the expansion of $\tanh(v)$ in the power series of the usual hyperbolic tangent,

$$\tanh(v) = \sum_{k=0}^{\infty} 4^k(4^k-1)\frac{B_{2k}}{(2k)!}(P(v))^k v = v - \frac{1}{3}P(v)v + \frac{2}{15}(P(v))^2 v + \ldots \tag{4.6}$$

(where B_{2k} are the Bernoulli-numbers), holding for v in some neighborhood of the origin, and of its local inverse

$$\operatorname{artanh}(v) = \sum_{k=0}^{\infty} \frac{1}{2k+1}(P(v))^k v. \tag{4.7}$$

Moreover, the relations

$$\sinh(2v) = 2\cosh(v)\cdot\sinh(v), \quad \cosh(2v) = 2\cosh(v)^2 - \mathbf{1} \tag{4.8}$$

hold for all $v \in V$; they are identities of power series which are proved using the corresponding identities from the real or complex case. Note that for the JTS $\mathbb{R}$ with $T(x,y,z) = 2xyz$ we have $P(x)v = x^2 v$, and thus $\sinh_T$ etc. are just the usual hyperbolic functions. However, for the c-dual JTS $\mathbb{R}$ with $T(x,y,z) = -2xyz$, $\sinh_T(v) = \sin(v)$ is the usual sine, and $\tanh_T(v) = \tan(v)$ is the usual tangent. In particular, it becomes singular for $v = \frac{\pi}{2}$.

Theorem X.4.1. *The exponential map of the global space M associated to a JTS T is given by* $\mathrm{Exp}(v) = \tanh_T(v)$ *for all $v \in V_f$.*

Proof. We use the notation $lv := l_0(v) = \mathbf{v} - \mathbf{p}_v \in \mathfrak{q}$ and $\widehat{l}w := \widehat{l}_o(w) = \mathbf{w} + \mathbf{p}_w \in \widehat{\mathfrak{q}}$. Then $(lv)(0) = v$ and, for all $v \in V_f$,

$$\mathrm{Exp}(v) = \exp(lv).0 = d_{\exp(lv)}(0)^{-1} \cdot n_{\exp(lv)}(0),$$

where

$$n_{\exp(lv)}(p) = ((\exp(lv))^* E)(p) = (e^{\mathrm{ad}(lv)}E)(p)$$

and

$$(d_{\exp(lv)}(p)) \cdot w = (\exp(lv)^* \mathbf{w})(p) = (e^{\mathrm{ad}(lv)}\mathbf{w})(p).$$

The theorem will be deduced from the following proposition:

Proposition X.4.2. *For all $v \in V$ and w belonging to the linear span $\langle v \rangle$ of the $P(v)^j v$, $j \in \mathbb{N}$,*

$$n_{\exp(lv)}(0) = \frac{1}{2}\sinh(2v) = \cosh(v)\cdot\sinh(v),$$

$$(d_{\exp(lv)}(0)) \cdot w = \cosh(v)^2 \cdot w.$$

Proof. In order to determine the series

$$n_{\exp(lv)}(0) = (e^{\mathrm{ad}(lv)}E)(0) = \sum_k \frac{1}{k!}(\mathrm{ad}(-lv)^k E)(0)$$

we note first that $[lv, E] = \widehat{l}v$. Next we prove that for all $u \in V$ and $w \in \langle u \rangle$,

$$\mathrm{ad}(lu)^2 \widehat{l}w = \widehat{l}(P(2u)w). \tag{4.9}$$

In fact, from the relation $[\mathfrak{q}, [\mathfrak{q}, \widehat{\mathfrak{q}}]] \subset \widehat{\mathfrak{q}}$ (Prop. VII.1.4) we see that the left hand side is contained in $\widehat{\mathfrak{q}}$. Since elements of $\widehat{\mathfrak{q}}$ are uniquely determined by their value at the base point, we only need to verify that

$$[lu, [lu, \widehat{l}w]](0) = 2T(u, w, u).$$

This is proved using the relation $[\mathbf{x}, \mathbf{p}_y] = T(x, y)$ (Eqn (VII.2.1)):

$$\begin{aligned}
[lu, [lu, \widehat{l}w]](0) &= [\mathbf{u} - \mathbf{p}_u, [lu, \widehat{l}w]](0) = [\mathbf{u}, [lu, \widehat{l}w]](0) \\
&= [lu, \widehat{l}w](u) \\
&= [\mathbf{u} - \mathbf{p}_u, \mathbf{p}_w + \mathbf{w}](u) \\
&= T(u, w, u) + T(w, u, u) = P(2u)w.
\end{aligned}$$

(For the last equality we need that $w \in \langle u \rangle$, cf. Prop. X.A.1.) From (4.9) we get $\mathrm{ad}(lu)^{2k} \widehat{l}w = \widehat{l}(P(2u)^k w)$ and therefore $\mathrm{ad}(lu)^{2k+1} \widehat{l}w \in [\mathfrak{q}, \widehat{\mathfrak{q}}] \subset \mathfrak{str}(T)$, whence $(\mathrm{ad}(lu)^{2k+1} \widehat{l}w)(0) = 0$. Thus the power series in question equals

$$\begin{aligned}
\sum_k \frac{1}{k!} (\mathrm{ad}(-lv)^k E)(0) &= \sum_k \frac{1}{(2k+1)!} P(2v)^k v \\
&= \frac{1}{2} \sum_k \frac{1}{(2k+1)!} (P(2v))^k 2v = \frac{1}{2} \sinh_T(2v).
\end{aligned}$$

Together with (4.8) this proves the first part of the proposition. Next we determine

$$(d_{\exp(lv)}(0)) \cdot w = (e^{\mathrm{ad}(lv)} \mathbf{w})(0) = (\sum_k \frac{1}{k!} \mathrm{ad}(lv)^k \mathbf{w})(0).$$

We notice first that $[lv, \mathbf{w}] \in \mathfrak{str}(T)$; evaluation at the origin yields zero. Next we claim that

$$\mathrm{ad}(lv)^2 \mathbf{w} = \frac{1}{2} \mathrm{ad}(lv)^2 \widehat{l}w. \tag{4.10}$$

In fact, writing $\mathbf{w} = \frac{1}{2}(lw + \widehat{l}w)$, we have $\mathrm{ad}(lv)^2 \mathbf{w} = \frac{1}{2}(\mathrm{ad}(lv)^2 lw + \mathrm{ad}(lv)^2 \widehat{l}w)$. The first of the two terms vanishes since $\mathrm{ad}(lv)^2 lw \in \mathfrak{q}$ and

$$[lv, [lv, lw]](0) = R(v, w, v) = T(v, w, v) - T(w, v, v) = 0$$

according to Prop. X.A.1 (using our assumption on w). This proves (4.10). From (4.10) together with (4.9) it follows that $\mathrm{ad}(lv)^{2k} \mathbf{w} = \frac{1}{2} \mathrm{ad}(lv)^{2k} \widehat{l}w = \frac{1}{2} \widehat{l}(P(2v)^k w)$. As above one deduces that $(\mathrm{ad}(lv)^{2k+1} \mathbf{w})(0) \in \mathfrak{str}(T).0 = 0$. Thus the power series in question is equal to

$$\begin{aligned}
(\sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}(lv)^k \mathbf{w})(0) &= w + \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{(2k)!} P(2v)^k w \\
&= \frac{1}{2}(w + \sum_{k=0}^{\infty} \frac{1}{(2k)!} P(2v)^k w) \\
&= \frac{1}{2}(\cosh(2v)w + w) = \cosh(v)^2 w,
\end{aligned}$$

proving the proposition. It follows that

$$\mathrm{Exp}(v) = d_{\exp(lv)}(0)^{-1} \cdot n_{\exp(lv)}(0) = \cosh(v)^{-2} \cosh(v) \sinh(v) = \tanh(v),$$

proving the theorem. ∎

Remark X.4.2. Using power associativity and the fact that the ordinary hyperbolic tangent satisfies the differential equation $\tanh' = 1 - \tanh^2$, one can directly deduce that the curve $\alpha(t) := \tanh(tv)$ is a solution of the differential equation $\alpha'(t) = v - P(\alpha(t))v$ with initial value $\alpha(0) = 0$ (cf. [Lo77, Lemma 4.3], or [FK94, p.216] for the special case of tube domains). This gives another proof of Th. X.4.1 which is of a more "local" nature. Since it does not use the everywhere converging series sinh and cosh, one then has to discuss convergence questions more carefully. Moreover, since $M \cap V$ and V_f need not be connected, this strategy requires additional arguments in order to deduce the global result. ■

Example X.4.3. In the case of the one-dimensional JTS $\mathbb{R}$ with $T(x,y,z) = 2xyz$ the function $\tanh_T$ is the ordinary hyperbolic tangent. It is a bijection $\mathbb{R} \to]-1,1[$. From the addition formula

$$\tanh(s+t) = \frac{\tanh t + \tanh s}{1 + \tanh s \tanh t}$$

we see that $M =]-1,1[$ is a group isomorphic to $\mathbb{R}$ with group multplication

$$m(x,y) = \frac{x+y}{1+xy} = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}.y.$$

The multiplication map of M is given by

$$\mu(x,y) = m(m(x,x),-y) = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}^2.(-y) = \frac{-x^2y + 2x - 1}{x^2 - 2xy + 1}$$

and the powers by

$$x^{n+1} = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}^n.x;$$

in particular $x^2 = \frac{2x}{1+x^2}$. ■

Remark X.4.4. Recall from the proof of Thm. II.2.15 that the exponential map of a quadratic prehomogeneous symmetric space is given by $\mathrm{Exp}(v) = \exp(L(v)).e = \sum_k \frac{v^k}{k}$. Using the real Cayley transform to be defined in Ch. XI (Def. XI.2.3), this formula is related to the one from Thm. X.4.1 via the classical relation

$$(e^x - 1)(e^x + 1)^{-1} = \tanh(\frac{x}{2}).$$

■

5. One-parameter subspaces and Peirce-decomposition

Definition X.5.1. A *one-parameter subspace* of the global space M is a non-trivial homomorphism of a real one-dimensional symmetric space M' with twist T' into M. The one-parameter subspace is called *hyperbolic* if $T'(x,y,z) = 2xyz$, it is called *elliptic* if $T'(x,y,z) = -2xyz$ and it is called *degenerate* if $T' = 0$. ■

In particular, a one-parameter subspace is a homomorphism of symmetric spaces, i.e. a geodesic. But not every geodesic is a one-parameter subspace.

Proposition X.5.2. *Hyperbolic one-parameter subspaces correspond to sub-JTS of T of the form $\mathbb{R}e$, where $e \in V$ satisfies $T(e,e,e) = 2e$, elliptic one-parameter subspaces correspond to sub-JTS $\mathbb{R}e$ with $T(e,e,e) = -2e$, and degenerate ones correspond to sub-JTS $\mathbb{R}e$ with $T(e,e,e) = 0$.*

Proof. The element e is the image of $1 \in \mathbb{R}$ under the injective homomorphism $\mathbb{R} \to V$. ∎

Definition X.5.3. An element $e \in V$ with $T(e,e,e) = 2e$ is called a *hyperbolic tripotent*, an element with $T(e,e,e) = -2e$ is called an *elliptic tripotent* and an element with $T(e,e,e) = 0$ is called *completely degenerate.* ∎

Remark X.5.4. An element e which is a hyperbolic tripotent w.r.t. the JTS T is an elliptic tripotent w.r.t. the JTS $T^{(-)}(x,y,z) = -T(x,y,z)$ and vice versa. Usually in the literature the notion of "tripotent" corresponds to our "hyperbolic tripotent". In contrast to the situation in Lie theory, it is a priori not clear whether one-parameter subspaces exist. If they exist, they can in general only start in very particular directions, namely those belonging to (hyperbolic or elliptic) tripotents or to completetely degenerate elements. On the other hand, this situation excludes pathological situations as the "dense wind" in Lie theory – one-parameter subspaces are always closed; cf. the remark after Prop. X.3.5. ∎

The twisted para-complexification (which exists by Th. X.3.4 (iii)) of a hyperbolic one-parameter subspace is a homomorphism

$$\mathbb{R}^{(+)}_{ph\mathbb{C}} = H^2(\mathbb{R}) = \mathrm{Sl}(2,\mathbb{R})/\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \mid t \in \mathbb{R}^*\} \to M_{ph\mathbb{C}},$$

and the twisted para-complexification of an elliptic one-parameter subspace is a homomorphism

$$\mathbb{R}^{(-)}_{ph\mathbb{C}} = H^2(\mathbb{R}) = \mathrm{Sl}(2,\mathbb{R})/\{\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} \mid t \in \mathbb{R}^*\} \to M_{ph\mathbb{C}}.$$

Since the group $\mathrm{Sl}(2,\mathbb{R})$ is semisimple, it follows from Th. V.1.9 that these homomorphisms are actually equivariant (on the Lie algebra level). In other words, they correspond to homomorphisms

$$\mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{co}(T)$$

which we are going to make explicit. However, as explained in Appendix I.A, we will rather obtain homomorphisms $\mathfrak{sl}(2,\mathbb{R})^{op} \to \mathfrak{co}(T)$, where for a Lie algebra $\mathfrak{g}$ we denote by $\mathfrak{g}^{op}$ the same underlying vector space with the negative of the Lie bracket from $\mathfrak{g}$. Let

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{5.1}$$

be the usual basis of $\mathfrak{sl}(2,\mathbb{R})$ with commutator relations $[H,X] = 2X$, $[H,Y] = -2Y$, $[X,Y] = -H$ in $\mathfrak{sl}(2,\mathbb{R})$.

Lemma X.5.5.

(i) *Let e be a hyperbolic tripotent. Then the map $\dot{f} : \mathfrak{sl}(2,\mathbb{R})^{op} \to \mathfrak{co}(T)$ defined by*

$$H \mapsto T(e,e), \quad X \mapsto \mathbf{e}, \quad Y \mapsto \mathbf{p}_e$$

is an injective Lie algebra homomorphism. If we define an involution $\theta = \theta^{(+)}$ on $\mathfrak{sl}(2,\mathbb{R})$ by $\theta(Z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then the relation $\Theta \circ \dot{f} = \dot{f} \circ \theta$ holds.

(ii) *Let e be an elliptic tripotent. Then the map $\dot f : \mathfrak{sl}(2,\mathbb{R})^{op} \to \mathfrak{co}(T)$ defined by*

$$H \mapsto T(e,e), \quad X \mapsto \mathbf{p}_e, \quad Y \mapsto -\mathbf{e}$$

is an injective Lie algebra homomorphism. If we define an involution $\theta = \theta^{(-)}$ on $\mathfrak{sl}(2,\mathbb{R})$ by $\theta(Z) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) Z \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, then the relation $\Theta \circ \dot f = \dot f \circ \theta$ holds.

Proof. (i) The first statement follows from the commutator relations

$$[\mathbf{e}, \mathbf{p}_e] = T(e,e), \quad [T(e,e), \mathbf{e}] = -\mathbf{T}(\mathbf{e},\mathbf{e},\mathbf{e}) = -2\mathbf{e}, \quad [T(e,e), \mathbf{p}_e] = \mathbf{p}_{T(e,e)e} = 2\mathbf{p}_e,$$

and the second by verifying that $\theta^{(+)}(X) = Y$, $\theta^{(+)}(H) = -H$ and $\Theta(\mathbf{e}) = \mathbf{p}_e$ and $\Theta(T(e,e)) = -T(e,e)$. Part (ii) follows by similar calculations, now with $T(e,e,e) = -2e$. ∎

Having fixed a tripotent e, for simplicity of notation we consider the homomorphism $\dot f$ as an inclusion $\mathfrak{sl}(2,\mathbb{R})^{op} \subset \mathfrak{co}(T)$. Via the adjoint representation, $\mathfrak{co}(T)$ can now be considered as an $\mathfrak{sl}(2,\mathbb{R})^{op}$-module.

Proposition X.5.6.

(i) *If e is a hyperbolic tripotent, then $T(e,e) : V \to V$ is diagonalizable and has at most three eigenvalues, namely $0, 1$ and 2.*

(ii) *If e is an elliptic tripotent, then $T(e,e) : V \to V$ is diagonalizable and has at most three eigenvalues, namely $0, -1$ and -2.*

Proof. (i) Weyl's theorem tells us that every finite-dimensional $\mathfrak{sl}(2,\mathbb{R})$-module is completely reducible, and standard $\mathfrak{sl}(2)$-theory (cf. e.g. [Hu72, p.31 ff]) that the restrictions $\mathrm{ad}_U(H)$, $\mathrm{ad}_U(X)$, $\mathrm{ad}_U(Y)$ to an irreducible submodule $U \subset \mathfrak{co}(T)$ are represented by the following matrices

$$\begin{pmatrix} m & & & 0 \\ & m-2 & & \\ & & \ddots & \\ 0 & & & -m \end{pmatrix}, \quad \begin{pmatrix} 0 & m & & 0 \\ & 0 & m-1 & \\ & & 0 & \ddots \\ 0 & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & 0 & \\ 0 & & m & 0 \end{pmatrix}.$$

(Note that in the standard textbooks one assumes that U is a complex representation. After complexifying and noticing that all eigenvalues turn out to be real, one sees that these results hold also for real $\mathfrak{sl}(2,\mathbb{R})$-modules.) Since $\mathrm{ad}_U(Y)^3 = (\mathrm{ad}(\mathbf{e})|_U)^3 = 0$ and $\mathrm{ad}_U(X)^3 = (\mathrm{ad}(\mathbf{p}_e)|_U)^3 = 0$, the matrices can at most have size 3×3, i.e. only the values $m = 0, 1$ or 2 can appear. Summing up, we have a decomposition

$$\mathfrak{co}(T) = \mathfrak{co}(T)_{-2} \oplus \mathfrak{co}(T)_{-1} \oplus \mathfrak{co}(T)_0 \oplus \mathfrak{co}(T)_1 \oplus \mathfrak{co}(T)_2$$

into eigenspaces w.r.t. $\mathrm{ad}(H) = \mathrm{ad}(T(e,e))$. Since $[E, T(e,e)] = 0$, it follows that all eigenspaces are $\mathrm{ad}(E)$-stable; therefore we have a similar decomposition of the homogeneous parts $\mathfrak{m}_+, \mathfrak{str}(T)$ and $\mathfrak{m}_-$ of $\mathfrak{co}(T)$. In particular, V can be decomposed into eigenspaces V_j of $\mathrm{ad}(H)$ with $j \in \{-2,-1,0,1,2\}$. We have to show that actually only non-negative j appear in the decomposition of $V = \mathfrak{m}_-$. But it is clear that $\mathbf{e}$ annihilates all elements in $V = \mathfrak{m}^-$; therefore V must contain the highest weight vector v of an irreducible $\mathfrak{sl}(2,\mathbb{R})$-module U intersecting V. But then the lowest weight vector $\mathrm{ad}(\mathbf{p}_e)^2 v$ is contained in $\mathfrak{m}^+$. In other words, only weight spaces $\mathfrak{co}(T)_j$ with $j \geq 0$ can have non-zero intersection with V.

(ii) We apply the same arguments, where now the role of highest and lowest weights is reversed. ∎

Definition X.5.7. The eigenspace decomposition $V = V_0 \oplus V_1 \oplus V_2$, resp. $V = V_0 \oplus V_{-1} \oplus V_{-2}$, of $T(e,e)$ is called the *Peirce-decomposition of* V *w.r.t.* e. ■

Lemma X.5.8. *For the Peirce-decomposition w.r.t. a tripotent the relation*

$$T(V_j, V_k, V_l) \subset V_{j-k+l}$$

holds. In particular, the V_j *are sub-JTS of* V.

Proof. This follows from the equivariance $T : V \otimes W \otimes V \to V$, where $T(e,e)$ acts on W by $\Theta(T(e,e)) = -T(e,e)$. ■

Note that $V_2 \neq 0$ if e is hyperbolic since then $e \in V_2$, and similarly $V_{-2} \neq 0$ if e is elliptic. We will see in the next section that V_2 is actually a Jordan algebra with unit element e.

Proposition X.5.9. *If* e *is a hyperbolic tripotent, we have*

$$\exp(tT(e,e)) = B(e, (1 - \exp(t))e),$$

and if e *is an elliptic tripotent, then*

$$\exp(-tT(e,e)) = B(e, (1 - \exp(t))e).$$

Proof. We specialize the identity (JT2) (cf. Prop. III.2.4) to the case $v = x = y = z = e$ and get

$$T(u, e, T(e,e,e)) = T(T(u,e,e),e,e) - T(e, T(u,e,e), e) + T(e,e,T(u,e,e)),$$

and since the left hand side equals $\pm 2T(e,e,u)$ (positive sign if e hyperbolic, negative sign if e elliptic), we have

$$\pm 2T(e,e) = 4P(e)^2 - 2T(e,e)^2.$$

It follows that $P(e)^2$ has eigenvalue 0 on V_0 and on $V_{\pm 1}$ and eigenvalue 1 on $V_{\pm 2}$. Therefore

$$B(e, (1-z)e) = \mathrm{id} - (1-z)T(e,e) + (1-z)^2 P(e)^2$$

has eigenvalue 1 on V_0, eigenvalue z on $V_{\pm 1}$ and eigenvalue $1 - 2(1-z) + (1-z)^2 = z^2$ on $V_{\pm 2}$. If $z = \exp(t)$, then $\exp(\pm tT(e,e))$ has the same eigenvalues and eigenspaces, and therefore both linear transformations are equal. ■

When integrating the homomorphism $\dot{f}$ from Lemma X.5.5, mind that $\mathfrak{co}(T)^{op}$ is the Lie algebra of $\mathrm{Co}(T)$.

Corollary X.5.10.

(i) *If* e *is a hyperbolic tripotent, then there is a unique homomorphism* $f : \mathrm{Sl}(2,\mathbb{R}) \to \mathrm{Co}(T)$ *determined by*

$$f\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = t_{te}, \quad f\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \Theta(t_{te}), \quad f\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = B(e, (1-t)e).$$

If we define an involution $\theta = \theta^{(+)}$ *on* $\mathrm{Sl}(2,\mathbb{R})$ *by* $\theta(Z) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) Z \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, *then the relation* $\Theta \circ f = f \circ \theta$ *holds.*

(ii) *If* e *is elliptic, then there is a unique homomorphism* $f : \mathrm{Sl}(2,\mathbb{R}) \to \mathrm{Co}(T)$

$$f\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \Theta(t_{te}), \quad f\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = t_{te}, \quad f\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = B(e, (1-t)e).$$

If we define an involution on $\mathrm{Sl}(2,\mathbb{R})$ *by* $\theta(g) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) g \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, *then the relation* $\Theta \circ f = f \circ \theta$ *holds.*

Proof. We complexify the homomorphism $\dot{f}$ to an injective homomorphism from $\mathfrak{sl}(2,\mathbb{C})^{op}$ into $\mathfrak{co}(T)_{\mathbb{C}}$. This induces a homomorphism of the simply connected group $\mathrm{Sl}(2,\mathbb{C})$ into $\mathrm{Co}(T)_{\mathbb{C}}$. This homomorphism commutes with the respective complex conjugations and therefore defines by restriction a homomorphism $f : \mathrm{Sl}(2,\mathbb{R}) \to \mathrm{Co}(T)$. For any $Z \in \mathfrak{sl}(2,\mathbb{R})$ we have the formula $f(\exp(Z)) = \exp(\dot{f}Z)$; thus in part (i)

$$f\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = f(\exp(-tY)) = \exp(-t\dot{f}(Y)) = \exp(-t\mathbf{p}_e) = \Theta(t_{te}),$$

and similarly for the other matrices. The statement about the compatibility of θ and Θ follows from Lemma X.5.5. ∎

Remark X.5.11. The homomorphism f is injective if $V_1 \neq 0$ (resp. $V_{-1} \neq 0$) and has kernel ± 1 otherwise. (In fact, $f(-\mathbf{1}) = B(e, 2e)$ has eigenvalues 1 on V_0, -1 on $V_{\pm 1}$, $(-1)^2 = 1$ on $V_{\pm 2}$; thus $f(-1) = \mathrm{id}$ iff $V_1 = 0$, resp. $V_{-1} = 0$.) If $\ker f = \{\pm 1\}$, then f passes to the quotient as a homomorphism $\mathbb{P}\,\mathrm{Sl}(2,\mathbb{R}) \to \mathrm{Co}(T)$. Since $\mathbb{P}\,\mathrm{Sl}(2,\mathbb{R}) = \mathrm{Co}(\mathbb{R})_o$, f is now a homomorphism $f : \mathrm{Co}(\mathbb{R})_o \to \mathrm{Co}(T)$. In the other case, it is indeed necessary to pass to the double covering $\mathrm{Sl}(2,\mathbb{R})$ of $\mathrm{Co}(\mathbb{R})_o$ in order to lift the Lie algebra homomorphism $\mathfrak{co}(\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{co}(T)$. ∎

6. Non-degenerate spaces

A symmetric space with twist is called *non-degenerate* iff its JTS T is non-degenerate. In this case we have for all $g \in \mathrm{Str}(T)$ the equation $\Theta(g) = (g^*)^{-1}$, where the adjoint is taken w.r.t. the trace form (cf. Remark after Prop. VIII.1.6). This in turn means that the $\mathrm{Str}(T)$-module W (cf. Section 2.1) is isomorphic to the dual V^* of V via $W \to V^*$, $w \mapsto (v \mapsto \mathrm{tr}\, T(v,w))$. Therefore we have isomorphisms $\mathrm{Hom}(W,V) \cong V \otimes V$ and $\Lambda^n W^* \otimes \Lambda^n V \cong (\Lambda^n V) \otimes (\Lambda^n V)$, and the sections S and s introduced in Section 2.1 can be interpreted as the duals of an invariant pseudo-metric (Thm. 6.2), resp. of the "square" of an invariant volume form (Prop. 6.3).

6.1. The invariant pseudo-metric. If T is non-degenerate, then

$$(x|y) = \mathrm{tr}\, T(x,y)$$

defines a non-degenerate bilinear form on V which is invariant and symmetric (Lemma V.2.4).

Definition X.6.1. A *dual pseudo-metric* on a manifold M is a symmetric (0,2)-tensor field g (i.e. a section of $S^2(TM)$) such that for all $p \in M$, g_p defines a non-degenerate bilinear form on $(T_pM)^*$. An n*-vector* on a manifold M is a nowhere vanishing section of the bundle $\Lambda^n(TM)$ ($n = \dim M$). ∎

If $M = L/Q$ is a homogeneous space and $\rho : Q \to \mathrm{Gl}(T_oM)$ the first isotropy representation, then we identify (0,2)-tensor fields with functions on L, given by elements of the induced representation $\mathrm{ind}_Q^L(\rho \otimes \rho)$, and (2,0)-tensor fields with elements of $\mathrm{ind}_Q^L(\rho^* \otimes \rho^*)$ (cf. Appendix VIII.B).

Theorem X.6.2. *Assume that* T *is a non-degenerate JTS and let* M *be the associated global space. For* $\varphi \in V^*$ *let* $v_\varphi \in V$ *be the unique vector with* $\varphi(w) = (v_\varphi|w)$ *for all* $w \in V$ *and define a non-degenerate symmetric bilinear form on* V^* *by* $(\varphi|\psi) := (v_\varphi|v_\psi)$.

(i) *There is a G-invariant $(0,2)$-tensor field g on V^c given by the formula*

$$(g(x))(\varphi \otimes \psi) := (\varphi \circ d_{\Theta(x)^{-1}x}(0)|\psi)$$

($x \in \mathrm{Co}(T)$, $\varphi, \psi \in V^*$). The bilinear form $g(x)$ is non-degenerate iff $x.0$ belongs to M. In particular, the restriction of g to M is an invariant dual pseudo-metric on M. The affine picture of g is given by

$$(g(t_x))(\varphi \otimes \psi) = (\varphi \circ B(x,x)|\psi).$$

(ii) *An invariant pseudo-metric on M is obtained by "lowering indices of g", i.e. by*

$$(\check{g}(x))(u \otimes v) := (d_{\Theta(x)^{-1}x}(0)^{-1}u|v)$$

if $x.0 \in M$. The affine picture of this invariant pseudo-metric is given by

$$(\check{g}(t_x))(u \otimes v) = (B(x,x)^{-1}u|v).$$

Proof. (i) Under the isomorphisms $W \to V^*$, $w \mapsto (v \mapsto \operatorname{tr} T(v,w))$, $\mathrm{Hom}(W,V)$ corresponds to $V \otimes V$, the section S defined in Prop. X.2.5 is identified with g. Therefore it follows from Prop. X.2.5 that g has the correct transformation property, and from Th. X.2. that $g(x)$ is non-degenerate iff $x.0$ belongs to M. The formula for the affine picture of g follows immediately from Prop. VIII.2.5 (i). It only remains to be shown that $g(x)$ is symmetric. By a density argument, it suffices to consider points where $g(x)$ is non-degenerate. But then, by Eqn. VIII.(2.3),

$$(d_{\Theta(x)^{-1}x}(0))^* = ((\mathrm{D}(\Theta(x)^{-1}x)(0))^{-1})^* = \Theta(\mathrm{D}(\Theta(x)^{-1}x)(0)) = d_{\Theta(x)^{-1}x}(0),$$

i.e. $d_{\Theta(x)^{-1}x}(0)$ is self-adjoint and therefore $g(x)$ is symmetric.

(ii) This is an obvious consequence of part (i). ■

6.2. Invariant density and invariant integral. It is clear that the section $s(x) = \det(d_{\Theta(x)^{-1}x}(0))$ defined in Section X.2 defines an invariant section of the line bundle induced from $\Lambda^n W^* \otimes \Lambda^n V \cong (\Lambda^n V) \otimes (\Lambda^n V)$. According to Cor. X.2.6, M is the set of points where s does not vanish. The section s behaves like the "square of an n-vector" on M. However, it is not always possible to define a "square root" of s on M since M need not be orientable (example: $M = \mathbb{RP}^2$). Recall (cf. e.g. [St83, p. 112]) that a *density* on a real manifold is a section of the line bundle whose transition functions are given by the *absolute value* of the Jacobian determinant of the maps induced by a change of coordinates. Thus, if $M = L/Q$ is homogeneous and ρ is the first isotropy representation of Q, densities are identified with elements of the induced representation

$$\mathrm{ind}_Q^L |\det \circ \rho|.$$

The group L acts on the density bundle in the natural way.

Proposition X.6.3.

(i) *The formula $d(x) := |s(x)|^{-1/2}$ ($x \in \mathrm{Co}(T)$, $x.0 \in M$) defines a G-invariant density on M with affine picture $d(t_x) = |\det B(x,x)|^{-1/2}$.*

(ii) *If the representation $H \mapsto \mathbb{R}^*$, $h \mapsto \det(\mathrm{D}h(0))$ is trivial, then there exists a G-invariant volume-form on $M_0 = G/H$.*

(iii) *If there is a representation $\pi : \mathrm{Str}(T) \to \mathbb{R}^*$ with $\pi \otimes \pi \cong \det$, then there exists a G-invariant volume-form on M whose affine picture is given by the function*

$$z \mapsto \det B(z,z)^{-1/2} := \pi(B(z,z))^{-1}.$$

Proof. (i) From Lemma X.2.5 it follows that $|\det s(x)|$ defines a G-invariant section of the bundle over V^c corresponding to $|\Lambda^n W^* \otimes \Lambda^n V| \cong |(\Lambda^n V) \otimes (\Lambda^n V)|$. Thus $|s(x)|^{1/2}$ defines a "dual density" on V^c which, according to Cor. X.2.6, does not vanish iff $x.0 \in M$. Thus we can take its inverse on M and obtain a G-invariant density there. The formula for the affine picture follows from $s(t_x) = \det B(x,x)$.

(ii) The assumption assures that we can define an n-form in a consistent way by transporting a volume form from $T_0 M$ to $T_{g.0}M$ by $g \in G$.

(iii) We define the element $\widetilde{d}$ of $\operatorname{ind}_P^{\mathrm{Co}(T)}(\pi \circ \rho)^{-1}$ by $\widetilde{d}(x) := \pi(S(x))$ and apply arguments similar to those proving part (i). ∎

Next recall (cf. e.g. [St83, p. 114/115]) that densities can be *integrated* in a natural way. The G-invariant integral on M belonging to the invariant density from Prop. X.6.3 (i) is given by the formula

$$I(f) := \int_{M \cap V} f(x) |\det B(x,x)|^{-1/2}\, dx, \tag{6.1}$$

where dx is Lebesgue measure on V, restricted to $M \cap V$ (the complement of $M \cap V$ in M is an algebraic set of lower dimension than M and is therefore of measure zero). If M is orientable, then the choice of a positive atlas yields an identification of the density bundle with the bundle of n-forms. Note that, since $M_0 = G/H$ is reductive, $H_1 := \{h \in H | \det(\mathrm{D}h(0)) = 1\}$ is a subgroup of index at most 2 in H, and thus G/H_1 is an oriented covering of order at most 2 of M_0. In the non-reductive case there are symmetric spaces with twist having no invariant density. An example is given by the space of upper triangular matrices: as is well-known, the group $S = AN$ of upper triangular matrices (which is the symmetric space with twist associated to the Jordan algebra of upper triangular matrices) is not unimodular; this means that as a symmetric space it has no invariant density.

6.3. The Cartan-involution.

Definition X.6.4. Let $M = G/H$ be a reductive symmetric space with associated involution σ. A Cartan-involution ϑ commuting with σ and stabilizing H is called a *Cartan-involution of* M. ∎

If ϑ is as in the definition, it passes to the quotient as an (equivariant) automorphism $\vartheta : M \to M$. If the center of G is finite, then M^ϑ is a maximal compact subspace of M (cf. [Lo69a, p.155]).

Theorem X.6.5. *If $M = G/H$ is a semisimple para-Hermitian symmetric space, then M admits a compact para-real form, i.e. a Cartan-involution which is a para-real form.*

Proof. Let σ be the involution associated to the symmetric pair (G,H) and $J \in \mathfrak{h}$ the element corresponding to the para-complex structure. Then J is a hyperbolic element in the semisimple Lie algebra $\mathfrak{g}$, i.e. $\mathrm{ad}(J)$ has only real eigenvalues (namely $0, 1, -1$). Therefore there exists a Cartan-involution ϑ of $\mathfrak{g}$ commuting with σ and such that $\vartheta(J) = -J$ (this is a refinement of a standard result – see e.g. [HN91, Satz III.6.17] – on semisimple Lie algebras, cf. [Ne79, p.285]). Thus $\vartheta|_{\mathfrak{q}}$ is a para-conjugation of the twisted polarized LTS $\mathfrak{q}$ belonging to M and thus is in particular a JTS-automorphism. It extends to a global para-conjugation (given in Jordan coordinates by a *linear* map) of the global space M. Now the group G, being isomorphic to the group $\mathrm{Co}(T)$, where T is the JTS of $\mathfrak{q}^\vartheta$, has a trivial center (cf. Thm. VIII.1.3), and as remarked above, it follows that M^ϑ is compact. ∎

Note that the preceding proof does not carry over to twisted complex symmetric spaces because in this case J is elliptic and not hyperbolic. In fact, Prop. V.5.2 implies that the only pseudo-Hermitian symmetric spaces admitting a compact real form are themselves compact.

Corollary X.6.6. *If M is a non-degenerate symmetric space with twist, then there exists a Cartan involution ϑ of the symmetric space M which is also an automorphism of the symmetric space with twist M. In other words, it is a JTS-automorphism.*

Proof. Let $\widetilde{\vartheta}$ be a Cartan-involution of $M_{ph\mathbb{C}}$ as constructed in the preceding theorem. It can be chosen such that it commutes with the conjugation τ w.r.t. M (cf. [Ne79, p.278]). Its restriction to M is a Cartan-involution of M and a JTS-automorphism since so is $\widetilde{\vartheta}$. ■

Following E. Neher ([Ne79]), we call the automorphism ϑ of the non-degenerate JTS T a *Cartan-involution of* T. The restriction of T to the ϑ-fixed space V^{ϑ} is negative and the restriction to $V^{-\vartheta}$ is positive in the sense of Def. V.5.1.

Corollary X.6.7. *Let T be a non-degenerate JTS and ϑ a Cartan-involution of T. Then the space $M_0^{(-\vartheta)}$ is compact. It is equal to V^c and hence V^c is compact.*

Proof. The space $M_0^{(-\vartheta)}$ is isomorphic to the compact para-real form of $M_{ph\mathbb{C}}$ from Th. X.6.5 (the isomorphism is given by $x \mapsto (x, \vartheta(x))$, cf. Section IV.2). Therefore it is compact. On the other hand, it is open in the conntected space V^c, and therefore it must be equal to V^c. Thus V^c itself is compact. ■

The preceding corollary allows to call V^c the *conformal compactification* of M. The fact that V^c is compact may also be proved by using root theory of $\mathrm{Co}(T)$ and showing that P is a (maximal) parabolic subgroup of $\mathrm{Co}(T)$. If we write $M^{(-\vartheta)} = U/K$ as compact connected symmetric space, we have the equality

$$U/K = \mathrm{Co}(T)/P$$

which says that U/K is a *symmetric R-space* in the sense of [Tak65]. Every symmetric R-space is obtained in this way (cf. [Lo85]). The classification of symmetric R-spaces is given in Table XII.3.3.

Remark X.6.8. Let $V = V^+ \oplus V^-$ be the eigenspace-decomposition w.r.t. the Cartan-involution ϑ. Then the preceding corollary implies that the ϑ-fixed subspace of M is isomorphic to $(V^+)^c$, and topologically M is a vector bundle over $(V^+)^c$ with fibers isomorphic to V^- (the Berger-fibration, cf. [Lo69a, Th. IV.3.5]). ■

Example X.6.9. Let V be a Euclidean Jordan algebra and α an involutive automorphism of V. (We will see in Section XI.3 that these spaces are certain *causal symmetric spaces*). Then α is a Cartan-involution of $M^{(\alpha)}$ (because $T^{(\alpha)}$ is negative on V^- and positive on V^+). ■

Appendix A: Power associativity

In this appendix we prove some identities for Jordan triple systems. Recall from Prop. III.2.4 the defining identity (JT2) of a JTS. For $x = z = u$ we obtain

$$T(x, v, T(x, y, x)) = 2T(T(x, v)x, y, x) - T(x, T(v, x)y, x)$$

which, with the notation $P(a) = \frac{1}{2}T(a, \cdot, a)$, is

$$P(x)T(v, x, y) = 2T(P(x)v, y, x) - T(P(x)y, v, x).$$

Since the left hand side is symmetric in v and y, we get $T(P(x)v,y,x) = T(P(x)y,v,x)$ (this is, by the way, equivalent to $[\mathbf{p}_y, \mathbf{p}_v] = 0$, which we already know) and thus

$$P(x)T(v,x,y) = T(P(x)v,y,x) = T(P(x)y,v,x).$$

(This the identity JP4 from [Lo75] and [Lo77].) With $v = x$ the identity

$$P(x)T(x,x)y = T(P(x)x,y,x) = T(x,x)P(x)y, \tag{A.1}$$

and thus in particular $[P(x), T(x,x)] = 0$, follows.

Proposition X.A.1. *For all $k \in \mathbb{N}$ and $x \in V$ the identity*

$$2^k P(x)^k x = T(x,x)^k x$$

holds.

Proof. For $k = 0$ the identity is trivial, and for $k = 1$ it follows immediately from the definition of P and T: $2P(x)x = T(x,x,x) = T(x,x)x$. Next we get from (JT2) with $x = y = z = u = v$ (equivalently, from (A.1) with $x = y$),

$$P(x)T(x,x)x = T(x,x)P(x)x.$$

Together with $T(x,x)x = P(x)x$ this implies $P(x)^2x = T(x,x)^2x$. Now we prove the claim for general k by induction. We assume that for all $j \leq k$ the identity $P(x)^j x = T(x,x)^j x$ holds. Then we get from (A.1)

$$\begin{aligned} 2^{k+1}P(x)^{k+1}x &= 2P(x)T(x,x)^k x = 2T(x,x)P(x)T(x,x)^{k-1}x \\ &= 2^k T(x,x)P(x)^k x = T(x,x)^{k+1}x, \end{aligned}$$

and the claim follows. ∎

Definition X.A.2. For each $x \in V$ we denote by $\langle x \rangle \subset V$ the subspace spanned by all $P(x)^k x$, $k = 0,1,\ldots$. The integer

$$\mathrm{rk}(T) := \max_{x \in V} \dim\langle x \rangle$$

is called the *rank of T*. ∎

Corollary X.A.3. *The operators*

$$T(x,x) : \langle x \rangle \to \langle x \rangle \qquad \textit{and} \qquad 2P(x) = T(x,\cdot,x) : \langle x \rangle \to \langle x \rangle$$

coincide.

Proof. This follows immediately from Prop. X.A.1. ∎

The next result is included for the sake of completeness; it is not used in this chapter.

Proposition X.A.4. *A JTS is power-associative in the sense that for all* $j, k, l \in \mathbb{N}$,

$$T(P(x)^j x, P(x)^k x, P(x)^l x) = 2P(x)^{j+k+l+1} x.$$

Proof. We will use the fundamental formula

$$T(P(x)u, v, P(x)v) = P(x)T(u, P(x)v, w)$$

(Prop. VIII.C.1). The claim is proved by induction on $j+k+l$. If $j+k+l=0$, then the claim is true by definition of P. Now we assume the claim to be true for $j+k+l \leq n$ and distinguish two cases: (a) $j>0, l>0$, (b) $j=0$ or $l=0$. In case (a) the fundamental formula gives us

$$T(P(x)^j x, P(x)^k x, P(x)^l x) = P(x)T(P(x)^{j-1} x, P(x)^{k+1} x, P(x)^{l-1} x),$$

and we can apply the induction hypothesis to the last term. In case (b) we assume w.l.o.g. that $j=0$ and get from the identity JP4

$$T(x, P(x)^k x, P(x)^l x) = P(x)T(P(x)^k x, x, P(x)^{l-1} x),$$

and again we can apply the induction hypothesis. ∎

Corollary X.A.5. *The space* $\langle x \rangle$ *is a flat subspace of the LTS given by* $R = R_T$ *on* V, *i.e. for all* $u, v, w \in \langle x \rangle$, $R(u,v)w = 0$.

Proof. It is enough to prove the condition $R(u,v)w = 0$ on the generators $P(x)^k x$ of $\langle x \rangle$. But

$$\begin{aligned} R(P(x)^j x, P(x)^k x, P(x)^l x) &= T(P(x)^j x, P(x)^k x, P(x)^l x) - T(P(x)^k x, P(x)^j x, P(x)^l x) \\ &= 0 \end{aligned}$$

by the preceding proposition. ∎

Notes for Chapter X.

X.1. Propositions X.1.1 and X.1.3 are generalizations of Prop. 2.2.1 in [Be96b] which summarizes the approach by A.A. Rivillis and B.O. Makarevič ([Ri70], [Ma73]) to open symmetric orbits in the conformal compactification of semisimple Jordan algebras.

X.2. Th. X.2.1 is a generalization of [Be98a, Th.2.1.1]. In [Ma79] a different, more root-theoretic description of the spaces $M^{(\alpha)}$ is given.

X.4. The formula $\mathrm{Exp} = \tanh_T$ is well-known for the case of bounded symmetric domains (cf. [Lo77, Lemma 4.3] and [FK94, p.216]). As explained in Remark X.4.2, our global result is new.

X.5. The Peirce-decomposition with respect to a hyperbolic tripotent is standard, cf. [Lo77, Ch.3] or [Sa80, p.242 ff]. For our treatment via $\mathfrak{sl}(2)$-theory cf. [Sa80, p.90]. The homomorphism $\mathrm{Sl}(2,\mathbb{R}) \to \mathrm{Co}(T)$ is well-known in the case of Hermitian symmetric spaces, cf. [Lo77, Lemma 9.7]. However, it seems that not much attention has been paid to the fact that much of the theory holds equally well for elliptic as for hyperbolic tripotents (which are usually the only ones considered).

X.6. It is well-known that the Bergman metric of a bounded symmetric domain is given by the polynomial $B(z,\overline{z})^{-1}$, cf. e.g. [Lo77, Th.2.10], [Sa80, Prop. II.6.2]. Our analog of this fact is a generalization of [Be98a, Th. 2.4.1]. A general theory of Cartan involutions of triple systems has been developed by E. Neher ([Ne79]).

Appendix A. The power associativity of a JTS has first been established by K. Meyberg [Mey70, Satz 4.3]. His proof uses power associativity of the Jordan algebras $A_x = T(\cdot, x, \cdot)$; we have avoided this argument since we have in general no geometric interpretation of the Jordan algebra A_x (cf. the next chapter).

Chapter XI: Spaces of the first and of the second kind

An important class of symmetric spaces with twist can be described by Jordan *algebras*; we call them *of the first kind*. They can be characterized by the fact that the involution Θ of the conformal group is an *inner* involution (Th. XI.1.4) which naturally leads to the *Jordan inverse* of a Jordan algebra (Prop. XI.1.5). Specializing the Peirce-theory from Section X.5 to this case, we define a *real Cayley transform* which in many cases yields *generalized tube domain realizations* (Th. XI.2.8) which are useful in the study of certain *causal symmetric spaces* (Section 3).

A natural further development of this approach would be a theory of *generalized Siegel domain realizations of symmetric spaces with twist*; we leave this general topic for later work. Instead, we focus here on relations between spaces of the first kind and the "extrinsic realization": on the one hand, every symmetric space with twist has a realization as "immersed symmetric orbit" (Prop. XI.4.1); on the other hand, using the Cayley transform, we realize all *Helwig spaces* (cf. Section II.4) as symmetric spaces with twist, appearing as subspaces in a space related to a Jordan algebra (Prop. XI.4.4). From work of K.H. Helwig it is known that many symmetric spaces have such a realization. It is an open problem how to characterize Helwig spaces among the immersed symmetric orbits. Algebraically, this problem translates to an "extension problem": why is there no analog in Jordan theory of the standard imbedding of a Lie triple system (Problem XI.4.5)?

1. Spaces of the first kind and Jordan algebras

Definition XI.1.1. A JTS T and an associated symmetric space with twist M is called *of the first kind* if the group $\mathrm{Co}(T)_{00} = \exp(\mathfrak{m}^+)$ has an open orbit in V^c. Otherwise it is called *of the second kind.* ∎

If a JTS is of the first kind, we have $\dim \mathfrak{m}^+ \geq \dim \mathfrak{m}^- = \dim M$. Thus T must be faithful and $\exp(\mathfrak{m}^+) = \Theta(t_V)$.

Definition XI.1.2. An element $x \in V$ is called *invertible* if $\det P(x) \neq 0$. ∎

It is clear that the set $V' = \{x \in V | \det P(x) \neq 0\}$ of invertible elements is either empty or open dense in V.

Lemma XI.1.3. *A JTS T is of the first kind if and only if it contains invertible elements.*

Proof. T is of the first kind if and only if there exists $x \in V$ such that the evaluation map

$$\mathfrak{m}^+ \to V, \quad \mathbf{p}_v \mapsto \mathbf{p}_v(x) = \frac{1}{2}T(x,v,x) = P(x)v$$

is surjective. This is equivalent to requiring that $P(x)$ is surjective or (by reasons of dimension) that it is bijective, i.e. x is invertible. ■

Theorem XI.1.4. *Assume that T is faithful. Then T is of the first kind if and only if Θ is an inner automorphism of* $\mathrm{Co}(T)$. *If this is the case, then Θ is the conjugation with the element $g \in \mathrm{Co}(T)$ given by the formula*

$$g(x) = P(x)^{-1}x.$$

Proof. Assume first that Θ is given by conjugation with an element $g \in \mathrm{Co}(T)$. Then $g(V) \subset V^c$ is an open $\Theta(t_V)$-orbit in V^c and hence T is of the first kind. Moreover, for all $x, v \in V$,

$$d_g(x)v = (g^*\mathbf{v})(x) = (\Theta(\mathbf{v}))(x) = -\mathbf{p}_v(x) = -P(x)v,$$

whence $d_g(x) = -P(x)$. Similarly,

$$n_g(x) = (g^*E)(x) = (\Theta(E))(x) = -E(x) = -x,$$

whence

$$g(x) = d_g(x)^{-1}n_g(x) = P(x)^{-1}x.$$

For the proof of the converse we proceed in several steps. We assume T to be of the first kind and fix a point $e \in V$ with $\det P(e) \neq 0$.

1. According to the fundamental formula (Prop. VIII.C.1), for all $x \in V$, $P(P(e)x) = P(e)P(x)P(e)$, i.e. $\alpha := P(e)$ satisfies Eqn. (III.4.2), and Lemma III.4.5 assures that

$$\widetilde{T}(x,y,z) := T^{P(e)^{-1}}(x,y,z) = T(x, P(e)^{-1}y, z)$$

is again a JTS on V. If $\widetilde{P}$ is the corresponding quadratic representation, we have

$$\widetilde{P}(e)x = \frac{1}{2}T(e, P(e)^{-1}x, e) = P(e)P(e)^{-1}x = x,$$

i.e. $\widetilde{P}(e) = \mathrm{id}_V$. Since $\widetilde{T}$ has the same conformal group as T with associated involution $\widetilde{\Theta} = \Theta \circ P(e)^{-1}$, we may w.l.o.g. assume that $T = \widetilde{T}$ and $P(e) = \mathrm{id}_V$. From Appendix X.A, Eqn. X.(A.1) we now deduce that

$$T(e,e)x = P(e)T(e,e)x = T(P(e)e, x, e) = T(e,x,e) = 2P(e)x = 2x,$$

i.e. $T(e,e) = 2\,\mathrm{id}_V$.

2. We have seen that e a is hyperbolic tripotent (having the particular property that $V = V_2$). Let $F := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{Sl}(2,\mathbb{R})$ and consider its image $\varphi := f(F) \in \mathrm{Co}(T)$ under the homomorphism $f : \mathrm{Sl}(2,\mathbb{R}) \to \mathrm{Co}(T)$ defined in Cor. X.5.10. Recall from X.5.5 that $\dot{f}(H) = T(e,e) = 2E$, where H is given by Eqn. X.(5.1). Since $FHF^{-1} = -H$, i.e. $\mathrm{Ad}(F)H = -H$, it follows that

$$\varphi_*E = f(F)_*\dot{f}(\frac{1}{2}H) = \frac{1}{2}\dot{f}(\mathrm{Ad}(F)H) = -\frac{1}{2}\dot{f}(H) = -E.$$

3. Now, since $\varphi_*(E) = -E$ and $\Theta(E) = -E$, the automorphism $\Upsilon := \Theta \circ \varphi_*$ satisfies $\Upsilon(E) = E$. According to Prop. VIII.1.5(ii), it follows that Υ is inner, and thus Θ is inner. ■

Proposition XI.1.5. *Let T be a JTS, $e \in V$ an element with $P(e) = \mathrm{id}_V$ and $j(x) := P(x)^{-1}x$. Then e is an isolated fixed point of j, and $\Omega := \mathrm{Str}(T).e$ is a quadratic prehomogeneous symmetric space with associated Jordan algebra*

$$xy = \frac{1}{2}T(x, e, y)$$

and Jordan inverse j.

Proof. Since $j(e) = P(e)^{-1}e = e$, e is a fixed point of j. It is isolated since $d_j(e) = -P(e) = -\mathrm{id}_V$. Any point $x \in V$ with $\det P(x) \neq 0$ belongs to an open $\mathrm{Str}(T)$-orbit in V: in fact,

$$\mathfrak{str}(T)x \supset T(V, V, x) \supset T(x, V, x) = P(x)V = V,$$

and thus $\mathrm{Str}(T) \to V$, $g \mapsto g.x$ is a submersion. In particular, the orbit Ω is open.

We are going to prove that Ω is symmetric. The decomposition of $\mathfrak{str}(T)$ associated to the involution $\Theta = j_*$ is $\mathfrak{str}(T) = \mathfrak{h} \oplus \mathfrak{q}$ with $\mathfrak{h} = \mathrm{Der}(T)$ and $\mathfrak{q}$ the space of $X \in \mathrm{End}(V)$ satisfying $XT(u, v, w) = T(Xu, v, w) - T(u, Xv, w) - T(u, v, Xw)$. If $X \in \mathrm{Der}(T)$, then

$$2Xe = XT(e, e, e) = T(Xe, e, e) + T(e, Xe, e) + T(e, e, Xe) = 6Xe,$$

whence $Xc = 0$. Conversely, if $X \in \mathfrak{q}$ satisfies $Xe = 0$, then we have for all $v \in V$,

$$2Xv = XT(e, v, e) = 2T(Xe, v, e) - T(e, Xv, e) = -2Xv,$$

whence $X = 0$. It follows that $\mathrm{Der}(T)$ is the Lie algebra of the stabilizer of e, and therefore Ω is symmetric with associated involution Θ.

In order to calculate the Lie triple algebra associated to the prehomogeneous symmetric space Ω (cf. Section II.1), recall that the elements $T(x, y)$ with $T(x, y) = T(y, x)$ belong to $\mathfrak{q}$. We claim that $T(e, v) = T(v, e)$ for all $v \in V$. Indeed, the identity JP4 (Appendix X.A) yields for all $v, y \in V$,

$$T(v, e)y = P(e)T(v, e, y) = T(P(e)y, v, e) = T(y, v, e) = T(e, v)y.$$

Moreover, the condition $\frac{1}{2}T(e, v)e = P(e)v = v$ shows that

$$L(v) := \frac{1}{2}T(e, v) = \frac{1}{2}T(v, e)$$

is the unique element in $\mathfrak{q}$ such that $L(v)e = v$. Thus the Lie triple algebra associated to (Ω, e) is given by $xy = L(x)y = \frac{1}{2}T(x, e, y)$. In order to prove that Ω is the open orbit of a quadratic prehomogeneous symmetric space, we verify condition (ii) of Th. II.2.6: the map $\Omega \to \mathrm{Hom}(V \otimes V, V)$, $g.e \mapsto \Theta(g).A_e$ has the same equivariance property as the linear map $x \mapsto \frac{1}{2}T(\cdot, x, \cdot)$; both agree on the base point e and therefore on all points of the open orbit Ω. Thus condition (ii) of Th. II.2.6 is verified, and moreover we have proved that T is the JTS associated to the Jordan algebra A_e, i.e. we have $T(x, y, z) = 2(x(yz) - y(xz) + (xy)z)$. ■

Corollary XI.1.6. *All JTS of the first kind are of the form $T = \widetilde{T}^{(\alpha)}$, where $\widetilde{T}$ is the JTS associated to a unital Jordan algebra and α belongs to the structure variety of $\widetilde{T}$.*

Proof. If T is of the first kind, we fix e and define $\widetilde{T} = T^{(\alpha^{-1})}$ with $\alpha = P(e)$ as in step 1 of the proof of Thm. XI.1.4; then $T = \widetilde{T}^{(\alpha)}$ and $\widetilde{P}(e) = \mathrm{id}_V$. The preceding proposition shows that $\widetilde{T}$ is associated to a unital Jordan algebra, and the calculation

$$\alpha\widetilde{T}(x, \alpha y, z) = \alpha T(x, y, z) = T(\alpha x, \alpha^{-1}y, \alpha z) = \widetilde{T}(\alpha x, y, \alpha z)$$

shows that α belongs to the structure variety of $\widetilde{T}$. ■

Note that $\alpha = P(e)$, but in general α is not of the form $\alpha = \widetilde{P}(d)$ for some $d \in V$. In fact, the condition $\alpha = \widetilde{P}(d) = P(d)\alpha^{-1}$ is equivalent to $P(e)^2 = \alpha^2 = P(d)$, and this equation may have no solution d.

Remark XI.1.7. According to Prop. X.1.3, the "integrated version" of the preceding corollary is: *All symmetric spaces with twist of the first kind are of the form* $M^{(\alpha)}$*, where* T *is the JTS associated to a unital Jordan algebra and* α *is a symmetric element of its structure group.* The corresponding zero-components

$$M_0^{(\alpha)} = \mathrm{Co}(V)_o^{(j\alpha)_*}.0 \subset V^c \tag{1.1}$$

are the most important class of "open symmetric orbits in symmetric R-spaces" considered by A.A. Rivillis and B.O. Makarevič ([Ri70],[Ma73]); in [Be96b] we have called them *Makarevič spaces.* ■

Corollary XI.1.8. *If* T *is a JTS of the first kind, then for all* $v \in V$*,* $T(\cdot, v, \cdot)$ *is a Jordan algebra (in general without unit element).*

Proof. Replacing T by a modification does not affect the claim. Thus according to Cor. XI.1.6 we may assume that T is associated to a unital Jordan algebra. Now the claim is true for all $v \in \Omega$ since $T(\cdot, v, \cdot)$ is the Jordan algebra associated to the point $j(v) \in \Omega$. It follows that the claim holds for all $v \in V$: in fact, it can be expressed by algebraic equations and it holds for all v in an open set of V. ■

2. Cayley transform and tube realizations

Proposition XI.2.1. *Let* T *be the JTS associated to a Jordan algebra with unit element* e*. Then the homomorphism* $f : \mathrm{Sl}(2,\mathbb{R}) \to \mathrm{Co}(T)$ *associated to* e (Cor. X.5.10) *is given by the formula*

$$(f \begin{pmatrix} a & b \\ c & d \end{pmatrix})(z) = (az + be)(cz + de)^{-1},$$

where the right-hand side is calculated in the Jordan algebra. It extends to a homomorphism $\mathbb{P}\,\mathrm{Gl}(2,\mathbb{R}) \to \mathrm{Co}(T)$*, given by the same formula.*

Proof. Since a Jordan algebra is power associative (Thm. II.2.15), we verify that the given formula indeed defines a homomorphism (from $\mathbb{P}\,\mathrm{Gl}(2,\mathbb{R})$ into $\mathrm{Co}(T)$) as in the classical case of the operation of $\mathbb{P}\,\mathrm{Gl}(2,\mathbb{R})$ on $\mathbb{RP}^1$. Then we check that this homomorphism coincides with f on the generators of $\mathrm{Sl}(2,\mathbb{R})$. This is clear for the matrices $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. For the matrices $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ it follows from the fact that $B(e, (1-t)e) = t^2 \,\mathrm{id}_V$ since under our assumptions we have $V = V_2$. Finally,

$$\begin{aligned}(f \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix})(x) &= \exp(\Theta(t\mathbf{e})).x = \Theta(t_e).x \\ &= j t_{te} j(x) = (te + x^{-1})^{-1} = x(tx+e)^{-1},\end{aligned}$$

where we have again used power associativity of a Jordan algebra. ■

Corollary XI.2.2. *Under the assumptions of the preceding proposition,*

$$\exp(\frac{\pi}{2}(\mathbf{e} + \mathbf{p}_e)).x = -x^{-1},$$

$$\exp(\frac{\pi}{4}(\mathbf{e} + \mathbf{p}_e)).x = (x+e)(e-x)^{-1}.$$

Proof.

$$\exp(\frac{\pi}{2}(\mathbf{e} + \mathbf{p}_e)) = f(\exp(\frac{\pi}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})) = f\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -j,$$

$$\exp(\frac{\pi}{4}(\mathbf{e} + \mathbf{p}_e)) = f(\exp(\frac{\pi}{4}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})) = f(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}),$$

and according to the preceding proposition, the last matrix operates as claimed. ■

Note that $j = f\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ does not belong to the identity component of $\mathbb{P}\,\mathrm{Gl}(2,\mathbb{R})$ and thus cannot be written as an exponential.

Definition XI.2.3. Let V be a unital Jordan algebra. The element $\gamma \in \mathrm{Co}(V)$ defined by

$$\gamma = \exp(\frac{\pi}{4}(\mathbf{e} + \mathbf{p}_e)),$$

i.e.

$$\gamma(x) = (x-e)^{-1}(x+e),$$

is called the *real Cayley transform.* ∎

Applying f to the relation

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

yields the relation

$$\gamma j \gamma^{-1} = -\,\mathrm{id}_V \tag{2.1}$$

between elements of $\mathrm{Co}(V)$. Similarly we get $\gamma(-\,\mathrm{id}_V)\gamma^{-1} = j$. If V is a complex Jordan algebra, then we define the usual *Cayley transform* by

$$C := i\gamma i$$

(where i is multiplication by $\sqrt{-1}$) and get, since $ij = -ji$,

$$C(-j)C^{-1} = i\gamma i j i^{-1}\gamma^{-1}i^{-1} = i\gamma(-\,\mathrm{id}_V)\gamma^{-1}i^{-1} = -\,\mathrm{id}_V\,. \tag{2.2}$$

Remark XI.2.4. For any hyperbolic tripotent e one can define by $\gamma_e = \exp(\frac{\pi}{4}(\mathbf{e} + \mathbf{p}_e))$ the *partial Cayley transform γ_e defined by e*, and then γ_e^2 may be called the *partial inverse defined by e* (cf. [Lo77, Ch.X]). Following the methods of [Lo77], it is possible to derive Jordan theoretic formulas for these transformations and to develop a general theory of “Siegel domain realizations” for symmetric spaces with twist. We will leave this topic for later work and content ourselves here with the important special case of a class of spaces “of generalized tube type”. They are associated to involutions of unital Jordan algebras. ∎

Lemma XI.2.5. *If α is an automorphism of the unital Jordan algebra V, then the relations $\gamma\alpha = \alpha\gamma$ and*

$$\gamma(j\alpha)\gamma^{-1} = -\alpha, \quad \gamma(-\alpha)\gamma^{-1} = j\alpha$$

hold. If V is complex and α a $\mathbb{C}$-conjugate linear automorphism, then we have

$$C(-j\alpha)C^{-1} = -\alpha, \quad C(-\alpha)C^{-1} = -j\alpha.$$

Proof. Since $\alpha(e) = e$,

$$\alpha((x-e)^{-1}(x+e)) = (\alpha(x)-e)^{-1}(\alpha(x)+e)$$

and hence $\gamma\alpha = \alpha\gamma$. It follows that $\gamma j \alpha\gamma = \gamma j \gamma\alpha = -\,\mathrm{id}\,\alpha$, and similarly for C. ∎

If α is an involutive automorphism of a unital Jordan algebra V, we denote by $V = V^+ \oplus V^-$ the associated eigenspace decomposition. It is clear that V^+ and V^- are sub-JTS of T, and moreover V^+ is a Jordan algebra with unit element e and Jordan inverse induced by the one of V. As usual, $\Omega = \mathrm{Str}(V)_o e$ is the open symmetric orbit associated to V. We denote by $\Omega^+ = \mathrm{Str}(V^+)_o e$ the open symmetric orbit associated to V^+. According to Lemma II.3.2, we have

$$\Omega^+ = (V^+ \cap \Omega)_e.$$

Definition XI.2.6. The open domain

$$V^- + \Omega^+ := \{x + y \mid x \in V^-, y \in \Omega^+\} \subset V$$

is called the *generalized tube associated to the involutive Jordan algebra* (V, α). ∎

For example, if α is complex conjugation of a semisimple complex Jordan algebra w.r.t. a Euclidean real form W, then $V^- + \Omega^+ = W + i\Omega$ is the classical tube domain over the symmetric cone Ω. We prove now that the generalized tubes, completed by points at infinity, are symmetric spaces with twist.

Theorem XI.2.7. *Let V be a unital Jordan algebra, α an involutive automorphism of V and $M^{(\alpha)}$ the global space of the first kind associated to the JTS $T^{(\alpha)}$. Then its Cayley-transformed realization is described by*

$$\gamma(M_0^{(\alpha)}) = \mathrm{Co}(V)_o^{(-\alpha)_*}.e \subset V^c$$

and

$$\gamma(M^{(\alpha)}) \cap V = \{x \in V \mid \det P(x + \alpha x) \neq 0\}.$$

The connected component of this set containing e is the generalized tube $V^- + \Omega^+$.

Proof. In order to prove the first claim, note that $\gamma(0) = e$. From Eqn. (1.1) we get

$$\gamma(M_o^{(\alpha)}) = \gamma\,\mathrm{Co}(V)_o^{(j\alpha)_*}.0 = \mathrm{Co}(V)_o^{(\gamma j \alpha \gamma)_*}.\gamma(0) = \mathrm{Co}(V)_o^{(-\alpha)_*}.e;$$

the last equation follows from Lemma XI.2.5. In order to prove the next claim, recall from Cor. X.2.6 that $M^{(\alpha)}$ is the set of points where the section s (seen as a function on $\mathrm{Co}(T)$) defined by $s(g) = \det(d_{(j\alpha)_* g^{-1} g}(0))$ does not vanish. The conformal group acts on sections by simple left translations; thus

$$\begin{aligned}(\gamma s)(g) &= s(\gamma^{-1} g)\\ &= \det(d_{j\alpha g^{-1}\gamma^{-1} j\alpha\gamma g}(0))\\ &= \det(d_{j\alpha g^{-1}(-\alpha) g}(0))\end{aligned}$$

where we have used Lemma XI.2.5. The affine picture of this section is (neglecting a factor $\det \alpha \in \{\pm 1\}$)

$$\begin{aligned}(\gamma s)(t_x) &= \det(d_{j\alpha t_{-x}(-\alpha) t_x}(0))\\ &= \det(d_{j t_{-\alpha x}(-\,\mathrm{id}) t_x}(0))\\ &= \det(d_{j t_{-(\alpha x + x)}}(0)) = \det(d_j(-(x + \alpha x))\\ &= \det(P(x + \alpha x)).\end{aligned}$$

Since $\gamma(M^{(\alpha)})$ is the set of points where the section γs does not vanish, this calculation proves the second claim. Finally,

$$\{x \in V \mid \det P(x + \alpha x) \neq 0\}_e = V^- + \Omega^+$$

because the condition $\det P(x + \alpha) \neq 0$ means precisely that the projection of x onto V^+ belongs to Ω. ∎

Many generalized tubes are essentially of the form G/H where G is the conformal group of the JTS V^-. This is partially explained by the following proposition.

Proposition XI.2.8. *Let the assumptions be as in the preceding theorem. Then the restriction to V^- defines a homomorphism*

$$r_- : \mathfrak{co}(V)^{(-\alpha)_*} \to \mathfrak{co}(V^-), \quad X \mapsto X|_{V^-}$$

where $\mathfrak{co}(V^-)$ is the conformal Lie algebra of the restriction of T to V^-. There is a similar restriction-homomorphism for inner conformal Lie algebras; it is surjective.

Proof. It is clear that, if $v \in V^-$ and $X \in \mathfrak{co}(V)^{(-\alpha)_*}$, then $X(v) \in V^-$, and therefore X can be restricted to V^-. We show that the restriction belongs to $\mathfrak{co}(V^-)$: since $(-\alpha)^*$ commutes with $(-\operatorname{id})_*$, we have the decomposition

$$\mathfrak{co}(V)^{(-\alpha)_*} = V^- \oplus \mathfrak{str}(V)^{\alpha_*} \oplus j_*(V^-).$$

If $X \in V^-$ (constant vector field), then its restriction to V^- is again constant and thus belongs to $\mathfrak{co}(V^-)$. If $X(p) = T(p,v,p)$ with $p \in V^-$, then Thm. VII.2.4 shows that again the restriction of X to V^- belongs to $\mathfrak{co}(V^-)$. Moreover, all constant, resp. all homogeneous quadratic vector fields of $\mathfrak{co}(V^-)$ are obtained in this way; this implies that the corresponding homomorphism on the level of inner conformal Lie algebras is surjective. ■

Remark XI.2.9. (i) The preceding statements can be lifted to the level of connected groups but not to the whole conformal group. Example: if V^- is of the second kind, then $-j$ belongs to $\mathrm{Co}(V)^{(-\alpha)_*}$, but it does not preserve $(V^-)^c$.
(ii) The restriction homomorphism is in general not injective: for example, if $\alpha = \mathrm{id}_V$, then V^- is a point and r_- is the zero homomorphism. However, in the "generic case" it is indeed bijective (cf. [BeHi98, Thm. 1.8.2]). ■

Realizations of the generalized cones Ω. For $\alpha = \mathrm{id}_V$ the generalized tube reduces to the open symmetric orbit Ω of which $M_0 = M_0^{(\mathrm{id}_V)}$ is the Cayley-transformed realization. Thus Ω has two distinguished realizations. Next we wish to describe *all* possible realizations of Ω under conformal transformations. For a JTS belonging to a Jordan algebra, the isomorphism $V^c \to W^c$, $gP \mapsto \Theta(g)P'$ can also be written, since $\Theta(g) = jgj$, $P' = jPj$,

$$V^c \to W^c, \quad g.P \mapsto jg.P.$$

Thus our base point in $V^c \times W^c$ is $(0_V, 0_W) = (0_V, j.0_V)$, not $(0_V, 0_V)$! In the case of the one-dimensional Jordan algebra $\mathbb{R}$, $j.0_V$ is "the" point at infinity, and therefore we will also use the notation

$$\infty := j(0) := j(0_V).$$

In order to avoid problems related to connected components, we will consider the algebraically connected space

$$\widetilde{\Omega} := \{x \in V \mid \det P(x) \neq 0\}$$

of invertible elements in a Jordan algebra (denoted by V' in Prop. II.2.9) which is the Cayley-transformed picture of the global space $M = M^{(\mathrm{id}_V)}$ associated to a Jordan algebra. The line bundle description of the "hyperplane at infinity" $H_\infty = V^c \setminus V$ (Prop. VIII.3.5 and Eqn. VIII.(3.1)) show that

$$\widetilde{\Omega} = V \cap j(V). \tag{2.3}$$

Theorem XI.2.10.
(i) *The translate of* $\widetilde{\Omega}$ *by the inverse of* $h \in \mathrm{Co}(T)$ *is*

$$h^{-1}(\widetilde{\Omega}) = \{g.0 \in V^c |\ \det(d_{hg}(0)) \det(d_{jhg}(0)) \neq 0\}.$$

Associating the point $h^{-1}(0_V, 0_W)$ *to* $h^{-1}(\Omega)$, *the space* $\mathrm{Co}(T).\widetilde{\Omega}$ *of translates of* Ω *is parametrized by the global polarized space* X *defined in* Section XIII.3.3*; if* V *is semisimple, then this parametrization is a bijection.*

(ii) *For all* $(a,b) \in V \times W$ *with* $\det B(a,b) \neq 0$, *there exists* $h \in \mathrm{Co}(T)$ *with* $h.a = 0_V$, $h.b = 0_W$, *and the affine picture of* $h^{-1}(\widetilde{\Omega})$ *is given by*

$$h^{-1}(\widetilde{\Omega}) \cap V = \{x \in V |\ \det(P(x-a)P(x-b)) \neq 0\}.$$

Proof. (i) Recall from Prop. VIII.3.4 the section given by $f(g) := \det(d_g(0))$ whose zero set is the complement of V. From Eqn. (2.3) it follows that

$$\widetilde{\Omega} = \{g.0 |\ \det(d_g(0)) \det(d_{jg}(0)) \neq 0\}.$$

Applying h^{-1} to this set yields the first claim. If $(a,b) \in X$, then there is $h \in \mathrm{Co}(T)$, uniquely determined up to an element s of $\mathrm{Str}(T)$, such that $a = h.0_V$, $b = h.0_W$ (Th. VIII.3.8). Since $s.\widetilde{\Omega} = \widetilde{\Omega}$, the set $h.\widetilde{\Omega}$ is uniquely determined by a and b, and by construction, every translate $h.\widetilde{\Omega}$ can be described in this manner. Thus we have a surjection

$$X \to \mathrm{Co}(T).\widetilde{\Omega}, \quad (a,b) = h.(0_V, 0_W) \mapsto h.\Omega. \tag{2.4}$$

For a general Jordan algebra it need not be injective: if $g \in \mathrm{Co}(T)$ satisfies $g.\widetilde{\Omega} = \widetilde{\Omega}$, then the equations of $\widetilde{\Omega}$ imply that either $g(H_\infty) = H_\infty$ and $g(jH_\infty) = jH_\infty$ or $g(H_\infty) = jH_\infty$. It follows that the group

$$G(\widetilde{\Omega}) := \{g \in \mathrm{Co}(V) |\, g.\widetilde{\Omega} = \widetilde{\Omega}\}$$

is the union $(G(V) \cap jG(V)j) \cup j(G(V) \cap jG(V)j)$ where

$$G(V) = \{g \in \mathrm{Co}(V) |\, g.V = V\}.$$

If V is semisimple, then $G(V) = P'$ (cf. remark after Def. VIII.3.3) and therefore $G(\widetilde{\Omega}) = (P \cap P') \cup j(P \cap P')) = \mathrm{Str}(T) \cup j\,\mathrm{Str}(T)$, and (2.4) is injective.

(ii) This is the affine version of part (i): the affine picture of X is given by the condition $\det B(a,b) \neq 0$ (cf. Th. VIII.3.7), the affine picture of $hj.V$ by the condition $\det P(x-b) \neq 0$ ($b = hj.0$) (cf. Eqn. VIII.(3.2)) and the affine picture of $h.V = hj.(jV)$ by $\det P(x-a) \neq 0$ ($a = hj.0$). ∎

We may summarize the result of the theorem by saying that the "hyperplane at infinity" H_∞ can be seen as "half of the boundary of Ω". Comparing with a general Makarevič space, the reader certainly has remarked that we have two descriptions of the Cayley-transformed realization of Ω: on the one hand by the equation $\det P(x-e) \det P(x-e) = 0$ (Th. XI.2.8 with $a = \gamma(0) = -e$, $b = \gamma(\infty) = e$), on the other hand by the equation $\det B(x,x) = 0$ (Th. X.2.1). In fact, the two expressions coincide already on the level of operators:

Lemma XI.2.11. *If T is the JTS associated to a unital Jordan algebra, then*

$$B(x,x) = P(x-e)P(x+e).$$

Proof. We use the notation $P(x,y) = 2(L(x)L(y)+L(y)L(x)-L(xy))$ and $P(x) = \frac{1}{2}P(x,x) = 2L(x)^2 - L(x^2)$ and keep in mind that $P(x,e) = 2L(x)$ and $T(x,x) = 2L(x^2)$:

$$\begin{aligned} P(x-e)P(x+e) &= (P(x) - P(x,e) + \mathbf{1})(P(x) + P(x,e) + \mathbf{1}) \\ &= P(x)^2 + 2P(x) + \mathbf{1} - P(x,e)^2 \\ &= \mathbf{1} + 2(P(x) - 2L(x)^2) + P(x)^2 \\ &= \mathbf{1} - T(x,x) + P(x)^2 = B(x,x). \end{aligned}$$

■

3. Causal symmetric spaces

The case where T is the JTS associated to a *Euclidean* Jordan algebra is of special interest because the Makarevič spaces associated to such algebras are *causal symmetric spaces*, i.e. symmetric spaces together with an invariant causal structure (cf. [HO96] for the general theory): recall from Section IX.2.3 that V and V^c carry a *causal structure* given by the *symmetric cone* Ω associated to V (cf. Section V.5.5). According to Thm. IX.2.4, the open subgroup $\mathrm{Cau}(T)$ of $\mathrm{Co}(T)$ generated by the translations, $G(\Omega)$ and $-j$ can be interpreted as the *causal group* of V and of V^c. All Makarevič spaces $G/H = M^{(\alpha)} \subset V^c$ inherit the causal structure from V^c, and this structure is invariant under the subgroup G of $\mathrm{Cau}(T)$. These spaces form an important class of causal symmetric spaces which we will call also *causal Makarevič spaces*.

To any causal structure one can associate its *causal pseudogroup* (cf. Section IX.2.3). In the framework of causal symmetric spaces there are no general results describing this pseudogroup, whereas for causal Makarevič spaces it is completely described by Th. IX.2.4: it can be imbedded into a *group* (namely the group $\mathrm{Cau}(T)$), and this group is much bigger than the automorphism group G of the symmetric space. (One may conjecture that this property characterizes causal Makarevič spaces among the causal symmetric spaces, but nothing precise is known.)

In this section we describe some basic features of causal Makarevič spaces; for further information see the literature mentioned in the Notes. First we give the main lemma leading to the *classification* of the irreducible spaces:

Lemma XI.3.1. *Every causal Makarevič space is isomorphic to one where α or $-\alpha$ is an involutive Jordan algebra automorphism of V.*

Proof. As explained in Section IV.2.3, the parameter α belongs to the *structure variety* of T; in case of a faithful JTS this is just the set of $\alpha \in \mathrm{Str}(T)$ with $\alpha^\sharp = \alpha^{-1}$. For a Euclidean Jordan algebra, $\alpha^\sharp = (\alpha^*)^{-1}$ (adjoint w.r.t. the trace form). Thus

$$\mathrm{Svar}(T) = \{\alpha \in \mathrm{Str}(T) \mid \alpha^* = \alpha\}$$

is the set of symmetric elements in the structure group. Since the structure group is defined by algebraic equations, the set $\mathrm{Svar}(T)$ has a polar decomposition: we can write $\alpha = k\exp(X)$ with X a symmetric element in $\mathfrak{str}(T)$ and $k \in \mathrm{Svar}(T)$ an orthogonal linear map, i.e. an element of $\mathrm{Str}(T) \cap \mathrm{O}(n) = \pm\mathrm{Aut}(T)$. But since k is also symmetric, the conditions $k^* = k^{-1}$ and $k = k^*$ together imply $k = k^{-1}$, i.e. $k^2 = \mathrm{id}$, and k is therefore either an involutive automorphism or the negative of one. It follows that $\alpha = k\exp(X)$ and k lie in the same connected component of $\mathrm{Svar}(T)$, and therefore the spaces $X^{(\alpha)}$ and $X^{(k)}$ are isomorphic (Section 2.3). ■

The lemma reduces the classification of causal Makarevič spaces associated to simple Euclidean Jordan algebras V to the classification of their involutions. This can be done by using some well-known structure theory of Euclidean Jordan algebras; for the classical Jordan algebras the result is contained in Tables XII.2.2 and XII.2.6 (cases i.1. with i=$1,\ldots,5$). The corresponding causal Makarevič spaces are listed in Tables XII.4.2 and XII.4.5 (cases i.1 with i as above). The interested reader should compare this classification with the one of simple causal symmetric spaces given in [HO96]: there are only two series of causal symmetric spaces (group case $\mathrm{SO}(n,2)$ and the spaces $\mathrm{SO}(p+q,2)/(\mathrm{SO}(p,1)\times \mathrm{SO}(q,1))$ with $\min(p,q)>1$) and four exceptional spaces which have no direct relation to causal Makarevič spaces. Thus also in the category of causal symmetric spaces there is a sort of "Jordan-Lie functor" which is not far from being bijective. However, the problem of understanding this becomes even more involved by the fact that the two series mentioned above can be associated to Jordan triple systems (cf. Table XII.4.5), but *not* to Euclidean Jordan algebras. It would be very interesting to find a geometric reason for this phenomenon.

Now let us describe some basic structure features of causal Makarevič spaces. In the theory of causal symmetric spaces one distinguishes two types: the *compactly causal* and the *non-compactly causal spaces*. Geometrically, the non-compact causal spaces are those which do not admit non-trivial closed causal curves, i.e. curves $\gamma(t)$ having derivative $\dot\gamma(t)$ in the cone $C_{\gamma(t)}\subset T_{\gamma(t)}M$ for all t. More technically, the compactly causal spaces are defined by the conditions $C_o\cap\mathfrak{q}_{\mathfrak{k}}\neq\emptyset$, $C_o\cap\mathfrak{q}_p=\emptyset$ and the non-compactly causal ones by the conditions $C_o\cap\mathfrak{q}_{\mathfrak{p}}\neq\emptyset$, $C_o\cap\mathfrak{q}_{\mathfrak{k}}=\emptyset$, where $\mathfrak{q}=\mathfrak{q}_k\oplus\mathfrak{q}_{\mathfrak{p}}$ is a Cartan-decomposition of the LTS $\mathfrak{q}\cong T_oM$ and $C_o\subset\mathfrak{q}$ is the (open) $\mathrm{Ad}(H)$-invariant cone defining the invariant causal structure on $M=G/H$ (cf. [HO96, Ch.3]).

Proposition XI.3.2. *Let α be an involution of the Euclidean Jordan algebra V. Then the space $M^{(-\alpha)}$ is compactly causal and the space $M^{(\alpha)}$ is non-compactly causal.*

Proof. The LTS $\mathfrak{q}=\mathfrak{q}^{(\alpha)}$ is given by

$$\mathfrak{q}=\{\mathbf{v}-\mathbf{p}_{\alpha v}|\, v\in V\}$$

(cf. proof of Prop. X.1.3), and our cone C_o, equivalent to Ω, is obtained by choosing $v\in\Omega$ in this description. A Cartan involution of $\mathfrak{q}$ is given by α (Ex. X.6.9); thus $\mathfrak{q}_{\mathfrak{k}}$ is the subspace corresponding to V^- and $\mathfrak{q}_{\mathfrak{p}}$ the subspace corresponding to V^+. Since $e\in(V^+\cap\Omega)$, it follows that $C_o\cap\mathfrak{q}_{\mathfrak{p}}\neq\emptyset$, and from $V^-\cap\Omega=\emptyset$ (if this were not so, Ω would contain a point x together with $\alpha(x)=-x$ since α is an automorphism; contradiction) we get $C_o\cap\mathfrak{q}_{\mathfrak{k}}=\emptyset$. Thus $M^{(\alpha)}$ is non-compactly causal.

For the LTS $\mathfrak{q}^{(-\alpha)}$ belonging to the c-dual space, the subspaces V^+ and V^- exchange their roles; thus the same arguments as above show that $M^{(-\alpha)}$ is compactly causal. ■

In particular, the compact symmetric space $V^c=M^{(-\mathrm{id}_V)}$ (Cor. X.6.7) is compactly causal, and its c-dual $M^{(\mathrm{id}_V)}$, the Cayley-transformed realization of the cone Ω, is non-compactly causal. There are also spaces which are both compactly and non-compactly causal; they are called *of Cayley type* (cf. [HO96]) and correspond precisely to twisted polarized LTS defined by Euclidean Jordan algebras (Tables XII.2.2 and XII.4.2).

From the point of view of harmonic analysis, the compactly and the non-compactly spaces show very different behaviors: the main feature of the compactly causal spaces is that one can define analogs of the classical *tube domains*. In the case of a compactly causal Makarevič space M these domains Ξ are in fact very closely related to the tube domain $T_\Omega=V+i\Omega$ associated to the Euclidean Jordan algebra V: they are just the intersection

$$\Xi=T_\Omega\cap M_{\mathbb{C}}$$

with the global complexification $M_{\mathbb{C}}$ (Prop. X.3.6) of M (cf. [Be98a, Th. 3.3.4]). Under the Cayley transform C, the tube T_Ω is transformed to the bounded symmetric domain D and $C(M_{\mathbb{C}})$ is described as in Th. XI.2.7, whence

$$\Xi' := C(\Xi) = \{z \in D |\ \det P(z + \alpha z) \neq 0\} = D \cap (V_{\mathbb{C}}^- + \Omega_{\mathbb{C}}^+).$$

The bounded symmetric domain

$$D^- := D \cap V_{\mathbb{C}}^- \subset D,$$

which lies on the boundary of Ξ', corresponds to the bounded symmetric domain G/K from [HO96]. It is this situation we mentioned in Remark XI.2.9 as an application of Prop. XI.2.8: in the "generic case" (essentially, this means that α is not a Peirce-involution), there is an isomorphism $G \to G(D^-)$ (cf. [BeHi99, Th. 1.8.2]). This gives the realization of G as a "Hermitian Lie group" predicted by the general theory of causal structures.

Turning now to the non-compactly causal symmetric spaces, their main feature is that they are *causally ordered*: let us write $x < y$ if there is a non-trivial causal curve (in the sense explained above) from x to y. If M is non-compactly causal, then this defines a partial order on M; the set of all y with $x < y$ can be interpreted as the "future of x". For instance, if C is the constant cone field on a vector space V given by a fixed cone $C_o \subset V$, then the future of x is simply the set $x + C_o$. Given a causal ordering, one defines *causal intervals* by

$$]x, z[:= \{y \in M |\ x < y < z\}$$

for $x, z \in M$ with $x < z$. In the example of a vector space with constant structure, $]x, z[= (x + C_o) \cap (z - C_o)$. If M is compactly causal, then there exist non-trivial closed causal curves, and these definitions no longer make sense. However, if M is the compactly causal Makarevič space V^c, then the flat interval structure on V has a unique extension to an "interval structure" on M:

Theorem XI.3.3. *Let $g \in \mathrm{Cau}(V)$ and $a = g(0)$, $b = g(j(0))$. Then the affine image of $g(\Omega)$ is of one of the following types:*

(1) *elliptic type: $a, b \in V$ and $a < b$ (in V); then $g(\Omega) \cap V = g(\Omega) = (a + \Omega) \cap (b - \Omega)$;*

(2) *hyperbolic type: $a, b \in V$ and not $a < b$; in this case $g(\Omega) \cap V$ is in general not connected;*

(3) *parabolic type: $b \in H_\infty$, $a \in V$; then $g(\Omega) \cap V = g(\Omega) = a + \Omega$, and similarly for $a \in H_\infty$, $b \in V$.*

Proof. If $a, b \in V$, then the affine picture of the algebraically connected space $g(\widetilde{\Omega})$ is given by the equation $\det(P(x-a)P(x-b)) \neq 0$ (Th. XI.2.10). Similarly as in the proof of that theorem, one shows that if $b \in H_\infty$, then the affine picture is given by the condition $\det P(x-a) \neq 0$, and vice versa for $a \in H_\infty$. The case that both a and b belong to H_∞ cannot occur since $(a, b) \in X$ (cf. Th. XI.2.10).

The space $g(\Omega)$ is a connected component of $g(\widetilde{\Omega})$. Assume $a, b \in V$ and $]a, b[:= (a + \Omega) \cap (b - \Omega)$ to be not empty; then because of the equations of $g(\widetilde{\Omega})$ the set $]a, b[$ is open and closed in $g(\widetilde{\Omega}) \cap V$. It is connected because Ω is convex and therefore is a connected component of $g(\widetilde{\Omega}) \cap V$ and thus also of $g(\widetilde{\Omega})$ (since no point of H_∞ belongs to its closure). By causality of g, all tangent vectors from $a + \Omega$ are directed into the interior of $g(\Omega)$, and therefore the connected component $g(\Omega)$ of $g(\widetilde{\Omega})$ must coincide with the connected component $]a, b[$. The third case is established by similar arguments.

In the second case, although the algebraic equations of $g(\Omega) \cap V$ are known, a concrete description in terms of connected components turns out to be fairly involved. (One has to distinguish the cases $b \in a - \Omega$ and $b \notin a - \Omega$, but already in the easier first case it can happen that $g(\Omega) \cap V$ has more than just the obvious components $a + \Omega$ and $b - \Omega$.) ∎

Already the simple example $V = \mathbb{R}$, $V^c = S^1$ gives a good illustration of the theorem: Ω is the open interval $]0, \infty[$, and the translates of Ω are precisely the open intervals of the circle. It is clear that we have three types of affine realizations of such intervals.

Restricting the "causal interval structure" of V^c defines such a structure on any causal Makarevič space M modelled on V. If M is non-compactly causal, then it follows easily from the arguments used in the preceding proof that the interval structure thus obtained is nothing but the interval structure coming from the global causal ordering of M. However, if M is compactly causal, then the existence of an "interval structure" seems to be a rather surprising feature: for a general causal manifold, one can always define an "interval structure" locally, but if it is not ordered, one runs into topological problems when trying to globalize it. The key property making causal Makarevič spaces exceptional in this regard is that they are *causally flat*, i.e. that the causal structure can in a chart (Jordan coordinates) be reduced to a constant one on a vector space.

The terminology used for the three types of realization is explained by the similarity with the affine realizations of a projective conic: for the elliptic realization, the affine image of the boundary of Ω is complete; for the parabolic realization the hyperplane at infinity is "tangent" to the boundary of Ω, and for the hyperbolic realization it intersects the boundary non-tangentially.

4. Helwig spaces and the extension problem

Many of the most important symmetric spaces with twist are of the first kind, but some are not, e.g. the orthogonal groups $\mathrm{SO}(m)$ for m odd and the Grassmannians $\mathrm{Gr}_{p,p+q}$ with $p \neq q$. On the other hand, we have a natural imbedding $\mathrm{SO}(m) \subset \mathrm{Gl}(m, \mathbb{R})$, which corresponds to the imbedding

$$\mathrm{Asym}(m, \mathbb{R}) = \{X \in M(m, \mathbb{R}) |\, X = -X^t\} \subset M(m, \mathbb{R}),$$

into a JTS of the *first* kind, and similarly we can imbed $M(p, q; \mathbb{R})$ into the JTS $\mathrm{Sym}(p + q, \mathbb{R})$ of the first kind. This motivates the following definitions which describe possible relations of Jordan triple systems with Jordan algebras.

Definition XI.4.1.
- (i) A JTS is said to be *associated to a Jordan algebra* V if it is of the form $T(x, y, z) = 2(x(yz) - y(xz) + (xy)z)$ for some Jordan algebra V.
- (ii) A JTS is said to be a -1*-space in an involutive Jordan algebra* (V, α) if it is given by restricting the JTS associated to the Jordan algebra V to the -1-eigenspace of some involutive automorphism α of V.
- (iii) A JTS is said to be a -1*-space in a modified involutive Jordan algebra* if it is given by restricting some JTS of the first kind to the -1-eigenspace of an involution. ■

We say that the corresponding symmetric spaces with twist have the properties just defined if their JTS have them. For instance, $\mathrm{SO}(m)$ is the -1-space in the involutive Jordan algebra $(M(m, \mathbb{R}), X \mapsto X^t)$. The -1-spaces in involutive Jordan algebras are interesting because they are nothing but Cayley-transformed Helwig spaces (cf. Section II.4):

Proposition XI.4.2. *Assume that α is an involution of a unital Jordan algebra V, M the global space associated to the JTS T of V and*

$$M^- := \{x \in M |\, \alpha(x) = -x\}$$

the corresponding -1-space. Then the Cayley transformed realization of M^-,

$$\gamma(M^-) = \{x \in V |\, \det P(x) \neq 0,\, j(x) = \alpha(x)\},$$

is a Helwig space, and every Helwig space is obtained in this way.

Proof. Recall first that $\gamma(M)$ is the space $V' = \{x \in V |\ \det P(x) \neq 0\}$ associated to the Jordan algebra V and thus has no points at infinity. Now the relation $\gamma j \alpha \gamma^{-1} = -\alpha$ shows that $\gamma(M^-) = \{x \in V' |\, j\alpha(x) = x\}$. This is precisely the definition of a Helwig space, and since the involution α used here is arbitrary, any Helwig space is obtained in this way. (We allow here Helwig spaces to be non-connected, thus slightly modifying the definition from Section II.4.) ■

In the example of $\mathrm{SO}(m)$, the Helwig space realization is the usual realization in $\mathrm{Gl}(m,\mathbb{R})$, and in the case of Grassmannians, the Helwig space realization is the one described in Ex. II.4.2 and II.4.4. Classification shows that many symmetric spaces have a Helwig space realization. In analogy with the situation in Lie-theory, one might guess that every JTS is a -1-space in an involutive Jordan algebra (just as any LTS is a -1-space in an involutive Lie algebra, namely in its standard imbedding), but this is false as shows a counterexample given by O. Loos and K. McCrimmon ([LoM77]). On the other hand, it is an open problem whether all JTS are -1-spaces in modified involutive Jordan algebras. The following *extension problem* summarizes these questions concerning the relations between triple systems and algebras:

Problem XI.4.3.

(a) Is every real finite-dimensional JTS a -1-space in a modified involutive Jordan algebra?

(b) Which real finite-dimensional JTS are -1-spaces in involutive Jordan algebras?

The geometric version is:

(a') Is every symmetric space a -1-space in a modified involutive Jordan algebra?

(b') Is there an intrinsic criterion allowing to decide which symmetric spaces with twist are Helwig spaces? ■

The second problem can be formulated in a more specific way. Let T be a faithful JTS (not necessarily of the first kind). We decompose the conformal Lie algebra under the involution Θ:

$$\mathfrak{co}(T) = \mathfrak{co}(T)^{\Theta} \oplus \mathfrak{co}(T)^{-\Theta}, \tag{4.1}$$

with

$$\mathfrak{co}(T)^{\Theta} = \mathfrak{q} \oplus \mathfrak{str}(T)^{\Theta} = \mathfrak{q} \oplus \mathrm{Der}(T) \tag{4.2}$$

and

$$\mathfrak{p} := \mathfrak{co}(T)^{-\Theta} = \widehat{\mathfrak{q}} \oplus \mathfrak{str}(T)^{-\Theta} = \widehat{\mathfrak{q}} \oplus \mathrm{Ader}(T), \tag{4.3}$$

where $\mathrm{Ader}(T)$ is the space of "skew-derivations" of T, i.e. the space of $X \in \mathrm{End}(V)$ with $XT(u,v,w) = T(Xu,v,w) - T(u,Xv,w) + T(u,v,Xw)$. The decomposition (4.1) is a decomposition of $\mathfrak{co}(T)$ into G-submodules, where $G = \mathrm{Co}(T)^{\Theta}_{o}$ acts by the adjoint representation $*$. Of course, the first term is nothing but the Lie algebra $\mathfrak{g}$ of G with its adjoint representation. The second term yields a new representation of G which is interesting because the representation space $\mathfrak{p} := \mathfrak{co}(T)^{-\Theta}$ has a natural base point, nameley the Euler operator E.

Proposition XI.4.4. *Let $M_0 = G/H$ be the connected space associated to the faithful JTS T. Then the orbit*

$$G_* E \subset \mathfrak{p} = \mathfrak{co}(T)^{-\Theta}$$

is isomorphic to the symmetric space $M = G/H$, and it is an (extrinsic) symmetric submanifold of $\mathfrak{p}$ in the sense of Def. II.4.1.

Proof. As remarked in the proof of Prop. X.1.1, H is the open subgroup $G \cap \mathrm{Aut}(T)$ of $\mathrm{Aut}(T)$. On the other hand, for $g \in G$ the condition $g_* E = E$ is equivalent to $g \in \mathrm{Str}(T) \cap G$ (Prop. VIII.1.5). Clearly $\mathrm{Str}(T) \cap G = \mathrm{Aut}(T) \cap G = H$, and therefore $G_* E \cong G/H \cong M_0$ as a symmetric space.

In order to prove that $G_*E \subset \mathfrak{p}$ is a symmetric submanifold, we note first that G acts via $*$ by *linear* maps on $\mathfrak{p}$. Next recall that the involution $(-\mathrm{id})_*$ commutes with Θ, and the corresponding decompositions of the eigenspaces of Θ are the ones indicated in (4.2) and (4.3). The Euler operator E belongs to $\mathrm{Ader}(T) = \mathfrak{p}^{-(-\mathrm{id})_*}$. We claim that $S_e = (-\mathrm{id})_*|_{\mathfrak{p}}$ satisfies the axioms (S1) – (S3) of Def. II.4.1. In fact, (S1) is clear, and (S2) follows immediately from the fact that H is open in $G^{(-\mathrm{id})_*}$: for all $g \in G$,

$$(-\mathrm{id})_*(g_*E) = ((-\mathrm{id})_*g)_*E.$$

Finally, from the decomposition $\mathfrak{g} = \mathrm{Der}(T) \oplus \mathfrak{q}$ we get

$$T_e(G_*E) = [\mathfrak{q}, E] = \widehat{\mathfrak{q}} = \mathfrak{p}^{-(-\mathrm{id})_*},$$

whence (S3). ∎

Now Problem XI.4.3 (b') can be formulated in the following way:

Problem XI.4.5.

(b1) Under which conditions is there a "natural" Jordan algebra structure with unit E on $\mathfrak{p}$ such that $(-\mathrm{id})_*|_{\mathfrak{p}}$ is turned into an involution?

(b2) Under which conditions is the orbit $G_*E \subset \mathfrak{p}$ a Helwig space? ∎

The most difficult point in Problem (b1) seems to be to turn $\mathrm{Ader}(T)$ into a Jordan algebra V^+ or, put in another way, to describe the obstruction for this in the general case. Here it would be helpful to have an intrinsic characterization of Helwig spaces among the symmetric submanifolds.

5. Examples

Finally we present the most important examples of global symmetric spaces with twist. The theory developed so far shows that it is natural to realize them in families

$$M_0^{(\alpha)} = \mathrm{Co}(T)_o^{\Theta\alpha_*}.0,$$

where T is a "fixed" JTS and α runs through the structure variety of T (i.e. α belongs to the structure group and $\Theta(\alpha) = \alpha^{-1}$). Then all spaces $M_0^{(\alpha)}$ are *conformally equivalent* in the sense that they have the same conformal group $\mathrm{Co}(T)$ and the same underlying Jordan pair $T_{ph\mathbb{C}}$. The Bergman-operator and the quasi-inverse for $M_0^{(\alpha)}$ are just given by $B(x, \alpha y)$, resp. by $\Theta(t_{\alpha y})(x)$, and thus all members of the family have very similar Jordan-theoretic descriptions (just replace y by αy).

The most natural candidates for α are *involutions*, i.e. automorphisms of order 2. We start with the case that T is associated to a (simple) Jordan algebra V. Then the spaces $M_0^{(\alpha)}$ are spaces of the first kind, also called *Makarevič spaces* (Remark XI.1.7).

5.1. The general linear group and its modifications. We have already seen that the general linear group is the space $M = M^{(\mathrm{id})}$ associated to $V = M(n, \mathbb{R})$ with the triple product $T(X, Y, Z) = XYZ + ZYX$ (Ex. IX.1.4, IX.2.9).

For $\alpha = -\mathrm{id}_V$, we obtain the c-dual space $M^{(-\mathrm{id}_V)} = \mathrm{Gl}(n, \mathbb{C})/\mathrm{Gl}(n, \mathbb{R})$ (cf. Ex. X.1.4).

For $\alpha(X) = -X^t$, we are led to the JTS $T^{(\alpha)}(X, Y, Z) = -(XY^tZ + ZY^tX)$ considered in Ex. X.1.5. As seen there, the associated space is the Grassmannian $\mathrm{O}(2n)/(\mathrm{O}(n) \times \mathrm{O}(n))$. This is the conformal compactification of V.

For $\alpha(X) = X^t$, we get the c-dual space $\mathrm{O}(n, n)/(\mathrm{O}(n) \times \mathrm{O}(n))$ of the Grassmannian.

More generally, let $\alpha(X) = A^{-1}X^tA$ be the adjoint w.r.t. the form given by an orthogonal matrix A.

Lemma XI.5.1. *For all* $g \in \mathrm{Co}(T)_o = \mathbb{P}\,\mathrm{Gl}(2n,\mathbb{R})_o$*,*

$$(j\alpha)_* = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} (g^t)^{-1} \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}^{-1},$$

$$(-j\alpha)_* = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} (g^t)^{-1} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}^{-1}.$$

Proof. We use notation of the proof of Prop. VIII.4.2. Equation VIII.(4.8) shows that the map α extends to the conformal compactification $\mathrm{Gr}_{n,2n}(\mathbb{R})$ by $E \mapsto E^{\perp,A_1}$ assigning to an n-dimensional subspace its orthocomplement w.r.t. the form on $\mathbb{R}^{2n}$ given by the matrix $A_1 := \left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right)$. Similarly, writing Eqn. VIII.(4.6) for invertible X in the form

$$(v|w) - (Xv|(X^*)^{-1}w) = 0,$$

we see that the map $-j\alpha$ extends to the conformal compactification by $E \mapsto E^{\perp,A_2}$ assigning to an n-dimensional subspace its orthocomplement w.r.t. the form on $\mathbb{R}^{2n}$ given the matrix $A_2 := \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. The same equation shows that $j\alpha$ extends to $\mathrm{Gr}_{n,2n}(\mathbb{R})$ by $E \mapsto E^{\perp,A_3}$ w.r.t. the matrix $A_3 := \left(\begin{smallmatrix} A & 0 \\ 0 & -A \end{smallmatrix}\right)$. As usual, we identify the birational maps $j\alpha$ etc. with their extension to the conformal compactification.

Now, if $E^\perp$ denotes the orthocomplement of $E \in \mathrm{Gr}_{n,2n}(\mathbb{R})$ w.r.t. an arbitrary non-degenerate form, the definition of the adjoint yields for all $g \in \mathrm{Gl}(2n,\mathbb{R})$

$$g.(E^\perp) = ((g^*)^{-1})^\perp.$$

Thus the map $\perp\colon \mathrm{Gr}_{n,2n}(\mathbb{R}) \to \mathrm{Gr}_{n,2n}(\mathbb{R})$ induces the map $g \mapsto (g^*)^{-1}$ of the conformal group $\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{R})$. Combining this observation with the expression of $j\alpha$ resp. of $-j\alpha$ in terms of orthocomplements yields the claim. ∎

The connected fixed point group of $(j\alpha)_*$ is $\mathrm{O}(\left(\begin{smallmatrix} A & 0 \\ 0 & -A \end{smallmatrix}\right),\mathbb{R})_o$, and the connected fixed point group of $(-j\alpha)_*$ is $\mathrm{O}(\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right),\mathbb{R})_o$. The automorphism $(-\mathrm{id})_*$ is given by conjugating with the matrix $I_{n,n}$. It follows that

$$M_0^{(\alpha)} \cong \mathrm{O}(\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix},\mathbb{R})/(\mathrm{O}(A,\mathbb{R}) \times \mathrm{O}(A,\mathbb{R})),$$

$$M_0^{(-\alpha)} \cong \mathrm{O}(\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},\mathbb{R})/(\mathrm{O}(A,\mathbb{R}) \times \mathrm{O}(A,\mathbb{R})).$$

Theorem XI.5.2. *Up to isomorphism, the spaces which are conformally equivalent to* $\mathrm{Gl}(n,\mathbb{R})$ *are classified as follows:*

(i) $\mathrm{Gl}(n,\mathbb{R})$ *and its c-dual space* $\mathrm{Gl}(n,\mathbb{C})/\mathrm{Gl}(n,\mathbb{R})$*,*

(ii) *the "pseudo-Grassmannians"* $\mathrm{O}(p+u,q+v)/(\mathrm{O}(u,v)\times \mathrm{O}(p,q))$ *with* $p+q=u+v=n$*,*

(iii) *if* $n=2m$ *is even, the spaces* $\mathrm{Sp}(2m,\mathbb{R})/(\mathrm{Sp}(m,\mathbb{R})\times\mathrm{Sp}(m,\mathbb{R}))$*.*

Proof. Type (i) has been discussed above. The space of type (iii) is obtained by choosing $A = J$; it is self c-dual. The space of type (ii) with $p=u, q=v$ is obtained by choosing $A = I_{p,q}$; the c-dual is a space of the same type with p and q exchanged. The other spaces of type (ii) are obtained from $\alpha(X) = I_{p,q}X^tI_{u,v}$ by a reasoning similar to the one given above. Finally, for the proof that the list is complete, we need the complete description of the structure group $\mathrm{Str}(T)$ (Table XII.1.3). From this one gets a description of $\mathrm{Svar}(T)$ (the self-adoint elements of $\mathrm{Str}(T)$ w.r.t. the trace form). Then one checks that the elements α leading to the spaces (i) – (iii) represent already all connected components of $\mathrm{Svar}(T)$ (cf. [Ma73]). ∎

Similarly, we get the list of spaces conformally equivalent to the group $\mathrm{Gl}(n,\mathbb{H})$:

(i) $\mathrm{Gl}(n,\mathbb{H})$ and its c-dual space $\mathrm{Gl}(2n,\mathbb{C})/\mathrm{Gl}(n,\mathbb{H})$,

(ii) $\mathrm{Sp}(p+u,q+v)/(\mathrm{Sp}(u,v)\times\mathrm{Sp}(p,q))$, $u+v=p+q=n$,

(iii) if $n=2m$, $\mathrm{O}^*(4m)/(\mathrm{O}^*(2m)\times\mathrm{O}^*(2m))$.

Theorem XI.5.3. *Up to isomorphism, the spaces which are conformally equivalent to* $\mathrm{Gl}(n,\mathbb{C})$ *(considered as real symmetric space with twist) are classified as follows:*

(i) *spaces of complex structures:* $\mathrm{Gl}(n,\mathbb{H})/\mathrm{Gl}(n,\mathbb{C})$ *and* $\mathrm{Gl}(2n,\mathbb{R})/\mathrm{Gl}(n,\mathbb{C})$,

(ii) *the "complex pseudo-Grassmannians"* $\mathrm{U}(p+u,q+v)/(\mathrm{U}(u,v)\times\mathrm{U}(p,q))$, $p+q=u+v=n$,

(iii) $\mathrm{Gl}(n,\mathbb{C})$,

(iv) $\mathrm{SO}(2n,\mathbb{C})/(\mathrm{SO}(n,\mathbb{C})\times\mathrm{SO}(n,\mathbb{C}))$,

(v) *if* $n=2m$, $\mathrm{Sp}(2m,\mathbb{C})/(\mathrm{Sp}(m,\mathbb{C})\times\mathrm{Sp}(m,\mathbb{C}))$.

Proof. The proof follows the same lines as the proof of Thm. XI.5.2. However, we have to distinguish the cases (a) that α is $\mathbb{C}$-conjugate linear and (b) that α is $\mathbb{C}$-linear. Case (b) leads as above the spaces of type (iii), (iv) and (v) which are straight complex and therefore self c-dual. As for case (b), the automorphism $\alpha(X)=I_{p,q}\overline{X}^t I_{u,v}$ leads to the spaces of type (ii) (just as for $\mathbb{F}=\mathbb{R}$ or $\mathbb{H}$).

We will show that the automorphism $\alpha(Z)=\overline{Z}$ leads to the spaces of type (i). The induced automorphism α_* of $\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{C})$ is again just complex conjugation of matrices, and since $(-j)_*$ is conjugation by the matrix J, we have $(-j\alpha)_*g=J\overline{g}J^{-1}$. The fixed point group is $\mathrm{Gl}(n,\mathbb{H})$ (cf. Eqn. (I.6.16)). Similarly, the fixed point group of $(j\alpha)_*$ is isomorphic to $\mathrm{Gl}(2n,\mathbb{R})$. Taking in these groups the group fixed under conjugation by $I_{n,n}$ we get $\mathrm{Gl}(n,\mathbb{C})$. We thus obtain the symmetric space $\mathrm{Gl}(2n,\mathbb{R})/\mathrm{Gl}(n,\mathbb{C})$ and $\mathrm{Gl}(n,\mathbb{H})/\mathrm{Gl}(n,\mathbb{C})$ which have been interpreted in Section I.6.3 as spaces of complex structures in $\mathrm{Gl}(2n,\mathbb{R})$, resp. in $\mathrm{Gl}(n,\mathbb{H})$.

For the proof of completeness of the classification the same remarks as in the proof of Thm. XI.5.2 apply. ■

The Grassmannians $\mathrm{Gr}_{p,p+q}(\mathbb{F})$ with $p\neq q$ are treated in a similar way. The spaces of type (i) (resp. types (i) and (iii) if $\mathbb{F}=\mathbb{C}$) are missing; for the other types there are obvious analogs of the series named above.

5.2. Orthogonal and unitary groups.

Lemma XI.5.4.

(i) *The group* $\mathrm{O}(A,\mathbb{R})$ *is the symmetric space with twist associated to the JTS* $\mathrm{Asym}(A,\mathbb{R})$ *with the triple product* $T(X,Y,Z)=XYZ+ZYX$.

(ii) *The group* $\mathrm{U}(A,\varepsilon,\mathbb{F})$ *is the symmetric space with twist associated to the JTS* $\mathrm{Aherm}(A,\varepsilon,\mathbb{F})$ *with the triple product* $T(X,Y,Z)=XYZ+ZYX$.

Proof. This can be proved both in the context of homogeneous spaces (Ex. X.1.4) and in the algebraic context (Ex. X.2.7). In the context of the homogeneous space $\mathrm{Gl}(n,\mathbb{F})$ consider the automorphism $\alpha(X)=-X^*$. The induced automorphism of the conformal group $\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{F})$ is given by taking the adjoint w.r.t. the form $\left(\begin{smallmatrix}0&A\\A&0\end{smallmatrix}\right)$ (cf. proof of Lemma XI.5.1). The subgroup of $\mathrm{Gl}(2n,\mathbb{F})$ fixed under this involution is

$$\{\begin{pmatrix}g&h\\h&g\end{pmatrix}\mid g,h\in\mathrm{U}(A,\varepsilon,\mathbb{F})\},$$

and the subgroup in this group fixed under $(-\mathrm{id})_*$ is isomorphic $\mathrm{U}(A,\varepsilon,\mathbb{F})$ (cf. Ex. X.1.4). This shows that $\mathrm{U}(A,\varepsilon,\mathbb{F})\times\mathrm{U}(A,\varepsilon,\mathbb{F})/\mathrm{U}(A,\varepsilon,\mathbb{F})$ is the sub-symmetric space of $\mathrm{Gl}(n,\mathbb{F})$ corresponding to the sub-JTS $\mathrm{Aherm}(A,\varepsilon,\mathbb{F})$ of $M(n,\mathbb{F})$.

In the algebraic context, note that the real Cayley transform R directly transforms graphs of operators from $\mathrm{Aherm}(A, \varepsilon, \mathbb{F})$ into graphs of A-unitary operators (cf. Prop. VIII.4.4). ■

Theorem XI.5.5. *Up to isomorphism, the spaces which are conformally equivalent to* $\mathrm{O}(n)$ *are classified as follows:*

(i) $\mathrm{O}(p, q)$ *and its c-dual space* $\mathrm{O}(n, \mathbb{C})/\mathrm{O}(p, q)$ $(p + q = n)$*,*

(ii) *if* $n = 2m$ *is even,* $\mathrm{Gl}(2m, \mathbb{R})/\mathrm{Sp}(m, \mathbb{R})$ *and its c-dual space* $\mathrm{U}(m, m)/\mathrm{Sp}(m, \mathbb{R})$*.*

Proof. The spaces $\mathrm{O}(p, q)$ are associated to the automorphism $\alpha(X) = I_{p,q} X I_{p,q}$ of $V = \mathrm{Asym}(n, \mathbb{R})$. If $n = 2m$ is even, then $X \mapsto JX$ is an isomorphism of $\mathrm{Asym}(2m, \mathbb{R})$ onto the JTS $\mathrm{Sym}(J, \mathbb{R})$ associated to the Jordan algebra $\mathrm{Sym}(J, \mathbb{R})$ (Lemma VIII.4.3). The spaces of type (ii) are just the prehomogeneous symmetric space associated to this Jordan algebra (cf. Section II.3.2) and its c-dual. For completeness of this classification, see remarks above. ■

In the same way one gets for the other orthogonal and unitary groups the classification of conformally equivalent spaces:

The group $\mathrm{Sp}(n, \mathbb{R})$:

(i) $\mathrm{Sp}(n, \mathbb{R})$ and its c-dual $\mathrm{Sp}(n, \mathbb{C})/\mathrm{Sp}(n, \mathbb{R})$,

(ii) $\mathrm{Gl}(2n, \mathbb{R})/\mathrm{O}(2n - p, p)$ and its c-dual space $\mathrm{U}(2n - p, p)/\mathrm{O}(2n - p, p)$ (for $p = 0$ this is a space of Lagrangians).

The group $\mathrm{Sp}(n)$:

(i) $\mathrm{Sp}(p, q)$ and its c-dual $\mathrm{Sp}(n, \mathbb{C})/\mathrm{Sp}(p, q)$ $(p + q = n)$,

(ii) $\mathrm{Gl}(n, \mathbb{H})/\mathrm{O}^*(2n)$ and its c-dual space $\mathrm{U}(n, n)/\mathrm{O}^*(2n)$.

The group $\mathrm{O}^*(2n)$:

(i) $\mathrm{O}^*(2n)$ and its c-dual space $\mathrm{O}(2n, \mathbb{C})/\mathrm{O}^*(2n)$,

(ii) $\mathrm{Gl}(n, \mathbb{H})/\mathrm{Sp}(p, q)$ and its c-dual space $\mathrm{U}(2p, 2q)/\mathrm{Sp}(p, q)$ $(p + q = n$; for $q = 0$ this is a space of Lagrangians).

The group $\mathrm{U}(n)$ needs a special treatment:

Theorem XI.5.6. *Up to isomorphism, the spaces which are conformally equivalent to* $\mathrm{U}(n)$ *are classified as follows:*

(i) $\mathrm{U}(p, q)$ *and its c-dual space* $\mathrm{Gl}(n, \mathbb{C})/\mathrm{U}(p, q)$ $(p + q = n)$*,*

(ii) $\mathrm{O}^*(2n)/\mathrm{O}(n, \mathbb{C})$ *and its c-dual space* $\mathrm{O}(n, n)/\mathrm{O}(n, \mathbb{C})$*,*

(iii) *if* $n = 2m$ *is even,* $\mathrm{Sp}(2m, \mathbb{R})/\mathrm{Sp}(m, \mathbb{C})$ *and its c-dual space* $\mathrm{Sp}(m, m)/\mathrm{Sp}(m, \mathbb{C})$*.*

Proof. Lemma XI.5.4 says that $\mathrm{U}(n)$ is associated to the JTS $\mathrm{Aherm}(n, \mathbb{C})$. Multiplication by i is an isomorphism onto $V = \mathrm{Herm}(n, \mathbb{C})$. The space of type (i) is associated to the involution $\alpha(X) = I_{p,q} X I_{p,q}$ of V. We claim that the space of type (i) is associated to $\alpha(Z) = Z^t = \overline{Z}$ and the space of type (ii) to $\alpha(Z) = J Z^t J^{-1} = J \overline{Z} J^{-1}$. Both are of the type $\alpha(Z) = A^{-1} Z^t A$ with an orthogonal matrix A. Recall that $\mathbb{P}\mathrm{U}(J, \mathbb{C})$ is the (identity component of the) conformal group $\mathrm{Co}(\mathrm{Herm}(n, \mathbb{C}))$. The map $\pm j\alpha_*$ is given by taking the adjoint in $\mathrm{U}(J)$ w.r.t. the form given by $\left(\begin{smallmatrix} A & 0 \\ 0 & \pm A \end{smallmatrix}\right)$. Thus

$$\mathrm{Co}(T)_o^{(\pm j\alpha)_*} = (\mathbb{P}\mathrm{U}(J, \mathbb{C}) \cap \mathbb{P}\mathrm{O}(\begin{pmatrix} A & 0 \\ 0 & \pm A \end{pmatrix}, \mathbb{C}))_o.$$

The subgroup fixed under $(-\mathrm{id})_*$ is

$$\{\begin{pmatrix} g & 0 \\ 0 & g^t \end{pmatrix} \mid g \in \mathrm{O}(A, \mathbb{C})\};$$

is is isomorphic to $\mathrm{O}(A, \mathbb{C})$.

If $A = \mathbf{1}_n$, then since $\mathrm{U}(J, \mathbb{C}) \cap \mathrm{O}(2n, \mathbb{C}) = \mathrm{O}^*(2n)$ we get the space $M^{(\alpha)} = \mathrm{O}^*(2n)/\mathrm{O}(n, \mathbb{C})$, and since $\mathrm{U}(J, \mathbb{C}) \cap \mathrm{O}(n, n; \mathbb{C}) \cong \mathrm{O}(n, n)$ (via the Cayley transform), we get the space $M^{(-\alpha)} = \mathrm{O}(n, n)/\mathrm{O}(n, \mathbb{C})$.

If $A = J$, then $\mathrm{U}(J_{2n}, \mathbb{C}) \cap \mathrm{U}(\left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right), \mathbb{C})$ is isomorphic to $\mathrm{Sp}(n, n)$ (via the Cayley transform), and we get $M^{(\alpha)} = \mathrm{Sp}(n, n)/\mathrm{Sp}(n, \mathbb{C})$. Using that $\mathrm{U}(J_{2n}, \mathbb{C}) \cap \mathrm{U}(\left(\begin{smallmatrix} J & 0 \\ 0 & -J \end{smallmatrix}\right), \mathbb{C})$ is isomorphic to $\mathrm{Sp}(2n, \mathbb{R})$ (again via Cayley transform), we get $M^{(-\alpha)} = \mathrm{Sp}(2n, \mathbb{R})/\mathrm{Sp}(n, \mathbb{C})$. ■

For the complex orthogonal groups, the more subtle modifications are again given by conjugate-linear involutions:

Theorem XI.5.7. *Up to isomorphism, the spaces which are conformally equivalent to the real symmetric space with twist* $\mathrm{O}(n, \mathbb{C})$ *are classified as follows:*

(i) $\mathrm{O}(n, \mathbb{C})$,

(ii) *if* $n = 2m$ *is even,* $\mathrm{Gl}(2m, \mathbb{C})/\mathrm{Sp}(m, \mathbb{C})$,

(iii) $\mathrm{O}^*(2n)/\mathrm{U}(p, q)$ *and its c-dual space* $\mathrm{O}(2p, 2q)/\mathrm{U}(p, q)$ *(*$p + q = n$*; for* $q = 0$ *this is a space of Lagrangians).*

Proof. (i) and (ii) are constructed as in the case of $\mathrm{O}(n)$. We claim that (iii) associated to the involution $\alpha(X) = I_{p,q} \overline{X} I_{p,q} = -A^{-1} \overline{X}^t A$ with $A = I_{p,q}$ of the JTS $V = \mathrm{Asym}(n, \mathbb{C})$. In fact, the same arguments as in the preceding proof show that

$$\mathrm{Co}(T)_o^{\pm j\alpha} = (\mathbb{P}\mathrm{O}(\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \mathbb{C}) \cap \mathbb{P}\mathrm{U}(\begin{pmatrix} A & 0 \\ 0 & \mp A \end{pmatrix}, \mathbb{C}),$$

and the subgroup fixed under $(-\mathrm{id})_*$ is isomorphic to $\mathrm{U}(A, \mathbb{C})$. Applying the Cayley-transform, we see that the spaces $M^{(\pm\alpha)} = \mathrm{Co}(V)^{(\pm j\alpha)_*}/\mathrm{U}(I_{p,q}, \mathbb{C})$ have the form given in the claim. ■

Similarly, we get for $\mathrm{Sp}(n, \mathbb{C})$ the classification of conformally equivalent spaces:

(i) $\mathrm{Sp}(n, \mathbb{C})$,

(ii) $\mathrm{Gl}(2n, \mathbb{C})/\mathrm{O}(2n, \mathbb{C})$,

(iii) $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(2n-p, p)$ and its c-dual space $\mathrm{Sp}(2n-p, p)/\mathrm{U}(2n-p, p)$ (for $p = 0$ this is a space of Lagrangians).

5.3. Spheres and hyperbolic spaces.

Theorem XI.5.8. *Up to isomorphism, the spaces which are conformally equivalent to the sphere S^n are classified as follows:*

(i) *the sphere S^n and its c-dual space, the real hyperbolic space $H^n(\mathbb{R})$,*

(ii) *the spaces $S^r \times H^{n-r}(\mathbb{R})$ with $0 < r < n$.*

Proof. We realize the sphere S^n and its conformal group $\mathbb{P}\,\mathrm{SO}(n+1,1)$ as in Section VIII.4.4. One verifies that then the spaces $S^r \times H^{n-r}(\mathbb{R})$ are associated to the involution of the JTS $V = \mathbb{R}^n$ given by $I_{r,n-r}$ and that these involutions represent all connected components of the structure variety of V (cf. [Ri69], [Ma73]). ■

Note that the spaces given in part (ii) are reducible as symmetric spaces, but not as symmetric spaces with twist (cf. Thm. V.3.4). Similarly, one gets the list of spaces which are conformally equivalent to the Euclidean Jordan algebra $\mathbb{R}^{n-1,1}$:

(i) the compact space $S^{n-1} \times S^1$ and its c-dual $H^{n-1}(\mathbb{R}) \times \mathbb{R}$,

(ii) the spaces $S^{n-r} \times \mathrm{SO}(r-1,2)/\,\mathrm{SO}(r-1,1)$ and their c-duals $H^{n-r}(\mathbb{R}) \times \mathrm{SO}(r,1)/\,\mathrm{SO}(r-1,1)$ $(r > 1)$. (For $r = n$ this space is irreducible as symmetric space, for $r < n$ it is not.)

The spaces equivalent to the conformal compactification $(S^p \times S^q)/(\mathbb{Z}/(2))$ of the JTS $\mathbb{R}^{p,q}$ are listed in Table XII.5.1. For $p = q$ we have the exceptional feature that the complex sphere $S^p_{\mathbb{C}}$ is conformally equivalent to $S^p \times S^p/(\mathbb{Z}/2)$. The conformal structure thus obtained is different from the usual one on $S^n_{\mathbb{C}}$ described in Section VIII.4.4. For the usual conformal structure on $S^n_{\mathbb{C}}$, we have the following classification of conformally equivalent spaces (cf. [Ma73]):

(i) $S^p_{\mathbb{C}} \times S^{n-p}_{\mathbb{C}}$ $(0 \leq p \leq n)$,

(ii) the pseudo-Hermitian spaces $\mathrm{SO}(p+1, n-p+1)/(\mathrm{SO}(p-1, n-p+1) \times \mathrm{SO}(2))$ $(0 < p < n+2)$ (for $p = n+1$ this space is compact Hermitian symmetric).

Notes for Chapter XI.

XI.1. Theorem XI.1.4 is due to Koecher ([Koe69a, Satz III.4.1]); our proof follows the one given there, but avoids the use of Koecher's theorem on "essential automorphisms" (loc.cit, Satz I.2.1). The statement of Cor. XI.1.8 holds for any JTS (not only the ones of the first kind); this is a result of K. Meyberg ([Mey70, Satz 4.2]). However, we have no geometric interpretation of this fact in case of a JTS of the second kind.

XI.2. Our treatment of the Cayley transform follows [Lo77, Ch.X]; as already mentioned, this approach can be generalized to any hyperbolic tripotent. The generalized tube domains (Def. XI.2.6) were introduced in [Be98a], where also Thm. XI.2.7 was proved.

XI.3. Causal Makarevič spaces have been introduced and classified in [Be96b]; further results on these spaces can be found in [Be98a] and [BeHi98]. From a purely Lie theoretic view point, these spaces and the corresponding causal compactifications have been investigated by F. Betten [Bet96]. The first time Makarevič spaces implicitly turned up in the study of causal symmetric spaces was in relation with the Cayley-type spaces, cf. [Fa95], [Cha98] and with compression semigroups, cf. [Kou95].

XI.4. In [Hw70], K.H. Helwig did not use conformal group techniques and therefore did not obtain the Cayley transformed realization of the spaces he introduced. Proposition XI.4.4 is essentially due to D. Ferus ([Fe80, Lemma 4]; see also the generalization by H. Naitoh [Nai83]). If M is compact, then Proposition XI.4.4 describes the "standard imbedding of the symmetric R-space M". To our knowledge, the relation between Helwig spaces and the immersed symmetric spaces of Ferus (Problem XI.4.5) has not yet been studied from a Jordan theoretic point of view.

XI.5. The presentations follows [Ma73], cf. [Be96b] and [Be98].

Chapter XII: Tables

The tables given in this chapter contain the classification and the most basic information about simple Jordan algebras, -triple systems and -pairs and about their associated symmetric spaces. We use the classification principle explained in Section IV.2. Most of the material given here has already been presented in form of examples (Sections I.6, IV.1, VIII.4, IX.2, X.1, X.2, XI.5); therefore we will not repeat calculations. The aim of this chapter is to provide reference material in a concise form; the results themselves are essentially all known from work of K.-H. Helwig, B.O. Makarevič, E. Neher and others (see the notes), but some of them are not easily accessible for non-specialists. We have not included the exceptional spaces and refer to the work of E. Neher ([Ne81], [Ne85]) for this.

Throughout this chapter we use notation introduced in Sections I.6, II.3 and VIII.4; recall in particular the matrices $J = J_n = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$, $F = F_n = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}$ and $I_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}$. The classical notation $\mathrm{O}(p,q) := \mathrm{O}(I_{p,q},\mathbb{R})$, $\mathrm{Sym}(n,\mathbb{R}) := \mathrm{Sym}(\mathbf{1}_n,\mathbb{R})$, $\mathrm{Sp}(n,\mathbb{R}) := \mathrm{O}(J_n,\mathbb{R})$ etc. will be used when there is no risk of ambiguity. Elements of $M(p,q;\mathbb{H})$ are considered as quaternionic $p\times q$-matrices and not as certain elements of $M(2p,2q;\mathbb{C})$. Recall finally that Gr denotes Grassmannians and Lag varieties of Lagrangians.

1. Simple Jordan algebras

Table XII.1.1. Complex simple Jordan algebras and associated open symmetric orbits Ω.

	underlying vector space $V_{\mathbb{C}}$	open symmetric orbit Ω
1.	$M(n,\mathbb{C})$	$\mathrm{Gl}(n,\mathbb{C})$
2.	$\mathrm{Sym}(n,\mathbb{C})$	$\mathrm{Gl}(n,\mathbb{C})/\mathrm{O}(n,\mathbb{C})$
3.	$\mathrm{Sym}(J,\mathbb{C})$	$\mathrm{Gl}(2n,\mathbb{C})/\mathrm{Sp}(n,\mathbb{C})$
4.	$\mathbb{C}^n$	$(\mathrm{SO}(n,\mathbb{C})\times\mathbb{C}^*)/\mathrm{SO}(n-1,\mathbb{C})$
5.	$\mathrm{Herm}(3,\mathbb{O}_{\mathbb{C}})$	$(E_6^{\mathbb{C}}\times\mathbb{C}^*)/F_4^{\mathbb{C}}$

In cases 1,2,3 and 5 the Jordan product is given by $XY = \frac{1}{2}(X\circ Y + Y\circ X)$. In case 4 it is given by $xy = b(x,e)y + b(y,e)x - b(x,y)e$, where $b(u,v) = u^t v$ and e is the last vector of the canonical basis (cf. Lemma II.3.3).

Table XII.1.2. Real simple Jordan algebras. The first class of these is given by the algebras from Table XII.1.1, considered as real ones. The second class is given by real forms V of complex algebras $V_{\mathbb{C}}$, i.e. by subalgebras fixed under a complex conjugation τ:

	underlying vector space V	conjugation τ of $V_{\mathbb{C}}$	open symmetric orbit Ω
1.1.	$\mathrm{Herm}(I_{p,q},\mathbb{C})$	$\tau(Z) = I_{p,q}\overline{Z}^t I_{p,q}$	$\mathrm{Gl}(n,\mathbb{C})/\mathrm{U}(p,q)$
1.2.	$M(n,\mathbb{R})$	$\tau(Z) = \overline{Z}$	$\mathrm{Gl}(n,\mathbb{R})^+$

1.3.	$M(m,\mathbb{H})$	$\tau(Z)=J\overline{Z}J^{-1}$	$\mathrm{Gl}(m,\mathbb{H})$
2.1.	$\mathrm{Sym}(I_{p,q},\mathbb{R})$	$\tau(Z)=I_{p,q}\overline{Z}I_{p,q}$	$\mathrm{Gl}(n,\mathbb{R})/\mathrm{O}(p,q)$
2.2.	$\mathrm{Herm}(m,\varphi,\mathbb{H})$	$\tau(Z)=J\overline{Z}J^{-1}$	$\mathrm{Gl}(m,\mathbb{H})/\,\mathrm{SO}^*(2m)$
3.1.	$\mathrm{Herm}(I_{p,q},\mathbb{H})$	$\tau(Z)=I_{p,q}\overline{Z}^t I_{p,q}$	$\mathrm{Gl}(n,\mathbb{H})/\,\mathrm{Sp}(p,q)$
3.2.	$\mathrm{Sym}(J,\mathbb{R})$	$\tau(Z)=\overline{Z}$	$\mathrm{Gl}(2n,\mathbb{R})/\,\mathrm{Sp}(n,\mathbb{R})$
4.1.	$\mathbb{R}^{p,q}$	$\tau(z)=I_{p,q}\overline{z}$	$(\mathrm{SO}(p,q)\times\mathbb{R}^+)/\,\mathrm{SO}(p-1,q)$
5.1.	$\mathrm{Herm}(3,\mathbb{O})$	$\tau(Z)=\overline{Z}$	$(E_{6(-26)}\times\mathbb{R}^+)/F_4$
5.2.	$\mathrm{Herm}(3,\mathbb{O}_s)$	$\tau(Z)=\overline{Z}^t$	$(E_{6(I)}\times\mathbb{R}^+)/F_{4(I)}$

(In 1.3 and 2.2, $n=2m$.) The Jordan algebra structure on V (line i.j.) is the one inherited from $V_{\mathbb{C}}$ (Table XII.1.1, line i.). In particular, in case 4.1 it is given by the formula for case 4., with b replaced by $b(x,y)=x^tI_{p,q}y$. For notation concerning case 5 cf. [Lo77, Section 4.14] or [FK94, Ch.V].

Table XII.1.3. Structure- and automorphism groups (non-exceptional cases). The identity components of $\mathrm{Str}(V)$ and $\mathrm{Aut}(V)$ can be read of the preceding table since $\Omega=(\mathrm{Str}(V)/\,\mathrm{Aut}(V))_e$. In the following table we describe also the other connected components of $\mathrm{Str}(V)$ and $\mathrm{Aut}(V)$. For simplicity of notation we denote by $[\mathrm{Gl}(n,\mathbb{C})\times\mathrm{Gl}(n,\mathbb{C})]$ the effective group of the natural action of $\mathrm{Gl}(n,\mathbb{C})\times\mathrm{Gl}(n,\mathbb{C})$ on $M(n,\mathbb{C})$, and similarly for other natural actions on matrix spaces (cf. Section II.3). In the following table, we give a set of generators of the relevant groups. A linear map α is denoted by $(X\mapsto\alpha(X))$. The identity component is usually given by the term in brackets.

Case	$\mathrm{Aut}_{\mathbb{R}}(V_{\mathbb{C}})$	$\mathrm{Str}_{\mathbb{R}}(V_{\mathbb{C}})$
1.	$[\mathrm{Gl}(n,\mathbb{C})],(Z\mapsto Z^t),(Z\mapsto\overline{Z})$	$[\mathrm{Gl}(n,\mathbb{C})\times\mathrm{Gl}(n,\mathbb{C})],(Z\mapsto Z^t),(Z\mapsto\overline{Z})$
2.	$[\mathrm{O}(n,\mathbb{C})],(Z\mapsto\overline{Z})$	$\pm[\mathrm{Gl}(n,\mathbb{C})],(Z\mapsto\overline{Z})$
3.	$[\mathrm{Sp}(n,\mathbb{C})],(Z\mapsto\overline{Z})$	$\pm[\mathrm{Gl}(2n,\mathbb{C})],(Z\mapsto\overline{Z})$
4.	$\mathrm{O}(n-1,\mathbb{C}),(z\mapsto\overline{z})$	$\mathrm{O}(n,\mathbb{C})\times\mathbb{C}^*,(z\mapsto\overline{z})$

Case	$\mathrm{Aut}(V)$	$\mathrm{Str}(V)$
1.1.	$[\mathrm{U}(p,q)],(Z\mapsto Z^t)$	$\pm[\mathrm{Gl}(n,\mathbb{C})],(Z\mapsto Z^t)$
1.2.	$[\mathrm{Gl}(n,\mathbb{R})],(X\mapsto X^t)$	$[\mathrm{Gl}(n,\mathbb{R})\times\mathrm{Gl}(n,\mathbb{R})],(X\mapsto X^t)$
1.3.	$[\mathrm{Gl}(m,\mathbb{H})],(Z\mapsto Z^t)$	$[\mathrm{Gl}(m,\mathbb{H})\times\mathrm{Gl}(m,\mathbb{H})],(Z\mapsto Z^t)$
2.1.	$[\mathrm{O}(p,q)]$	$\pm[\mathrm{Gl}(n,\mathbb{R})]$
2.2.	$[\mathrm{O}^*(2m)]$	$\pm[\mathrm{Gl}(m,\mathbb{H})]$
3.1.	$[\mathrm{Sp}(p,q)]$	$\pm[\mathrm{Gl}(m,\mathbb{H})]$
3.2.	$[\mathrm{Sp}(n,\mathbb{R})]$	$\pm[\mathrm{Gl}(2n,\mathbb{R})]$
4.1.	$\mathrm{O}(p-1,q)$	$\mathrm{O}(p,q)\times\mathbb{R}^*$

Remark XII.1.4. The Jordan algebras 1.1, 2.1, 3.1 (whith $p=n,q=0$) 4.1 (with $p=1,q=n-1$) and 5.1 are *Euclidean* and the associated open orbits Ω are precisely the *irreducible symmetric cones.*

Table XII.1.5. Low-dimensional isomorphisms.
Complex Jordan algebras:

$$\mathbb{C}\cong M(1,\mathbb{C})\cong\mathrm{Sym}(1,\mathbb{C})\cong\mathrm{Sym}(J_1,\mathbb{C}) \tag{1.1}$$

$$\mathbb{C}^2\cong\mathbb{C}\times\mathbb{C}\quad(\text{not simple!}) \tag{1.2}$$

$$\mathbb{C}^3\cong\mathrm{Sym}(2,\mathbb{C}) \tag{1.3}$$

$$\mathbb{C}^4\cong M(2,\mathbb{C}) \tag{1.4}$$

$$\mathbb{C}^6\cong\mathrm{Sym}(J_2,\mathbb{C}) \tag{1.5}$$

Real Jordan algebras:

$$\mathbb{R} \cong M(1,\mathbb{R}) \cong \mathrm{Sym}(1,\mathbb{R}) \cong \mathrm{Herm}(1,\mathbb{C}) \cong \mathrm{Herm}(1,\mathbb{H}) \cong \mathrm{Sym}(J_1,\mathbb{R}) \tag{1.6}$$

$$\mathbb{R}^2 = \mathbb{R}^{2,0} \cong \mathbb{C} \tag{1.7}$$

$$\mathbb{R}^{1,1} \cong \mathbb{R} \times \mathbb{R} \quad \text{(not simple!)} \tag{1.8}$$

$$\mathbb{R}^{1,2} \cong \mathrm{Sym}(2,\mathbb{R}); \quad \mathbb{R}^{1,2} \cong \mathrm{Sym}(I_{1,1},\mathbb{R}), \tag{1.9}$$

$$\mathbb{R}^3 \cong \mathrm{Herm}(1,\varphi,\mathbb{H}) \tag{1.10}$$

$$\mathbb{R}^{1,3} \cong \mathrm{Herm}(2,\mathbb{C}); \quad \mathbb{R}^{3,1} \cong \mathrm{Herm}(I_{1,1},\mathbb{C}), \tag{1.11}$$

$$\mathbb{R}^4 = \mathbb{R}^{4,0} \cong \mathbb{H} \tag{1.12}$$

$$\mathbb{R}^{2,2} \cong M(2,\mathbb{R}) \tag{1.13}$$

$$\mathbb{R}^{1,5} \cong \mathrm{Herm}(2,\mathbb{H}); \quad \mathbb{R}^{5,1} \cong \mathrm{Herm}(I_{1,1},\mathbb{H}) \tag{1.14}$$

$$\mathbb{R}^{3,3} \cong \mathrm{Sym}(J_2,\mathbb{R}) \tag{1.15}$$

The isomorphisms (1.1) and (1.6) are trivially verified since there is just one one-dimensional complex, resp. real Jordan algebra. The isomorphisms (1.8), (1.9), (1.11) and (1.14) of Euclidean Jordan algebras are well-known (cf. [FK94, p.98]) and are easily established using some elementary structure theory of Euclidean Jordan algebras (Peirce-decomposition). Complexifying, they yield the isomorphisms (1.2) – (1.5). The other isomorphisms are realized by identifying the corresponding other real forms. However, the real forms $\mathbb{R}^{2,4}$ and $\mathbb{R}^{4,2}$ of $\mathbb{C}^6$ appear as "exceptional" when related to the matrix algebra $\mathrm{Sym}(J_2,\mathbb{C})$.

2. Simple Jordan triple systems

Our classification simple real Jordan triple systems follows the classification principle outlined in Section IV.2. The information on structure groups needed for this classification can be found in Tables XII.1.3 and XII.2.3.

Table XII.2.1. Simple complex Jordan pairs. The simple complex Jordan pairs are all of the form $T_{ph\mathbb{C}}$ where T is one of the following simple complex JTS:

	underlying vector space $V_{\mathbb{C}}$	Jordan triple product on $V_{\mathbb{C}}$
1.	$M(n,\mathbb{C})$	$T(X,Y,Z) = XYZ + ZYX$
2.	$\mathrm{Sym}(n,\mathbb{C})$	$T(X,Y,Z) = XYZ + ZYX$
3.	$\mathrm{Sym}(J,\mathbb{C})$	$T(X,Y,Z) = XYZ + ZYX$
4.	$\mathbb{C}^n$	$T(x,y,z) = x(yz) - y(xz) + (xy)z$
5.	$\mathrm{Herm}(3,\mathbb{O}_{\mathbb{C}})$	$T(X,Y,Z) = XYZ + ZYX$
6.	$M(p,q;\mathbb{C})$, $p > q$	$T(X,Y,Z) = XY^tZ + ZY^tX$
7.	$\mathrm{Asym}(2m+1,\mathbb{C})$	$T(X,Y,Z) = XYZ + ZYX$
8.	$M(1,2;\mathbb{O}_{\mathbb{C}})$	$T(X,Y,Z) = X\widetilde{Y}^tZ + Z\widetilde{Y}^tX$

In the last line $Y \to \widetilde{Y}$ denotes the canonical ($\mathbb{C}$-linear) involution of $\mathbb{O}_{\mathbb{C}}$ (cf. [Lo77, 4.14]). In cases 1. – 5. the Jordan triple product is the one associated to the Jordan algebra structure from Table XII.1.1. In cases 6. – 8. the Jordan triple product is induced from imbeddings as -1-eigenspaces into Jordan algebras: $\mathrm{Asym}(2m+1,\mathbb{C}) \subset M(2m+1,\mathbb{C})$ is the -1-eigenspace of $Z \mapsto Z^t$; $M(p,q;\mathbb{C}) \supset \mathrm{Sym}(p+q,\mathbb{C})$ is the -1-eigenspace of $Z \mapsto I_{p,q}ZI_{p,q}$, and similarly for $M(1,2;\mathbb{O}_{\mathbb{C}}) \subset \mathrm{Herm}(3,\mathbb{O}_{\mathbb{C}})$ (cf. [Lo77, 4.14]).

Table XII.2.2. Simple real Jordan pairs. The first class of these is given by complex Jordan pairs considered as real ones (see above). The second class is given by real forms of complex pairs. They are described by real forms V of the complex JTS given above.

real form $V = V_{\mathbb{C}}^{\tau}$		conjugation τ of $V_{\mathbb{C}}$
1.1.	$\mathrm{Herm}(n,\mathbb{C})$	$\tau(Z) = \overline{Z}^t$
1.2.	$M(n,\mathbb{R})$	$\tau(Z) = \overline{Z}$
1.3.	$n = 2m$: $M(m,\mathbb{H})$	$\tau(Z) = J\overline{Z}J^{-1}$
2.1.	$\mathrm{Sym}(n,R)$	$\tau(Z) = \overline{Z}$
2.2.	$n = 2m$: $\mathrm{Herm}(m,\varphi,\mathbb{H})$	$\tau(Z) = J\overline{Z}J^{-1}$
3.1.	$\mathrm{Herm}(m,\mathbb{H})$	$\tau(Z) = \overline{Z}^t = J\overline{Z}J^{-1}$
3.2.	$\mathrm{Sym}(J,\mathbb{R})$ ($\cong \mathrm{Asym}(2m,\mathbb{R})$)	$\tau(Z) = \overline{Z}$
4.1.	$\mathbb{R}^{p,q}$, $p+q=n, p \geq 1$	$\tau(z) = I_{p,q}\overline{z}$
5.1.	$\mathrm{Herm}(3,\mathbb{O})$	$\tau(Z) = \overline{Z}$
5.2.	$\mathrm{Herm}(3,\mathbb{O}_s)$	$\tau(Z) = \widetilde{Z}$
6.1.	$M(p,q;\mathbb{R})$, $p > q$	$\tau(Z) = \overline{Z}$
6.2.	$p = 2r, q = 2s$: $M(r,s;\mathbb{H})$	$\tau(Z) = J_r\overline{Z}J_s^{-1}$
7.1.	$\mathrm{Asym}(2m+1,\mathbb{R})$	$\tau(Z) = \overline{Z}$
8.1.	$M(1,2;\mathbb{O})$	$\tau(Z) = \overline{Z}$
8.2.	$M(1,2;\mathbb{O}_s)$	$\tau(Z) = \widetilde{Z}$

In all cases, the JTS-structure is the one induced from the corresponding complex space in Table XII.2.1.

Table XII.2.3. Structure groups of simple Jordan pairs. The structure groups of the pairs 1. – 5. and 1.1. – 5.2. are precisely the ones of the corresponding Jordan algebras (Table XII.1.3). For the other classical pairs, they are as follows:

Case	$\mathrm{Str}(T)$
6.	$[\mathrm{Gl}(p,\mathbb{C}) \times \mathrm{Gl}(q,\mathbb{C})], (Z \mapsto \overline{Z})$
7.	$\pm[\mathrm{Gl}(2n+1,\mathbb{C})], (Z \mapsto \overline{Z})$
6.1.	$[\mathrm{Gl}(p,\mathbb{R}) \times \mathrm{Gl}(q,\mathbb{R})]$
6.2	$[\mathrm{Gl}(p,\mathbb{H}) \times \mathrm{Gl}(q,\mathbb{H})]$
7.1	$\pm[\mathrm{Gl}(2m+1,\mathbb{R})]$

Table XII.2.4. Simple Hermitian Jordan triple systems. The first class of these is given by complex Jordan pairs. The second class is given by the JTS of the form $T^{(\alpha)}$ where T is a complex JTS from Table XII.2.1 (line i.) and α one of the following conjugate-linear elements of the structure variety (line i.X.):

1. A. $\alpha(X) = \overline{X}$
1. B. $\alpha(X) = I_{p,q}\overline{X}^t I_{u,v}$ $(p+q = u+v = n)$
2. A. $\alpha(X) = I_{p,q}\overline{X}I_{p,q}$
3. A. $\alpha(X) = JI_{p,q}\overline{X}^t JI_{p,q}$ $(p+q = 2m)$
4. A. $\alpha(z) = I_{p,q}\overline{z}$
4. B. (n even) $\alpha(z) = J\overline{z}$
6. A. $\alpha(X) = I_{k,l}\overline{X}I_{i,j}$ $(k+l = p, i+j = q)$
7. A. $\alpha(X) = I_{p,q}\overline{X}I_{p,q}$ $(p+q = 2m+1)$

Table XII.2.5. Simple complex Jordan triple systems. The first class is given by complex Jordan pairs. The second class is given by JTS of the form $T^{(\alpha)}$ where T is from Table XII.2.1 and α is one of the following $\mathbb{C}$-linear elements of the structure variety:

1. a. $\alpha = \mathrm{id}$
1. b. $\alpha(X) = X^t$
1. c. $n = 2m$: $\alpha(X) = JX^tJ^{-1}$
2. a. $\alpha = \mathrm{id}$
2. b. $n = 2m$: $\alpha(X) = JXJ^{-1}$ (i.e., $T^{(\alpha)} \cong \mathrm{Asym}(J_m, \mathbb{C})$)
3. a. $\alpha = \mathrm{id}$
3. b. $\alpha(X) = JXJ^{-1}$ (i.e., $T^{(\alpha)} \cong \mathrm{Asym}(2n, \mathbb{C})$)
4. a. $\alpha = I_{q+1,p-1}$, $p > 0$
6. a. $\alpha = \mathrm{id}$
6. b. ($p = 2r$, $q = 2s$): $\alpha(X) = J_rXJ_s^{-1}$
7. a. $\alpha = \mathrm{id}$

Table XII.2.6. Simple real Jordan triple systems. The first four classes are given by Tables XII.2.1, XII.2.2, XII.2.4 and XII.2.5. The last class is given by JTS of the form $T^{(\alpha)}$ where T is a JTS from Table XII.2.2 (line i.j.) and α one of the following elements of the structure variety (line i.j.x.). In cases 1.3, 2.2, 3.1 and 6.2, X is a quaternionic matrix and j denotes also the matrix $j\mathbf{1}_n$ with the usual element $j \in \mathbb{H}$.

1.1. a. $\alpha(X) = I_{p,q}XI_{p,q}$
1.1. b. $\alpha(X) = X^t$
1.1. c. $n = 2m$: $\alpha(X) = JX^tJ^{-1}$
1.2. a. $\alpha = \mathrm{id}$
1.2. b. $n = 2m$: $\alpha(X) = JX^tJ^{-1}$
1.2. c. $\alpha(X) = I_{p,q}X^tI_{u,v}$
1.3. a. $\alpha = \mathrm{id}$
1.3. b. $\alpha(X) = X^t$
1.3. c. $\alpha(X) = I_{p,q}\overline{X}^tI_{u,v}$
2.1. a. $\alpha(X) = I_{p,q}XI_{p,q}$
2.1. b. $n = 2m$: $\alpha = JXJ^{-1}$
2.2. a. $\alpha = \mathrm{id}$
2.2. b. $\alpha(X) = I_{r,s}jXj^{-1}I_{r,s}$ ($r + s = m$) (i.e., $T^{(\alpha)} \cong \mathrm{Aherm}(I_{r,s}, \mathbb{H})$)
3.1. a. $\alpha(X) = I_{p,q}XI_{p,q}$
3.1. b. $\alpha(X) = jXj^{-1}$ (i.e. $T^{(\alpha)} \cong \mathrm{Aherm}(m, \varphi, \mathbb{H})$)
3.2. a. $\alpha = \mathrm{id}$
3.2. b. $\alpha(X) = JI_{p,q}XJI_{p,q}$ (i.e. $T^{(\alpha)} \cong \mathrm{Asym}(I_{p,q}, \mathbb{R})$)
4.1. a. $\alpha(x + y) = I_{r,p-r}x + I_{s,q-s}y$ ($x \in \mathbb{R}^p, y \in \mathbb{R}^q$)
4.1. b. n even, $p = q$: $\alpha = F$
6.1. a. $\alpha(X) = I_{k,l}XI_{i,j}$
6.1. b. $p = 2r$, $q = 2s$: $\alpha(X) = J_rXJ_s^{-1}$
6.2. a. $\alpha(X) = I_{k,l}XI_{i,j}$
6.2. b. $\alpha(X) = jXj^{-1}$
7.1. a. $\alpha(X) = I_{p,q}XI_{p,q}$

3. Conformal groups and conformal completions

Table XII.3.1. Conformal groups and conformal completions of simple real Jordan triple systems. The conformal group and the conformal completion depend only on the underlying Jordan pair. They are determined as described in Section VIII.4. The normal forms corresponding to the Jordan pairs from Tables XII.2.1 (line i.), resp. XII.2.2. (line i.j) are given in the following table. The full conformal group $\mathrm{Co}(T)$ is generated by the group given in the table (which is in most cases the identity component) and the elements of $\mathrm{Str}(T)$ which are given in Tables XII.1.3 and XII.2.3. Note that if T belongs to a *Euclidean Jordan algebra* (cases 1.1, 2.1, 3.1, 4.1 and 5.1) the conformal group can also be interpreted as a *causal group* (cf. Sections VIII.2.3, X.5.3).

complex Jordan pair $V_{\mathbb{C}}$	$(V_{\mathbb{C}})^c$	$\mathrm{Co}(T)$
1. $M(n,\mathbb{C})$	$\mathrm{Gr}_{n,2n}(\mathbb{C})$	$\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{C})$
2. $\mathrm{Sym}(n,\mathbb{C})$	$\mathrm{Lag}(J,\mathbb{C})$	$\mathbb{P}\,\mathrm{Sp}(n,\mathbb{C})$
3. $\mathrm{Sym}(J,\mathbb{C}) \cong \mathrm{Asym}(2n,\mathbb{C})$	$\cong \mathrm{Lag}(I_{2n,2n},\mathbb{C})$	$\cong \mathbb{P}\,\mathrm{SO}(4n,\mathbb{C})$
4. $\mathbb{C}^n$	$(S^n)_{h\mathbb{C}}$	$\mathrm{SO}(n+2,\mathbb{C})$
5. $\mathrm{Herm}(3,\mathbb{O}_{\mathbb{C}})$		E_7
6. $M(p,q;\mathbb{C})$	$\mathrm{Gr}_{p,p+q}(\mathbb{C})$	$\mathbb{P}\,\mathrm{Gl}(p+q,\mathbb{C})$
7. $\mathrm{Asym}(2n+1,\mathbb{C})$	$\cong \mathrm{Lag}(I_{2n+1,2n+1},\mathbb{C})$	$\cong \mathbb{P}(\mathrm{SO}(4n+2,\mathbb{C}))$
8. $\mathrm{Herm}(1,2;\mathbb{O}_{\mathbb{C}})$		E_6

real Jordan pair V	V^c	$\mathrm{Co}(T)$
1.1. $\mathrm{Herm}(n,\mathbb{C}) = i\,\mathrm{Aherm}(n,\mathbb{C})$	$\cong \mathrm{Lag}(I_{n,n},\tau,\mathbb{C})$	$\cong \mathbb{P}(\mathrm{U}(n,n))$
1.2. $M(n,\mathbb{R})$	$\mathrm{Gr}_{n,2n}(\mathbb{R})$	$\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{R})$
1.3. $M(m,\mathbb{H})$	$\mathrm{Gr}_{m,2m}(\mathbb{H})$	$\mathbb{P}\,\mathrm{Gl}(2m,\mathbb{H})$
2.1. $\mathrm{Sym}(n,\mathbb{R})$	$\mathrm{Lag}(J,\mathbb{R})$	$\mathbb{P}\,\mathrm{Sp}(n,\mathbb{R})$
2.2. $\mathrm{Herm}(m,\varphi,\mathbb{H}) \cong \mathrm{Aherm}(m,\mathbb{H})$	$\cong \mathrm{Lag}(I_{m,m},\tau,\mathbb{H})$	$\cong \mathbb{P}\,\mathrm{Sp}(m,m)$
3.1. $\mathrm{Herm}(m,\mathbb{H}) \cong \mathrm{Aherm}(m,\varphi,\mathbb{H})$	$\cong \mathrm{Lag}(I_{m,m},\varphi,\mathbb{H})_0$	$\cong \mathbb{P}\,\mathrm{SO}^*(4m)$
3.2. $\mathrm{Sym}(J,\mathbb{R}) \cong \mathrm{Asym}(2n,\mathbb{R})$	$\cong \mathrm{Lag}(I_{2n,2n},\mathbb{R})$	$\cong \mathbb{P}\,\mathrm{SO}(2n,2n)$
4.1. $\mathbb{R}^{p,q}$	$S^p \times S^q$	$\mathrm{SO}(p+1,q+1)$
5.1. $\mathrm{Herm}(3,\mathbb{O})$		$E_{7(-25)}$
5.2. $\mathrm{Herm}(3,\mathbb{O}_s)$		$E_{7(7)}$
6.1. $M(p,q;\mathbb{R})$	$\mathrm{Gr}_{p,p+q}(\mathbb{R})$	$\mathbb{P}\,\mathrm{Gl}(p+q,\mathbb{R})$
6.2. $M(p,q;\mathbb{H})$	$\mathrm{Gr}_{p,p+q}(\mathbb{H})$	$\mathbb{P}\,\mathrm{Gl}(p+q,\mathbb{H})$
7.1. $\mathrm{Asym}(2n+1,\mathbb{R})$	$\cong \mathrm{Lag}(I_{2n+1,2n+1},\mathbb{R})$	$\cong \mathbb{P}\,\mathrm{SO}(2n+1,2n+1)$
8.1. $M(1,2;\mathbb{O})$		$E_{6(6)}$
8.2. $M(1,2;\mathbb{O}_s)$		$E_{6(-26)}$

Here τ is the standard-conjugation of $\mathbb{H}$, resp. of $\mathbb{C}$, and φ is τ composed with conjugation by the quaternion j. In the exceptional cases a geometric description of the conformal compactification is missing.

Table XII.3.2. Low dimensional isomorphisms of conformal groups. The isomorphisms of Jordan algebras from Table XII.1.4 induce the following isomorphisms of conformal groups (the isomorphism (3.x) comes from Eqn. (1.x)). (In fact, we only need that the corresponding Jordan pairs are isomorphic. However, all relevant isomorphisms of Jordan pairs are in fact induced by isomorphisms of Jordan algebras.)

$$\mathrm{SO}(3,\mathbb{C}) \cong \mathbb{P}\,\mathrm{Sl}(2,\mathbb{C}) \cong \mathbb{P}\,\mathrm{Sp}(1,\mathbb{C}) \tag{3.1}$$

$$\mathrm{SO}(4,\mathbb{C}) \cong \mathrm{SO}(3,\mathbb{C}) \times \mathrm{SO}(3,\mathbb{C}) \tag{3.2}$$

$$\mathrm{SO}(5,\mathbb{C}) \cong \mathbb{P}\,\mathrm{Sp}(2,\mathbb{C}) \tag{3.3}$$

$$\mathrm{SO}(6,\mathbb{C}) \cong \mathbb{P}\,\mathrm{Sl}(4,\mathbb{C}) \tag{3.4}$$

$$\mathrm{SO}(2,1) \cong \mathbb{P}\,\mathrm{Sl}(2,\mathbb{R}) \cong \mathbb{P}\,\mathrm{Sp}(1,\mathbb{R}) \cong \mathbb{P}\,\mathrm{SU}(1,1) \tag{3.6}$$

$$\mathrm{SO}(3,1) \cong \mathbb{P}\,\mathrm{Sl}(2,\mathbb{C}) \tag{3.7}$$

$$\mathrm{SO}(2,2) \cong \mathrm{SO}(2,1) \times \mathrm{SO}(2,1) \tag{3.8}$$

$$\mathrm{SO}(2,3) \cong \mathbb{P}\,\mathrm{Sp}(2,\mathbb{R}) \tag{3.9}$$

$$\mathrm{SO}(4,1) \cong \mathbb{P}\,\mathrm{Sp}(1,1) \tag{3.10}$$

$$\mathrm{SO}(2,4) \cong \mathbb{P}\mathrm{U}(2,2) \tag{3.11}$$

$$\mathrm{SO}(5,1) \cong \mathbb{P}\,\mathrm{Sl}(2,\mathbb{H}) \tag{3.12}$$

$$\mathrm{SO}(3,3) \cong \mathbb{P}\,\mathrm{Sl}(4,\mathbb{R}) \tag{3.13}$$

$$\mathrm{SO}(2,6) \cong \mathbb{P}\,\mathrm{SO}^*(8) \tag{3.14}$$

One has to be careful when relating the isomorphisms (1.5) $\mathbb{C} \cong \mathrm{Sym}(J_1,\mathbb{C})$ and (1.6) $\mathbb{R} \cong \mathrm{Sym}(J_1,\mathbb{R}) \cong \mathrm{Herm}(1,\mathbb{H})$ to conformal groups because in this low-dimensional case the groups $\mathbb{P}\mathrm{O}(4,\mathbb{C})$, resp. $\mathbb{P}\mathrm{O}(2,2)$ and $\mathbb{P}\mathrm{O}^*(4)$ do not act faithfully on the latter spaces. In fact, both groups are direct products (cf. the isomorphism (3.8); and according to [Hel78, p.520] we have $\mathfrak{o}^*(4) \cong \mathfrak{su}(2)\times\mathfrak{sl}(2,\mathbb{R})$), and only one of the factors acts faithfully. One can also obtain interesting group isomorphisms by comparing structure groups of isomorphic Jordan algebras. For instance, the isomorphism $\mathbb{H} \cong \mathbb{R}^4$ yields the well-known homomorphism $\mathrm{SU}(2)\times\mathrm{SU}(2) \to \mathrm{SO}(4)$. It is remarkable that in fact all isomorphisms of low-dimensional classical Lie groups can in this way be naturally explained by Jordan theory (cf. [Hel78, p. 519/20]).

Table XII.3.3. Symmetric R-spaces (non-exceptional).
The first class is given by compact Hermitian symmetric spaces:

complex Jordan pair $V_{\mathbb{C}}$	$(V_{\mathbb{C}})^c$ as symmetric R-space
1. $M(n,\mathbb{C})$	$\mathrm{U}(2n)/(\mathrm{U}(n)\times\mathrm{U}(n))$
2. $\mathrm{Sym}(n,\mathbb{C})$	$\mathrm{Sp}(n)/\mathrm{U}(n)$
3. $\mathrm{Sym}(J,\mathbb{C}) \cong \mathrm{Asym}(2n,\mathbb{C})$	$\mathrm{SO}(4n)/\mathrm{U}(2n)$
4. $\mathbb{C}^n$	$\mathrm{SO}(n+2)/(\mathrm{SO}(n)\times\mathrm{SO}(2))$
6. $M(p,q;\mathbb{C})$	$\mathrm{U}(p+q)/(\mathrm{U}(p)\times\mathrm{U}(q))$
7. $\mathrm{Asym}(2n+1,\mathbb{C})$	$\mathrm{SO}(4n+2)/\mathrm{U}(2n+1)$

The second class is given by real forms of compact Hermitian symmetric spaces:

real Jordan pair V	V^c as symmetric R-space
1.1. $\mathrm{Herm}(n,\mathbb{C}) = i\,\mathrm{Aherm}(n,\mathbb{C})$	$\mathrm{U}(n)$
1.2. $M(n,\mathbb{R})$	$\mathrm{O}(2n)/(\mathrm{O}(n)\times\mathrm{O}(n))$
1.3. $M(m,\mathbb{H})$	$\mathrm{Sp}(2m)/(\mathrm{Sp}(m)\times\mathrm{Sp}(m))$
2.1. $\mathrm{Sym}(n,\mathbb{R})$	$\mathrm{U}(n)/\mathrm{O}(n)$
2.2. $\mathrm{Herm}(m,\varphi,\mathbb{H}) \cong \mathrm{Aherm}(m,\mathbb{H})$	$\mathrm{Sp}(n)$
3.1. $\mathrm{Herm}(m,\mathbb{H}) \cong \mathrm{Aherm}(m,\varphi,\mathbb{H})$	$\mathrm{U}(2n)/\mathrm{Sp}(n)$
3.2. $\mathrm{Sym}(J,\mathbb{R}) \cong \mathrm{Asym}(2n,\mathbb{R})$	$\mathrm{SO}(2n)$
4.1. $\mathbb{R}^p\times\mathbb{R}^q$	$S^p\times S^q$
6.1. $M(p,q;\mathbb{R})$	$\mathrm{O}(p+q)/(\mathrm{O}(p)\times\mathrm{O}(q))$
6.2. $M(p,q;\mathbb{H})$	$\mathrm{Sp}(p+q)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$
7.1. $\mathrm{Asym}(2n+1,\mathbb{R})$	$\mathrm{SO}(2n+1)$

The symmetric R-spaces also appear in the tables of the next section, where the corresponding Cartan-involution of the conformal group is specified.

4. Classification of simple symmetric spaces with twist

In this section we give a list of the symmetric spaces with twist $M_0^{(\alpha)}$ (as homogeneous spaces $G^{(\alpha)}/H$) belonging to the triple systems $T^{(\alpha)}$ classified in the preceding section, and of their complexification diagrams. In principle, this means that we calculate the standard imbedding of the LTS $R_{T^{(\alpha)}}(X,Y) = T^{(\alpha)}(Y,X) - T^{(\alpha)}(X,Y)$. For most of the cases, the calculations are explicitly carried out in Section XI.5, based on the theory of conformal groups of Jordan algebras. As explained in Section IV.2.4, most calculations can also be done without using conformal groups or Jordan algebras. The reader who is interested in special cases may choose the method he prefers. The approach using Jordan algebras and Jordan inverse is in most cases clearer and less calculatory. When using this approach, it is often useful to note the isomorphism $\mathrm{Co}(T)^{(j\alpha)_*} \cong \mathrm{Co}(V^-)$ if T belongs to a Jordan algebra V and α is a "generic" Jordan algebra automorphism (cf. Prop. XI.2.8). By "generic" we mean essentially that α is not the identity and not a Peirce-involution (cf. [BeHi98, Section 1.8]). If V is a Jordan algebra and $\alpha = \mathrm{id}_V$, the complexification diagram from Prop. X.3.7 reduces to

$$\begin{array}{ccccc} & \nearrow & \mathrm{Str}(V_{\mathbb{C}})/\,\mathrm{Aut}(V_{\mathbb{C}}) & \searrow & \\ \mathrm{Str}(V)/\,\mathrm{Aut}(V) & \to & \mathrm{Co}(V)/(\mathrm{Str}(V_{\mathbb{C}}))^{j_*\tau_*} & \to & \mathrm{Co}(V_{\mathbb{C}})/\,\mathrm{Str}(V_{\mathbb{C}}). \\ & \searrow & \mathrm{Co}(V)/\,\mathrm{Str}(V) & \nearrow & \end{array}$$

The special feature to be noted here is that $M_{h\mathbb{C}}$ and $M_{ph\mathbb{C}}$ are homogeneous under the same group.

Notation: A symmetric space of group type is denoted by $X = (H \times H)/H$ where a Lie group H is identified with the diagonal in $H \times H$; its c-dual space is given by $H_{\mathbb{C}}/H$.

Table XII.4.1. Spaces having invariant structures of all three types. These are the spaces $\mathrm{Co}(T_{\mathbb{C}})/\,\mathrm{Str}(T_{\mathbb{C}})$, where $T_{\mathbb{C}}$ is a complex JTS from Table XII.2.1:

$V_{\mathbb{C}}$	$G = \mathrm{Co}(V_{\mathbb{C}})_o$	$H = \mathrm{Str}(V_{\mathbb{C}})_o$
1.	$\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{C})$	$\mathbb{P}(\mathrm{Gl}(n,\mathbb{C}) \times \mathrm{Gl}(n,\mathbb{C}))$
2.	$\mathbb{P}\,\mathrm{Sp}(n,\mathbb{C})$	$\mathrm{Gl}(n,\mathbb{C})$
3.	$\mathbb{P}\,\mathrm{SO}(4m,\mathbb{C})$	$\mathrm{Gl}(2m,\mathbb{C})$
4.	$\mathrm{SO}(n+2,\mathbb{C})$	$\mathrm{SO}(n,\mathbb{C}) \times \mathrm{SO}(2,\mathbb{C})$
5.	E_7	$E_6 \times \mathbb{C}^*$
6.	$\mathbb{P}\,\mathrm{Gl}(p+q,\mathbb{C})$	$\mathbb{P}(\mathrm{Gl}(p,\mathbb{C}) \times \mathrm{Gl}(q,\mathbb{C}))$
7.	$\mathbb{P}\,\mathrm{SO}(4m+2,\mathbb{C})$	$\mathrm{Gl}(2m+1,\mathbb{C})$
8.	E_6	$\mathrm{SO}(10,\mathbb{C}) \times \mathbb{C}^*$

Any of the three complexification functors, applied to one of these spaces, yields a direct product of the space with itself. The space of type 4 carries a second structure of symmetric space with twist, given by Table XII.4.4, 6.a. ($q = 2$).

Table XII.4.2. Para-Hermitian symmetric spaces without straight complex structure. These are the spaces $\mathrm{Co}(V)/\,\mathrm{Str}(V)$ with V a Jordan pair from Table XII.2.2. They are obtained as straight real forms of the spaces given in the preceding table.

V	$G = \mathrm{Co}(V)_o$	$H = \mathrm{Str}(V)_o$
1.1.	$\mathbb{P}\,\mathrm{SU}(n,n)$	$\mathrm{Gl}(n,\mathbb{C})/\mathbb{R}^*$
1.2.	$\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{R})$	$\mathbb{P}(\mathrm{Gl}(n,\mathbb{R}) \times \mathrm{Gl}(n,\mathbb{R}))$
1.3.	$n = 2m$: $\mathbb{P}\,\mathrm{Gl}(2m,\mathbb{H})$	$\mathbb{P}(\mathrm{Gl}(m,\mathbb{H}) \times \mathrm{Gl}(m,\mathbb{H}))$
2.1.	$\mathbb{P}\,\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{Gl}(n,\mathbb{R})$
2.2.	$n = 2m$: $\mathbb{P}\,\mathrm{Sp}(m,m)$	$\mathrm{Gl}(m,\mathbb{H})$
3.1.	$\mathrm{SO}^*(4m)$	$\mathrm{Gl}(m,\mathbb{H})$
3.2.	$n = 2m$: $\mathbb{P}\,\mathrm{SO}(n,n)$	$\mathrm{Gl}(n,\mathbb{R})$
4.1.	$\mathrm{SO}(p+1,q+1)$	$\mathrm{SO}(p,q) \times \mathrm{SO}(1,1)$, $p+q=n$
5.1.	$E_{7(-25)}$	$E_{6(-25)} \times \mathbb{R}^+$
5.2.	$E_{7(7)}$	$E_{6(6)} \times \mathbb{R}^+$
6.1.	$\mathbb{P}\,\mathrm{Gl}(p+q,\mathbb{R})$	$\mathbb{P}(\mathrm{Gl}(p,\mathbb{R}) \times \mathrm{Gl}(q,\mathbb{R}))$
6.2.	$p = 2r$, $q = 2s$: $\mathbb{P}\,\mathrm{Gl}(r+s,\mathbb{H})$	$\mathbb{P}(\mathrm{Gl}(r,\mathbb{H}) \times \mathrm{Gl}(s,\mathbb{H}))$
7.1.	$n = 2m+1$: $\mathbb{P}\,\mathrm{SO}(n,n)$	$\mathrm{Gl}(n,\mathbb{R})$
8.1.	$E_{6(6)}$	$\mathrm{SO}(5,5) \times \mathbb{R}^+$
8.2.	$E_{6(-26)}$	$\mathrm{SO}(1,9) \times \mathbb{R}^+$

The straight and the twisted complexification of the space labelled by $i.j.$ is given by the space in Table XII.4.1. labelled by i. The space of type 4.1. has another structure of symmetric space with twist given by Table XII.4.5, 6.1.a ($i = j = 1$). The spaces 1.1., 2.1., 3.1., 4.1. and 5.1. have invariant causal structures; they are known as causal symmetric spaces *of Cayley type*.

Table XII.4.3. Pseudo-Hermitian symmetric spaces.

α	$G^{(\alpha)}$	$G^{(-\alpha)}$	H
1. A.	$\mathrm{Gl}(n,\mathbb{H})$	$\mathrm{Gl}(2n,\mathbb{R})$	$\mathrm{Gl}(n,\mathbb{C})$
1. B.	$\mathrm{U}(p+v,q+u)$	$\mathrm{U}(p+u,q+v)$	$\mathrm{U}(u,v)\times\mathrm{U}(p,q)$
2. A.	$\mathrm{Sp}(p,n-p)$	$\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{U}(p,n-p)$
3. A.	$\mathrm{SO}(2p,4m-2p)$	$\mathrm{SO}^*(4m)$	$\mathrm{U}(p,2m-p)$
4. A.	$\mathrm{SO}(p-1,q+3)$	$\mathrm{SO}(p+1,q+1)$	$\mathrm{SO}(p-1,q+1)\times\mathrm{SO}(2)$
4. B.	$\mathrm{SO}^*(2+2q)$	$\mathrm{SO}^*(2+2q)$	$\mathrm{SO}^*(2)\times\mathrm{SO}^*(2q)$
6. A.	$\mathrm{U}(l+i,k+j)$	$\mathrm{U}(l+j,k+i)$	$\mathrm{U}(k,l)\times\mathrm{U}(i,j)$
7. A.	$\mathrm{SO}^*(4m-2)$	$\mathrm{SO}(2p,4m+2-2p)$	$\mathrm{U}(p,2m+1-p)$

The straight complexification and twisted para-complexification of space $i.X.$ is given by space $i.$ in Table XII.4.1. The space of type 4.A. carries a second structure of symmetric space with twist, given by Table XII.4.5, 6.1.a ($i = 2, j = 0$), and the space of type 4.B. carries a second structure given by Table XII.4.5, 6.2.b ($p = 1$). The following are *pseudo-Hermitian symmetric spaces of tube type* in the sense of [FG96]: 1.A.+, 1.A.-, 1.B.+($p = u, q = v$), 2.A.+, 2.A.-, 3.A.+, 3.A.-, 4.A.+ .

Table XII.4.4. Straight complex spaces having no twisted complex structure.

α	$G^{(\alpha)}\cong G^{(-\alpha)}$	H
1. a.	$H\times H$	$H=\mathrm{Gl}(n,\mathbb{C})$
1. b.	$\mathrm{O}(2n,\mathbb{C})$	$\mathrm{O}(n,\mathbb{C})\times\mathrm{O}(n,\mathbb{C})$
1. c.	$\mathrm{Sp}(2m,\mathbb{C})$	$\mathrm{Sp}(m,\mathbb{C})\times\mathrm{Sp}(m,\mathbb{C})$
2. a.	$\mathrm{Gl}(n,\mathbb{C})$	$\mathrm{O}(n,\mathbb{C})$
2. b.	$H\times H$	$H=\mathrm{Sp}(m,\mathbb{C})$
3. a.	$\mathrm{Gl}(2m,\mathbb{C})$	$\mathrm{Sp}(m,\mathbb{C})$
3. b.	$H\times H$	$H=\mathrm{SO}(2m,\mathbb{C})$
4. a.	$\mathrm{SO}(p+1,\mathbb{C})\times\mathrm{SO}(q+1,\mathbb{C})$	$\mathrm{SO}(p,\mathbb{C})\times\mathrm{SO}(q,\mathbb{C})$
6. a.	$\mathrm{O}(p+q,\mathbb{C})$	$\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(q,\mathbb{C})$
6. b.	$p=2r, q=2s:$ $\mathrm{Sp}(r+s,\mathbb{C})$	$\mathrm{Sp}(r,\mathbb{C})\times\mathrm{Sp}(s,\mathbb{C})$
7. a.	$H\times H$	$H=\mathrm{SO}(2m+1,\mathbb{C})$

The twisted complexification and para-complexification of space $i.x.$ is given by space $i.$ in Table XII.4.1. The complex spheres carry *three* structures of symmetric space with twist: they appear in the above table as types 6.a. $p = 1$ and 4.a. $q = 0$, and also in Table XII.4.5, 4.1.b. Further, the spaces of type 4.a. carry also structures of *reducible* symmetric spaces with twist, namely those given by direct product of the structures on the complex spheres. Finally, the space of type 6.a. $q = 2$ carries a second structure of symmetric space with twist given by Table XII.4.1, 4.

Table XII.4.5. Spaces without invariant complex or paracomplex structure. These are the spaces $M^{(\alpha)}$ and $M^{(-\alpha)}$ corresponding to the JTS from Table XII.2.6. In the last column we list the twisted complexification of the space: it is given by the line of Table XII.4.3 with the indicated label. The additional parameter "$-$" means that in the corresponding line of Table XII.4.3, α and $-\alpha$ have to be exchanged. For example, the twisted complexification of the group $\mathrm{Sp}(m,\mathbb{R})$ (line 2.1.b, first and third column) can be found in Table XII.4.3, line 2.A, second and third column with the indicated parameters: it is the space $\mathrm{Sp}(2m,\mathbb{R})/\mathrm{U}(m,m)$. For some symplectic series one has to take twice the parameters as given in Table XII.4.3 in order to get the twisted complexification; this is indicated by "double param."

The twisted para-complexification of space $i.j.x.$ is given by space $i.j.$ of Table XII.4.2, and the "double complexification" is given by space $i.$ of Table XII.4.1.

Most of the resulting complexification diagrams are given in explicit form in Section IV.1.

α	$G^{(\alpha)}$	$G^{(-\alpha)}$	H	twisted compl.
1.1.a.	$H \times H$	$H_{\mathbb{C}} = \mathrm{Gl}(n, \mathbb{C})$	$H = \mathrm{U}(p, q)$	1.B, $v = p, u = q$
1.1.b.	$\mathrm{SO}^*(2n)$	$\mathrm{SO}(n, n)$	$\mathrm{SO}(n, \mathbb{C})$	1.A
1.1.c.	$\mathrm{Sp}(2m, \mathbb{R})$	$\mathrm{Sp}(m, m)$	$\mathrm{Sp}(m, \mathbb{C})$	1.A.-
1.2.a.	$H_{\mathbb{C}} = \mathrm{Gl}(n, \mathbb{C})$	$H \times H$	$H = \mathrm{Gl}(n, \mathbb{R})$	1.A
1.2.b.	$\mathrm{Sp}(2m, \mathbb{R})$	$\mathrm{Sp}(2m, \mathbb{R})$	$\mathrm{Sp}(m, \mathbb{R}) \times \mathrm{Sp}(m, \mathbb{R})$	1.B.-, $p = q = u = v = m$
1.2.c.	$\mathrm{O}(p + v, q + u)$	$\mathrm{O}(p + u, q + v)$	$\mathrm{O}(u, v) \times \mathrm{O}(p, q)$	1.B
1.3.a.	$H_{\mathbb{C}} = \mathrm{Gl}(2m, \mathbb{C})$	$H \times H$	$H = \mathrm{Gl}(m, \mathbb{H})$	1.A.-
1.3.b.	$\mathrm{O}^*(4m)$	$\mathrm{O}^*(4m)$	$\mathrm{O}^*(2m) \times \mathrm{O}^*(2m)$	1.B.-, $p = q = u = v = m$
1.3.c.	$\mathrm{Sp}(p + v, q + u)$	$\mathrm{Sp}(p + u, q + v)$	$\mathrm{Sp}(u, v) \times \mathrm{Sp}(p, q)$	1.B, double param.
2.1.a.	$\mathrm{U}(p, q)$	$\mathrm{Gl}(n, \mathbb{R})$	$\mathrm{O}(p, q)$	2.A
2.1.b.	$H \times H$	$H_{\mathbb{C}} = \mathrm{Sp}(m, \mathbb{C})$	$H = \mathrm{Sp}(m, \mathbb{R})$	2.A.-, $p = m = n/2$
2.2.a.	$\mathrm{U}(m, m)$	$\mathrm{Gl}(m, \mathbb{H})$	$\mathrm{SO}^*(2m)$	2.A.-, $p = m = n/2$
2.2.b.	$H \times H$	$H_{\mathbb{C}} = \mathrm{Sp}(m, \mathbb{C})$	$H = \mathrm{Sp}(r, m - r)$	2.A, $p = 2r, n = 2m$
3.1.a.	$\mathrm{U}(2p, 2q)$	$\mathrm{Gl}(m, \mathbb{H})$	$\mathrm{Sp}(p, q)$	3.A, double param.
3.1.b.	$H \times H$	$H_{\mathbb{C}} = \mathrm{SO}(2m, \mathbb{C})$	$H = \mathrm{SO}^*(2m)$	3.A.-, $p = m$
3.2.a.	$\mathrm{U}(m, m)$	$\mathrm{Gl}(2m, \mathbb{R})$	$\mathrm{Sp}(m, \mathbb{R})$	3.A.-, $p = m$
3.2.b.	$H \times H$	$H_{\mathbb{C}} = \mathrm{SO}(2m, \mathbb{C})$	$H = \mathrm{SO}(p, 2m - p)$	3.A
4.1.b.	$\mathrm{SO}(p + 1, \mathbb{C})$	$\mathrm{SO}(p + 1, \mathbb{C})$	$\mathrm{SO}(p, \mathbb{C})$	4.B, $p = q$
6.1.a.	$\mathrm{O}(l + i, k + j)$	$\mathrm{O}(l + j, k + i)$	$\mathrm{O}(k, l) \times \mathrm{O}(i, j)$	6.A
6.1.b.	$\mathrm{Sp}(r + s, \mathbb{R})$	$\mathrm{Sp}(r + s, \mathbb{R})$	$\mathrm{Sp}(r, \mathbb{R}) \times \mathrm{Sp}(s, \mathbb{R})$	6.A, $k = l = r, i = j = s$
6.2.a.	$\mathrm{Sp}(l + i, k + j)$	$\mathrm{Sp}(l + j, k + i)$	$\mathrm{Sp}(k, l) \times \mathrm{Sp}(i, j)$	6.A, double param.
6.2.b.	$\mathrm{O}^*(2p + 2q)$	$\mathrm{O}^*(2p + 2q)$	$\mathrm{O}^*(2p) \times \mathrm{O}^*(2q)$	6.A, $k = l = 2p, i = j = 2q$
7.1.a.	$H \times H$	$H_{\mathbb{C}} = \mathrm{SO}(2m + 1, \mathbb{C})$	$H = \mathrm{SO}(p, 2m + 1 - p)$	7.A.-

Finally, for the space of type 4.1.a we define

$$H^{p,q} := H^{p,q}(\mathbb{R}) := \mathrm{SO}(p, q + 1) / \mathrm{SO}(p, q);$$

for $q = 0$ these are the real hyperbolic spaces. Then we have in case 4.1.a

$$M^{(\alpha)} \cong H^{r-1,s} \times H^{p-s,q-r+1}, \quad M^{(-\alpha)} \cong H^{s,r-1} \times H^{q-r+1,p-s},$$

and the twisted complexification is given by 4.A.

The following spaces have several structures of symmetric spaces with twist:

6.2.b. (cf. Table XII.4.3),

6.1.a. for $i = 2, j = 0$ (cf. Table XII.4.3) and for $i = j = 1$ (cf. Table XII.4.2),

4.1.b. (cf. Table XII.4.4),

4.1.a. for $r = 1, s = 0$ (cf. 6.1.a. with $i = 1, j = 0$). Among the latter are the real hyperbolic spaces.

4.1.a. also have structures of *reducible* symmetric spaces with twist.

The reader may have noticed that all multiple structures come out of the relation between types 4 ("classical" conformal space) and 6, $p = 1$ (projective space). Finally, the one-dimensional LTS has a *continuous* family of Jordan-extensions – all scalar multiples of $T(x, y, z) = xyz$. The corresponding complexifications of the real line can be understood as the hyperbolic plane which is deformed into the flat complex plane as the positive scalar factor tends to zero, and similarly with the Riemann sphere if the scalar factor is negative. The corresponding para-complexification is the one-sheeted hyperboloid.

Remark XII.4.6. (Causal symmetric spaces with twist) If the underlying Jordan pair of T is the Jordan pair belonging to a *Euclidean Jordan algebra*, then the corresponding symmetric space with twist is a *causal symmetric space*, cf. Section XI.3. These are the spaces 1.1.a,b,c; 2.1.a,b; 3.1.a,b and 4.1.a ($p=1$) in Table XII.4.5 and 1.1; 2.1; 3.1; 4.1; 5.1 in Table XII.4.2. ■

Remark XII.4.7. (A semi-exceptional space.) According to [Ma73, p.416], the symmetric spaces of classical type $M = \mathrm{Sl}(4,\mathbb{H})/\mathrm{Sp}(3,1)$ and $M_{\mathbb{C}} = \mathrm{Sl}(8,\mathbb{C})/\mathrm{Sp}(4,\mathbb{C})$ have exceptional structures of symmetric spaces with twist, namely

$$M_{h\mathbb{C}} = E_{7(2)}/(E_{6(2)} \times S^1)$$

and

$$M_{ph\mathbb{C}} = \mathrm{Co}(\mathrm{Herm}(3,\mathbb{O}))/\mathrm{Str}(\mathrm{Herm}(3,\mathbb{O}))$$

(cf. Table XII.4.2). Central extensions of M and $M_{\mathbb{C}}$ have classical structures of Makarevič space, cf. Table XII.4.5, 3.1.a and XII.4.4, 3.a, corresponding to the equation of dimensions $\dim \mathrm{Herm}(3,\mathbb{O}) = \dim \mathrm{Herm}(4,\mathbb{H}) - 1$. The fact that classical LTS's may have exceptional Jordan-extensions has also been observed by Loos [Lo77, 11.19] and Neher [N85].

Remark XII.4.8. (Berger's list.) Comparison with Berger's list ([B57]) of irreducible symmetric spaces shows that the symmetric spaces appearing in Tables XII.4.1 – XII.4.5 exhaust the irreducible symmetric spaces (of non-exceptional type) in the following sense: Some spaces from our tables (e.g. the group $\mathrm{Gl}(n,\mathbb{R})$) are not irreducible as symmetric spaces; in these cases the simple part ($\mathrm{Sl}(n,\mathbb{R})$ in the example) appears in Berger's list. This observation has already been mentioned in the introduction to Ch. IV (Observation A). We have indicated the spaces which appear several times in our tables (only the sphere $S^n_{\mathbb{C}}$ appears three times, most spaces of rank 1 appear twice, and the other spaces only once), thus making the Observation B from Ch. IV more precise. ■

Notes for Chapter XII.

XII.1. The classification of complex simple Jordan algebras is equivalent to the classification of simple Euclidean Jordan algebras; the result (Table XII.1.1) goes back to Jordan, von Neumann and Wigner; cf. [BK66] and [FK94]. The classification of simple real Jordan algebras (Table XII.1.2) has been carried out by K.-H. Helwig ([Hw67]); however, his results have not been completely published; cf. [Kay94]. The methods used by Helwig have been generalized by E. Neher in order to classify simple real JTS (cf. notes for Chapter IV). The precise description of structure and automorphism groups (Table XII.1.3) is also due to E. Neher ([Ne80]). Note that the structure group only depends on the underlying Jordan pair of T whereas the automorphism group depends really on the JTS; this explains that the automorphism groups listed here are not always the same as the ones listed in [Ne80].

XII.2. For the principles of classification, see Chapter IV.2 and Notes for Chapter IV.

XII.3. Symmetric R-spaces have been introduced by M. Takeuchi ([Tak65]). In [Lo85] it is proved that they coincide with the spaces $\mathrm{Co}(T)/P$ considered as symmetric spaces which are classified in Table XII.3.3.

XII.4. We follow the classification by B.O. Makarevič ([Ma73]). When comparing with [Ma73], it should be noted that in case of the projective spaces $\mathrm{Gr}_{1,n+1}(\mathbb{F})$ not all of the "open symmetric orbits" considered by B.O. Makarevič are of the form $M^{(\alpha)}$; in fact, if V is a semisimple Jordan algebra, then the semisimple part of the cone $\Omega = (\mathrm{Str}(V)/\,\mathrm{Aut}(V))_e$ yields an open orbit $\mathbb{P}(\Omega)$ of the subgroup $\mathbb{P}(\mathrm{Str}(V))$ of $\mathbb{P}\,\mathrm{Gl}(V)$ in the projective space $\mathbb{P}(V)$ such that $\mathbb{P}(\Omega)$ is a symmetric space, but it is not an orbit of the form $M^{(\alpha)}$. The non-trivial fact that, if V^c is not isomorphic to a projective space, the open symmetric orbits are indeed all of the form $M^{(\alpha)}$ is due to A.A. Rivillis ([Ri70]).

Chapter XIII: Further topics

The aim of these notes was to introduce and to define the geometric objects corresponding to (real) Jordan structures, to present a fairly exhaustive description of the main examples and to give in this way a self-contained "geometric" introduction to Jordan theory. However, we have not even started to develop a "structure theory" in the usual sense which should contain the following elements:

- A *root theory for symmetric spaces with twist*: relate the Peirce decomposition w.r.t. a complete system of orthogonal tripotents (cf. [Lo75], [Ne82]) with the root theory for symmetric spaces – see [Lo85] for the case of compact spaces. We have developed the theory as far as possible without root theory in order to convince the reader that in Jordan theory much can be done before root theory really becomes necessary – in contrast to Lie theory where root theory is an almost indispensable tool.
- A *representation theory*: relate representations of Lie groups and symmetric spaces to representations of Jordan objects. With this aim in mind, we have in our presentation paid some attention on functorial questions – they show the way one has to go if one wants to understand what a geometric representation theory of Jordan objects should look like. Although there exists not yet even the outline of a theory, we can already see that (at least in the semisimple case) homomorphisms quite often have "good" properties in the sense that they carry over to Jordan-extensions, to conformal Lie algebras or even -groups, etc. The representation theory of Hermitian symmetric spaces is treated by Lie theoretic methods in [Sa80]; for some interesting results on representations of Jordan algebras cf. [Cl92].
- A *theory of invariant differential operators and enveloping algebras*: there exist concepts of enveloping algebras in Jordan theory (cf. [Lo75]). What is their geometric counterpart? Can they be realized by certain spaces of invariant differential operators on symmetric spaces with twist? An explicit description of the algebra of invariant differential operators is very important in harmonic analysis.
- A *theory of Siegel-domain realizations of symmetric spaces with twist*: cf. Introduction to Ch. XI and Remark XI.2.4.
- A *geometric theory of alternative algebras, triple systems and pairs*: As explained in [Lo75], *alternative* triple systems and pairs correspond to Peirce-1-spaces in Jordan structures, and these play an important role in the classification. What is the geometric meaning of the additional structure given by the alternative law?
- *Other base fields:* It would be nice if the geometric methods which turn out to be so successful in the real and complex case could be generalized to other base fields. When we try to do this, the question arises: what do we mean by "geometric"? For people working in harmonic analysis, "geometry" and "differential geometry" are more or less the same,

and at least the first half of our presentation follows this way of thinking. However, the more it becomes clear that Jordan theory already contains the global and not only the local geometry, one is already in the real case lead to stress the analogy of Jordan theory with classical projective geometry. Therefore a geometric Jordan theory for general base fields will be based on a more classical notion of "geometry" (work in progress).

This list of further topics is not complete. We hope that working on a structure theory in this sense will also lead to answers to the problems we consider to be central in the whole theory: the problem of the Jordan-Lie functor (cf. Section 0.6), and the problem of the relation between extrinsic and intrinsic theory (cf. Section 0.9).

The reader interested in applications in harmonic analysis can find some literature in our list of references; in most cases, the use of Jordan theory there is somewhat hidden or even not mentioned at all, but the reader will find that methods belonging to Jordan theory often play an important role in analyzing "concrete" cases, so that one might be tempted to call Jordan theory "the general theory of special cases". We even would like to include Weyl's classic [Wey39] into this remark since it seems not be an accident that the classical groups are precisely those which are in the image of the Jordan-Lie functor (cf. Section 0.6). The most systematic study of the interplay between Jordan theory and harmonic analysis is the monograph [FK94] (where many more references are given). An interesting outlook on a vast area of open problems in harmonic analysis related to "generalized conformal geometry" can be found in the overview article [Gi98] by S. Gindikin.

References

[A74] Arnold, V., *Équations différentielles ordinaires*, Editions MIR, Moscow 1974.

[Ar66] Artin, E., *Geometric Algebra*, Interscience, New York 1966.

[B57] Berger, M., "Les Espaces Symétriques non Compacts", Ann. Ec. Norm. Sup. (3) 74 (1997), p. 85 - 177.

[B87] Berger, M., *Geometry* (2 volumes), Springer-Verlag, Berlin 1987.

[Be94] Bertram, W., "Dualité des espaces riemanniens symétriques et analyse harmonique", thesis, Université Paris 6, 1994.

[Be96a] Bertram, W., "Un théorème de Liouville pour les algèbres de Jordan", Bull. Soc. Math. Francaise **124** (1996), 299–327.

[Be96b] Bertram, W., "On some Causal and Conformal Groups", J. Lie Theory **6** (1996), 215 - 244.

[Be97a] Bertram, W., "Ramanujan's Master Theorem and Duality of Symmetric Spaces", J. of Funct. An. **148** (1997), 117 – 151.

[Be97b] Bertram, W., "Complexifications of Symmetric Spaces and Jordan Theory. I." Preprint, Clausthal 1997 (submitted).

[Be98a] Bertram, W., "Algebraic Structures of Makarevič Spaces. I", Transformation Groups, Vol. **3**, No.1, (1998), 3 – 32.

[Be98b] Bertram, W., "Jordan algebras and conformal geometry", in: Hilgert, Vinberg (eds), *Positivity in Lie Theory – Open Problems*, de Gruyter, 1998.

[Be98c] Bertram, W., "Complexifications of Symmetric Spaces and Jordan Theory. Part II: Integrability and Conformal Group." Preprint, Clausthal 1998 (submitted).

[BeHi97] Bertram, W. and J. Hilgert, "Reproducing Kernels on Vector Bundles", Preprint, Clausthal 1997.

[BeHi98] Bertram, W. and J. Hilgert, "Hardy Spaces and Analytic Continuation of Bergman Spaces", Bull. Soc. Math. Française **126**, 435 – 482, 1998.

[BeHi99] Bertram, W. and J. Hilgert, "Geometric Bergman and Hardy spaces", preprint, Clausthal 1999 (submitted).

[Bet96] Betten, F., "Kausale Kompaktifizierung kompakt-kausaler Räume", thesis, Göttingen 1996 ("Causal Compactification of Compactly Causal Spaces", Report No. 39 1995/96 of the Mittag-Leffler Institute, Stockholm).

[Bir98] Birkenhake, C., "Complex Structures contained in classical groups", J. Lie Theory **8** (1), 139 – 152, 1998.

[BK66] Braun, H. and M. Koecher, *Jordan-Algebren*, Springer, New York 1966.

[Bou59] Bourbaki, N., *Algèbre, chapitre 9 - Formes sesquilinéaires et formes quadratiques*, Hermann, Paris 1959.

[BtD85] Bröcker, T. and T. tom Dieck, *Representations of Compact Lie Groups*, Springer, New York 1985.

[Cartan] Cartan, É, *Oeuvres complètes, Partie II, Vol.2: Groupes infinis, systèmes différentielles, théorie d'equivalence*, Gauthier-Villars, Paris 1953.

[Cha98] Chadli, M., "Espace de Hardy d'un espace symétrique de type Cayley", Ann. Inst. Fourier **48**, 97 – 132, 1998.

[Ch49] Chow, W.L., "On the geometry of algebraic homogeneous spaces", Ann. Math. **50**, no.1, 32 – 75 , 1949.

[Cl92] Clerc, J.-L., "Representation d'une algèbre de Jordan, polynômes invariants et harmoniques de Stiefel", J. reine und angew. Math. **423**, 47 – 71, 1992.

[D63] Dieudonné, J., *La géométrie des groupes classiques* Springer, New York, 1963.

[DNF84] Doubrovine, B.A., S.P. Novikov and A.T. Fomenko, *Modern Geometry - Methods and Applications*, Springer, New York 1984.

[Ei27] Eisenhart, L.P., *Non-Riemannian Geometry*, AMS Coll. Publ. VIII, New York 1927.

[Fe80] Ferus, D., "Symmetric Submanifolds of Euclidean Space", Math. Ann. **247**, 81 – 93, 1980.

[Fa92] Faraut, J., "Fonctions de Legendre sur une algèbre de Jordan", CWI Quarterly **5**, 309 - 320, 1992.

[Fa95] Faraut, J., "Fonctions sphériques sur un espace symétrique ordonné de type Cayley", Contemp. Math. **191**, 41 - 55, 1995.

[Fa96] Faraut, J., "Opérateurs différentiels invariants hyperboliques sur un espace symétrique de type Cayley" J. Lie Theory **6**, 271 – 289, 1996.

[FK94] Faraut, J., and A. Korányi, *Analysis on symmetric cones*, Clarendon Press, Oxford, 1994.

[FG96] Faraut, J., and S. Gindikin, "Pseudo-Hermitian Spaces of Tube Type", Progress in Nonlinear Diff. Equat. **20** (1996), 123 - 154.

[Gam91] Gamkrelidze, R., (ed.) *Geometry I*, Springer EMS, New York 1991.

[Gi92a] Gindikin, S., "Generalized conformal structures on classical real Lie groups and related problems on the theory of representations", C.R.Acad. Sci. Paris, **315**, 675-79, 1992.

[Gi92b] Gindikin, S., "Fourier transform and Hardy spaces of $\overline{\partial}$-cohomology in tube domains", C.R.Acad.Sci. Paris **315**, 1139 – 1143, 1992.

[Gi98] Gindikin, S., "Tube domains in Stein symmetric spaces", in: *Positivity in Lie Theory: Open Problems* (Eds. J. Hilgert et al.), de Gruyter, Berlin 1998.

[GiKa96] Gindikin, S., and S. Kaneyuki, "On the automorphism group of the generalized conformal structure of a symmetric R-space", to appear in: Differential Geometry and its Applications.

[Go87] Goncharov, A.B., "Generalized conformal structures on manifolds", Selecta Math. Soviet. **6**, 307-340, 1987.

[GY97] Gong, S. and Q. Yu, "The Schwarzian Derivative in Kähler Manifolds", preprint, 1997.

[Gui65] Guillemin, V., "The integrability problem for G-structures", Trans. AMS **116**, 544 – 560, 1965.

[Hw67] Helwig, K.-H., "Halbeinfache reelle Jordan-Algebren," Habilitationsschrift, München 1967.

[Hw69] Helwig, K.-H., "Involutionen von Jordan-Algebren", manuscripta math. **1**, 211 – 229, 1969.

[Hw70] Helwig, K.-H., "Jordan-Algebren und symmetrische Räume.I," Math.Z. **115**, 315-349, 1970.

[He78] Helgason, S., *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York 1978.

[HHL89] Hilgert, J., H.Hofmann and J. Lawson, *Lie groups, convex cones, and Semigroups*, Oxford University Press, 1989.

[HN91] Hilgert, J. and K.H. Neeb, *Lie-Gruppen und Lie-Algebren*, Vieweg-Verlag, Braunschweig 1991.

[HO96] Hilgert, J. and G. Ólafsson, *Causal Symmetric Spaces - Geometry and Harmonic Analysis*, Perspectives in Mathematics Vol. 18, Academic Press, San Diego 1996.

[HOØ91] Hilgert, J., G. Ólafsson and B. Ørsted, "Hardy spaces on affine symmetric spaces", J. reine angew. Math. 415, p. 189-218, 1991.

[Hua45] Hua, L.-K., "Geometries of matrices I. Generalization of von Staudt's Theorem", Trans. AMS **57**, 441 – 481, 1945.

[Hu72] Humphreys, J.E., *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York 1972.

[Io90] Iordanescu, R., "Jordan structures with applications.III.IV. Jordan algebras in differential geometry. Jordan triple systems in differential geometry", Preprint, Bucharest, 1990.

[Io98] Iordanescu, R., "Application of Jordan structures to differential geometry and to physics (Recent advances and new open problems)", Preprint, Bucharest, 1998.

[Jac49] Jacobson, N., "Lie and Jordan triple systems", Amer.J.Math **71**, 149 – 170, 1949.

[Jac51] Jacobson, N., "General Representation Theory of Jordan Algebras", Trans. A.M.S. **70**, 509 – 530, 1951.

[Jaf75] Jaffee, H.A., "Real forms of Hermitian Symmetric Spaces", Bull. AMS **81** (2), 456 – 458, 1975.

[JV77] Jakobsen, H.P. and M. Vergne, "Wave and Dirac operators, and representations of the Conformal Group", J.Funct.An. **24**, 52 – 106, 1977.

[Kan85] Kaneyuki, S., "On classification of parahermitian symmetric spaces", Tokyo J.Math **8**, 473–482, 1985.

[Kan87] Kaneyuki, S., "On orbit structure of compactifications of parahermitian symmetric spaces", Japan J.Math. **13**, 333–370, 1987.

[KanKo85] Kaneyuki, S. and M.Kozai, "Paracomplex structures and affine symmetric spaces", Tokyo J.Math **8**, 81–98, 1985.

[Kan89] Kaneyuki, S., "On the causal structures of the Shilov boundaries of symmetric bounded domains", in: Prospects in Complex Geometry, Springer LN 1468, New York 1989.

[Kan92] Kaneyuki, S., "Homogeneous Symplectic Manifolds and Dipolarization in Lie Algebras", Tokyo J. of Math. **15**, no.2, 313 – 325, 1992.

[Kau94] Kaup, W., "Hermitian Jordan triple systems and the automorphisms of bounded symmetric domains", 204 – 214 in *Non-associative algebra and its applications*, Ed. S. Gonzales, Kluwer Acad. Publ., Dordrecht 1994.

[KaV79] Kashiwara, M. and M. Vergne, "Functions on the Shilov boundary of the generalized half plane", in: *Noncommutative harmonic analysis*, Springer Lecture Notes in Mathematics **728**, 136 – 176, 1979.

[Kay94] Kayoya, J.B., "Analyse sur les algèbres de Jordan simples," thesis, Université Paris 6, 1994.

[Koe58] Koecher, M., "Die Geodätischen von Positivitätsbereichen", Math. Ann. **135**, 192 – 202, 1958.

[Koe67] Koecher, M., "Über eine Gruppe von rationalen Abbildungen", Inv. Math. **3** (1967), 136-171.

[Koe68] Koecher, M., "Imdedding of Jordan algebras into Lie algebras.I.", Am. J. Math **89**, p. 787-816, 1967, "–.II", Am. J. Math. **90**, 467 – 510, 1968.

[Koe69a] Koecher, M., *An elementary Approach to Bounded Symmetric Domains*, Lecture Notes, Rice University, Houston, Texas 1969.

[Koe69b] Koecher, M., "Gruppen und Lie-Algebren von rationalen Funktionen", Math. Z. **109** (1969), 349 – 392.

[Koe78] Koecher, M., "Eine Konstruktion von Jordan-Algebren", manuscripta math. **23**, 387 - 425, 1978.

[Koh65] Koh, S., "On affine symmetric spaces", Trans.AMS **119**, 291–309, 1965.

[Ko72] Kobayashi, S., *Transformation Groups in Differential Geometry*, Springer, New York 1972.

[KoNa64] Kobayashi, S. and T. Nagano, "On filtered Lie Algebras and Geometric Structures.I.", J.Math.Mech 13 (1964), 875 - 908.

[KoNo63] Kobayashi, S., and Nomizu, *Foundations of Differential Geometry I*, New York 1963.

[KoNo69] Kobayashi, S., and Nomizu, *Foundations of Differential Geometry II*, New York 1969.

[Kou95] Koufany, K., "Semi-groupe de Lie associé à un cône symétrique", Ann. Inst. Fourier **45**, 1 – 29, 1995.

[Kow80] Kowalski, O., *Generalized symmetric spaces*, Springer LNM **805**, Berlin 1980.

[KSS67] Kantor, I.L., A.I. Sirota and A.S. Solodovnikov, "A class of symmetric spaces with extendable group of motions and a generalization of the Poincaré model," Soviet Math. Dokl. **8**, 423 – 426, 1967.

[LaMi89] Lawson, H.B. and M.-L. Michelson, *Spin Geometry*, Princeton Univ. Press, Princeton 1989.

[L1850] Liouville, J., "Théorème sur l'équation $dx^2 + dy^2 + dz^2 = \lambda(d\alpha^2 + d\beta^2 + d\gamma^2)$," J. Math. Pures et Appl., **15**, p.103, 1850.

[Li52] Lister, W.G., "A structure theory of Lie triple systems", Trans. A.M.S., **72** (1952), 217 – 242.

[Lo67] Loos, O., "Spiegelungsräume und homogene symmetrische Räume", Math.Z. **99** (1967), 141 – 170.

[Lo69a] Loos, O., *Symmetric Spaces I*, Benjamin, New York 1969.

[Lo69b] Loos, O., *Symmetric Spaces II*, Benjamin, New York 1969.

[Lo72] Loos, O., "An Intrinsic Characterization of Fibre Bundles Associated with Homogeneous Spaces by Lie Group Automorphisms," Abh. Math. Sem. Univ. Hamburg **37** 3/4, 160 – 179, 1972.

[Lo75] Loos, O., *Jordan pairs*, Springer LN 460, New York 1975.

[Lo77] Loos, O., *Bounded symmetric domains and Jordan pairs*, Lecture Notes, Irvine 1977.

[Lo78] Loos, O., "Homogeneous Algebraic Varieties defined by Jordan pairs", Monatshefte Mathematik **86**, 107 - 129, 1978.

[Lo79] Loos, O., "On algebraic groups defined by Jordan pairs," Nagoya Math. J. **74**, 23 - 66, 1979.

[Lo85] Loos, O., "Charakterisierung symmetrischer R-Räume durch ihre Einheitsgitter", Math. Z. **189** (1985), 211–226.

[LoM77] Loos, O. and K. Mc.Krimmon, “Speciality of Jordan Triple Systems”, Comm. in Algebra **5**(10), 1057 - 1082, 1977.

[Ma73] Makarevič, B.O., “Open symmetric orbits of reductive groups in symmetric R-spaces”, Math. USSR Sbornik 20 (1973), No 3, 406-418.

[Ma79] Makarevič, B.O., “Jordan algebras and orbits in symmetric R-spaces”, Trudy Moskov. Mat. Obshch. **39**, 157 – 179, 1979 (english translation: Trans. Moscow. Math. Soc. **39**, 169 – 193, 1979).

[Mey70] Meyberg, K., “Jordan-Tripelsysteme und die Koecher-Konstruktion von Lie-Algebren”, Math.Z. **115** (1970), 58–78.

[MnT86] Mneimné, R. and F. Testard, *Introduction à la theéorie des groupes de Lie classiques*, Hermann, Paris 1986.

[Nai83] Naitoh, H., “Pseudo-Riemannian Symmetric R-spaces”, Osaka J. Math. **21**, 733 - 764, 1984.

[Ne79] Neher, E., “Cartan-Involutionen von halbeinfachen reellen Jordan-Tripelsystemen”, Math.Z. **169**, 271 – 292, 1979.

[Ne80] Neher, E., “Klassifikation der einfachen reellen speziellen Jordan-Tripelsysteme”, manuscripta math. **31**, 197–215, 1980.

[Ne81] Neher, E., “Klassifikation der einfachen reellen Ausnahme-Jordan-Tripelsysteme”, J. reine angew. Math. **322**, 145–169, 1981.

[Ne82] Neher, E., *Jordan Triple Systems by the Grid Approach*, Springer Lecture Notes **1280**, New York 1982.

[Ne85] Neher, E., “On the Classification of Lie and Jordan Triple Systems”, Comm. in algebra **13**, 2615–2667, 1985.

[Os92] Osgood, B., “The Schwarzian Derivative and Conformal Mappings of Riemannian Manifolds”, Duke Math. J **67**, 57 – 99, 1992.

[Po62] Pohl, W.F., “Differential geometry of higher order”, Topology **1**, 169 - 211, 1962.

[Pe67] Petersson, H., “Zur Theorie der Lie-Tripel-Algebren”, Math. Z. **97**, 1-15, 1967.

[Pev98] Pevsner, M., “Analyse conforme sur les algèbres de Jordan”, thesis, Paris 1998.

[Ri69] Rivillis, A.A., “Homogeneous locally symmetric regions in a conformal space”, Soviet. Math. Dokl. **10**, 147-150, 1969..

[Ri70] Rivillis, A.A., “Homogeneous locally symmetric regions in homogeneous spaces associated with semisimple Jordan-algebras”, Math. USSR Sbornik **11** (1970), No 3..

[Ry98] Ryan, J., “Dirac Operators, Conformal Transformations and Aspects of Classical Harmonic Analysis”, J.Lie Theory **8**, 67 – 82, 1998.

[Sch85] Schwarz, T., “Special Simple Jordan Triple Systems”, Algebras-Groups-Geometries **2**, 117 – 128, 1985.

[Sa80] Satake, I., *Algebraic Structures of symmetric domains*, Iwanami Shoten, Princeton 1980.

[Se76] Segal, I.E., *Mathematical cosmology and extragalactic astronomy*, Academic Press, 1976.

[Sp73] Springer, T.A., *Jordan Algebras and Algebraic Groups*, Springer Verlag, New York 1973.

[Shi75] Shima, H., “On locally homogeneous domains of completeley reductive linear Lie groups”, Mathematische Annalen **217**, 93-95, 1975.

[Sch54] Schouten, J.A., *Ricci-calculus*, Springer-Verlag, Berlin 1954.

[St83] Sternberg, S., *Lectures on Differential Geometry*, Scd. Ed., Chelsea Pub. Comp., New York 1983.

[Tak65] Takeuchi, M., “Cell decompositions and Morse equalities on certain symmetric spaces”, J. Fac. Sci. Univ. Tokyo, Sect. IA **12**. 81 – 192, 1965.

[Tan85] Tanaka, N., “On affine symmetric spaces and the automorphism group of product manifolds”, Hokkaido Math. J. **14**, 277 - 351, 1985.

[Vdb98] Vandebrouck, F., “Analyse fonctionelle sur les domaines bornés symétriques et systèmes triples de Jordan”, thesis, Poitiers 1998.

[Vi63] Vinberg, E.B., “The theory of convex homogeneous cones”, Trans. Moscow Math. Soc. **12**, 340 - 402, 1963.

[Wey23] Weyl, H., *Mathematische Analyse des Raumproblems*, Springer, Berlin 1923.

[Wey39] Weyl, H., *The classical groups*, Princeton 1939.

[Ze92] Zelikin, M.I., “On the theory of the matrix Riccati equation”, Math. USSR Sbornik **73**, no.2, 341 – 354, 1992.

[ZSSS82] Zhevlakov, K.A., A.M. Slin’ko, I.P. Shestakov and A.I. Shirshov, *Rings that are nearly associative*, Academic Press, New York 1982.

Index of notation

$\mathfrak{q}$ – -1-eigenspace in an involutive Lie algebra; LTS,	I.1
Q – quadratic map of a symmetric space	I.5
R – curvature tensor, Lie triple product	I.1, I.2
Ric – Ricci-tensor,	V.1
$\widetilde{r}_x, \widetilde{r}_{x,y}$ – exponential of a twisted polarization,	VI.2
$r^{\pm}_{x,y}, r_{x,y}$ – "multiplication by $r \in \mathbb{R}^*$ w.r.t. a point (x,y)",	VI.3
$\mathrm{Str}(T)$ – structure group of a JTS T	VIII.1
$\mathrm{Str}(V)$ – structure group of a Jordan algebra V	II.3
$\mathrm{Svar}(T)$ – structure variety of a JTS T	III.4
$\mathrm{Sym}(A, \mathbb{F})$ – space of symmetric matrices,	I.6
s_x – Symmetry w.r.t. a point x in a symmetric space,	I.3
Θ – involution of $\mathrm{Co}(T)$ and of $\mathfrak{co}(T)$,	VII.2, VIII.2
T – structure tensor, Jordan triple product	III.2, III.3, III.4
$T^{(\alpha)}$ – α-modification of a JTS T	III.4
$T^2_p M$ – second order tangent space,	I.B
t_x – translation by a vector x,	I.A
$\widetilde{t}_x$ – exponential of the quadratic vector field $\mathbf{p}_x$,	VIII.2
$\mathrm{U}(A, \varepsilon, \mathbb{F})$ – A-unitary group,	I.6
$\mathbf{v}$ – constant vector field,	I.A.1
V' – set of invertible elements in a Jordan algebra V,	II.2
V^c – conformal completion of V	VIII.3
W – vector space V with $\mathrm{Str}(T)$-action by $g.w = \Theta(g)(w)$ –	II.2
W – subspace $\{T(\cdot, v, \cdot) \mid v \in V\}$ of $\mathrm{Hom}(S^2V, V)$	VII.2
W^c – conformal completion of W,	VIII.3
$\mathbf{W} = (\mathbf{W}_p)_{p \in V^c}$ – structure bundle	VIII.3
$\mathfrak{X}(M)$ – Lie algebra of vector fields on M,	I.A.1
$[X, Y, Z]$ – Lie triple product,	I.1
∇ – affine connection,	I.B

Differential calculus: We distinguish in our notation three different kinds of "differential": by $T_p\varphi : T_pM \to T_{\varphi(p)}N$ we denote the tangent map at p of a smooth map φ between manifolds M and N. By $\mathrm{D}\varphi : V \supset M \to \mathrm{Hom}(V, W)$, $x \mapsto \mathrm{D}\varphi(x)$ we denote the total differential of a smooth map φ having domain M in a vector space V and range in a vector space W. (The second total differential is then a map $\mathrm{D}^2\varphi : M \to \mathrm{Hom}(S^2V, W) \subset \mathrm{Hom}(V, \mathrm{Hom}(V, W))$, where $\mathrm{Hom}(S^2(V, W)$ is the space of symmetric bilinear maps from V to W.) Finally, $\mathrm{d}\omega$ is the exterior derivative of a p-form ω; in particular, $\mathrm{d}f$ for a function f is a one-form. – For further notation concerning differential calculus and transformation groups cf. Appendix I.A.

Quadratic maps: to a quadratic map $q : E \to F$ we associate the symmetric bilinear map $q : E \times E \to F$ defined by $q(x, y) = q(x + y) - q(x) - q(y)$.

Index