

Generalized projective geometries

by
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Abstract

This is a survey paper. After reviewing some features of “ordinary” projective geometry over a commutative base field, generalized projective (resp. polar) geometries (over a commutative base field or ring \mathbb{K} in which 2 is invertible) as well as the symmetric space (over \mathbb{K}) associated to a generalized polar geometry are defined. Examples of such geometries are given. The equivalence of connected generalized projective (resp. polar) geometries over \mathbb{K} (with base point) with Jordan pairs (resp. triple systems) over \mathbb{K} is described. The importance of the Lie-Jordan functors is explained. Some open problems are pointed out.

1 Introduction: Projective geometry revisited

Before explaining the concept of a *generalized* projective geometry, I would like to recall quickly some features of “ordinary” projective geometry (over a commutative base field \mathbb{K}) which I consider to be fundamental and which I would not like to miss in later generalizations:

- (1) Duality. As a matter of principle, a projective space $X = \mathbb{K}P^n = P(W)$ ($W \cong \mathbb{K}^{n+1}$) should always be considered together with its *dual space*

$$X' = P(W^*)$$

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which can be seen as the *space of hyperplanes in X* (the class $[\lambda]$ of a non-zero linear form can be identified with the hyperplane $[\ker \lambda]$). Duality defines *incidence*: a point $[x] \in X$ and an element $[\lambda] \in X'$ are *incident* if $\lambda(x) = 0$. Let us say that, in general, a *pair geometry* is given by two sets X, X' and a subset $M \subset X \times X'$, called the *set of non-incident* or *remote pairs*, such that, for all $x \in X, a \in X'$, the sets

$$V_a := \{z \in X \mid (z, a) \in M\}, \quad V'_x := \{b \in X' \mid (x, b) \in M\}$$

are non-empty. In the case of projective geometry, the V_a , respectively the V'_x are the complements of hyperplanes, and they clearly are non-empty.

- (2) The affine-projective relationship. It is a classical exercise in linear algebra that the set V_a , i.e., the complement of a given hyperplane a in $\mathbb{K}P^n$, has a canonical structure of an affine space over \mathbb{K} . Let us say that an *affine pair geometry* is a pair geometry (as defined in (1)) such that, for any $x \in X$ and $a \in X'$, the sets V_a , respectively V'_x , have the structure of an affine space over \mathbb{K} . In other words, for any $(x, a) \in M$, V_a is a vector space over \mathbb{K} with zero vector x , and for x varying in V_a , these vector space structures are related among each other in the usual way, and similarly for V'_x with origin a . Thus $(x, a) \in M$ defines a *vectorialization* of X , and M can be seen as the *space of vectorializations of X (and of X')*.

- (3) The “fundamental identities”. Let us write, for $r \in \mathbb{K}$ and $x, y \in V_a$,

$$\mu_r(x, a, y) := r_{x,a}(y) := r \cdot y,$$

where the product $r \cdot y$ is the usual multiplication by scalars in the vector space V_a with origin x . The map μ_r thus defined can be seen as a *ternary product map*

$$\mu_r : X \times X' \times X \supset D \rightarrow X, \quad (x, a, y) \mapsto \mu_r(x, a, y)$$

defined on the set D given by the conditions $(x, a), (y, a) \in M$ (for $X = \mathbb{K}P^n$, this is a Zariski-dense open subset). In the same way one defines a map $\mu'_r : D' \rightarrow X'$. One may ask whether the “product maps” (μ_r, μ'_r) satisfy algebraic identities such as, e.g., associativity or commutativity. It is fairly obvious that μ_r is non-associative and non-commutative (however, for $r = \frac{1}{2}$, μ_r is weakly commutative in the sense that it is symmetric in x and y), but there are indeed other identities: first of all, there is a set of “easy” identities which express

just the fact that $\pi_r := \mu_r(\cdot, a, \cdot)$ describes the affine structure of V_a – in fact, identifying V_a with a standard vector space $V = \mathbb{K}^n$, the map π_r is nothing but the binary map

$$\pi_r(x, y) = (1 - r)x + ry.$$

The reader may, as an elementary exercise in linear algebra, try to find some algebraic identities for π_r which in turn are sufficient to recover the structure of an affine space on V – a solution can be found in [Be01a] where a (of course non-unique) set (Af1)-(Af4) of such identities is given. Moreover, as outlined in loc. cit., this approach to affine geometry has some advantages compared with the usual approach – in many regards it is easier and more conceptual. Coming back to projective geometry, the maps (μ_r, μ'_r) satisfy two other identities which call the “fundamental identities of projective geometry”, denoted by (PG1), (PG2). Roughly, (PG1) says that, if r is invertible in \mathbb{K} , the map $r_{x,a} : V_a \rightarrow V_a$ extends to a bijection of X which is an automorphism of all product maps (μ_s, μ'_s) , $s \in \mathbb{K}$. Indeed, $r_{x,a}$ is a linear bijection of V_a and is therefore induced by an element of the general projective group $G = \text{PGl}(n + 1, \mathbb{K})$; but clearly all elements of G act as automorphisms of the product maps (μ_s, μ'_s) . The identity (PG2) is similar in nature. Using the formalism explained in [Be01a,b], the fundamental identities can be written in the short form

$$(L_{x,a}^{(r)})^t = L_{a,x}^{(r)}, \quad (R_{a,x}^{(r)})^t = R_{x,a}^{(r)}, \quad (M_{x,y}^{(r)})^t = M_{y,x}^{(r)}$$

where t stands for “transposed” and L and M are defined as “operators of left, right and middle multiplication by

$$\mu_r(x, a, y) = L_{x,a}^{(r)}(y) = R_{a,y}^{(r)}(x) = M_{x,y}^{(r)}(a).$$

- (4) Scalar extension. The real projective space can be embedded into the complex projective space: $\mathbb{R}\mathbb{P}^n \subset \mathbb{C}\mathbb{P}^n$ – see [B87, Ch. 7] for a conceptual, but rather complicated construction of this inclusion. Similarly, we have inclusions like $\mathbb{Q}\mathbb{P}^n \subset \mathbb{R}\mathbb{P}^n$. More generally, if $\mathbb{K} \subset \bar{\mathbb{K}}$ is a base field extension, then we have an “extension”

$$(X, X') \subset (X_{\bar{\mathbb{K}}}, X'_{\bar{\mathbb{K}}})$$

which is compatible with the ternary products (μ_r, μ'_r) living on these spaces. Taking some care in the definitions, this carries over to the

case of projective spaces defined over commutative base *rings*. Here an important special case is given by the extension

$$\overline{\mathbb{K}} = \mathbb{K} \oplus \epsilon\mathbb{K}, \quad \epsilon^2 = 0,$$

called the *dual numbers over* \mathbb{K} and constructed in the same way as the complex numbers from the real numbers, but replacing the condition $i^2 = -1$ by the condition $\epsilon^2 = 0$. In this case, the projective space $(X_{\overline{\mathbb{K}}}, X'_{\overline{\mathbb{K}}})$ can be interpreted as the *tangent bundle* (TX, TX') of (X, X') . In this way we can introduce differential geometric terms in classical projective geometry, even if the base field in question is different from the field of real or complex numbers.

- (5) Polar geometries. In geometry, one is interested in metric or pseudo-metric structures or in their analogues. However, a projective space $P(W)$ does, à priori, not carry such a structure; it depends on additional choices. More precisely, what one needs is a way to identify X with its dual space X' , usually called a *correlation*: if $(p : X \rightarrow X', p' : X' \rightarrow X)$ is a pair of bijections of an affine pair geometry, we say that (p, p') is
- an *anti-automorphism* if it is compatible with all product maps μ_r, μ'_r ,
 - a *correlation* if it is an anti-automorphism of order 2 (i.e. $p' = p^{-1}$),
 - a *null-system* if it is a correlation having only *isotropic points* (a point $x \in X$ is called *isotropic* if $(x, p(x))$ is incident),
 - a *polarity* if it is a correlation admitting some non-isotropic point.

2 Generalized projective geometries and symmetric spaces

Now I will take the properties (1)–(5) just explained as starting point of an axiomatic definition (see [Be01b] for the exact formulation): a *generalized projective geometry* (over a commutative base field or ring \mathbb{K} in which 2 is invertible) is an affine pair geometry (X, X') such that the fundamental identities (PG1) and (PG2) hold in all scalar extensions of \mathbb{K} . A *generalized polar geometry* is a generalized projective geometry (X, X') together with a polarity (p, p') .

Apart from the case of ordinary projective geometry over \mathbb{K} , explained in Section 1, there are many other examples of generalized projective geometries (cf. [Be01a]):

- (1) *Grassmannian geometries* $(X, X') = (\text{Gras}_p(\mathbb{K}^{p+q}), \text{Gras}_q(\mathbb{K}^{p+q}))$, which can be defined more generally in infinite dimensions and over rings,
- (2) *Lagrangian geometries*; here $X = X'$ is the space of Lagrangian subspaces of some symplectic or (neutral) symmetric or Hermitian form; correspondingly, there are two main types of such geometries, namely *symplectic* and *orthogonal* Lagrangian geometries,
- (3) *conformal geometries*; here $X = X'$ is a projective quadric; the structure of generalized projective geometry is defined via a generalized stereographic projection – see [Be01a];
- (4) two types of *exceptional geometries*, one of them related to the *Cayley plane*.

Moreover, in each of these cases there are several polarities, leading to many interesting polar geometries. The reader having some knowledge in symmetric spaces will recognize that, for $\mathbb{K} = \mathbb{C}$, our list of geometries (1) – (4) above is precisely the list of simple *compact Hermitian symmetric spaces*. For $\mathbb{K} = \mathbb{R}$, it corresponds to the list of simple *symmetric R-spaces* which are nothing but real forms of compact Hermitian symmetric spaces. The relationship with symmetric spaces is a general feature of generalized projective and polar geometries: if $p : X \rightarrow X'$ is polarity, then the set $M^{(p)}$ of non-isotropic points [i.e. $(x, p(x))$ non-incident] is non-empty and is stable under the binary map

$$\mu : M^{(p)} \times M^{(p)} \rightarrow M^{(p)}, \quad (x, y) \mapsto \mu_{-1}(x, p(x), y).$$

From the axioms of a generalized projective geometry it follows easily that the map μ satisfies the following identities, where we let $\sigma_x(y) := \mu(x, y) = (-1)_{x, p(x)}(y)$:

- (M1) $\mu(x, x) = x$,
- (M2) $\sigma_x^2 = \text{id}$, i.e. $\mu(x, \mu(x, y)) = y$,
- (M3) σ_x is an automorphism of μ , i.e. $\sigma_x \mu(y, z) = \mu(\sigma_x y, \sigma_x z)$,
- (M4) there is a “neighborhood” of x (namely $V_{p(x)} \cap M^{(p)}$) on which x is the only fixed point of σ_x .

3 Equivalence with Jordan structures

As is by now well-known from work of Koecher and Loos (see [Lo77]), Hermitian symmetric spaces are closely related to *Hermitian Jordan triple systems*, and more generally, any Jordan triple system gives rise to a Lie triple system and thus to a symmetric space. Such symmetric spaces are called “with twist” in [Be00] – see also the notes [BeHi99] for a short account of the theory and some of its applications.

The concept of generalized projective and polar geometries is an attempt to generalize this theory: the main result from [Be01b] says that the following objects are in bijection:

- (1) *connected generalized projective geometries over \mathbb{K} with base point,*
- (2) *Jordan pairs over \mathbb{K} ,*
- (3) *3-graded Lie algebras $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (with minimal \mathfrak{g}_0) over \mathbb{K}*

and similarly, we have bijections between

- (1') *connected generalized polar geometries over \mathbb{K} with base point,*
- (2') *Jordan triple systems over \mathbb{K} ,*
- (3') *3-graded Lie algebras over \mathbb{K} with minimal \mathfrak{g}_0 and with an involution exchanging \mathfrak{g}_1 and \mathfrak{g}_{-1} .*

Moreover, the bijections of (1) and (2), respectively of (1') and (2'), are essentially equivalences of categories. In other words, Jordan theoretic objects can always (in arbitrary dimension and over almost arbitrary base rings) be “integrated” to geometric objects – here the situation is certainly much better than in Lie theory !

The equivalence of (2) and (3) (resp. of (2') and (3')), which is purely algebraic, is well-known; it is, however, not an equivalence of categories. Let me briefly describe how to go from (1) to (3) (see [Be01b] for the details): as is done in [Lo95] for symmetric spaces, we look at the space $\text{Der}(X, X')$ of “derivations”, that is, vector fields (sections of the tangent bundle (TX, TX') , which is defined using point (4) of Section 1) which at the same time are homomorphisms of generalized projective geometries. One can prove that, in any affine chart V , this space is represented by quadratic polynomial functions from V to V and thus has a gradation into homogeneous parts. Moreover, it is stable under the natural Lie bracket on the space $\text{Pol}(V, V)$ of V -valued polynomials on V and hence $\mathfrak{g} = \text{Der}(X, X')$ is the 3-graded Lie algebra we were looking for.

4 The Jordan-Lie functor

As mentioned above, any Jordan triple system $T : V \times V \times V \rightarrow V$ gives rise to a Lie triple system $R = R_T$, defined by antisymmetrization:

$$R_T(X, Y)Z = -(T(X, Y, Z) - T(Y, X, Z)).$$

We call the correspondence $T \mapsto R_T$ the *algebraic Jordan-Lie functor*. Its geometric version is the functor, explained above, associating to a generalized polar geometry $(X, X'; p)$ the “symmetric space over \mathbb{K} ” $(M^{(p)}, \mu)$. In the finite-dimensional case over $\mathbb{K} = \mathbb{R}$, Lie triple systems are in bijection with (connected simply connected) symmetric spaces (see [Lo69]); this “Lie functor” has been used in [Be00] in an essential way. In the general case, the “Lie functor” is no longer available, and there is no hope to define an analogue by the usual Lie theoretic methods: for instance, the rational projective space $\mathbb{Q}P^n$ is a nice symmetric space over $\mathbb{K} = \mathbb{Q}$ in our sense (the map μ being defined exactly as in the real case), but it is very far from being homogeneous under its automorphism group $O(n+1, \mathbb{Q})$, and hence the usual “Lie theoretic” methods do not apply (if one is not willing to replace simply \mathbb{K} by its algebraic closure). In the infinite dimensional situation, even over the real or complex numbers, similar problems arise. Therefore symmetric spaces which are “in the image of the Jordan-Lie functor” have the great advantage that Lie theory (which not always exists) can be replaced by Jordan theory.

5 Open problems

The following list is not complete – see also [Be00] and [Be01b].

- (1) Classification shows that, for simple finite-dimensional objects over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , the Jordan-Lie functor is “almost bijective”. In other words, most of the simple symmetric spaces can be treated by Jordan methods, and mostly in one way only (see [Be00] for more precise statements). The central problem of the theory, in my opinion, is to give a conceptual explanation of this fact, that is, to describe intrinsically fibers and image of the Jordan-Lie functor.
- (2) *Jordan algebras* play an important rôle in Jordan theory, and their relation with Jordan pairs and Jordan triple systems has many interesting aspects. What are the geometric objects corresponding to Jordan algebras? For Jordan algebras with unit element and under some restrictions in the infinite dimensional case, the problem is solved

in [Be01c]: they correspond to *generalized projective geometries with a central null-system*. However, in the case of Jordan algebras without unit and in some infinite dimensional cases the problem remains open.

- (3) Generalized projective geometries have an interesting and highly non-trivial “incidence structure”. It is determined by the algebraic structure. One would like to understand how algebra determines “incidence structures”, and if this could be interpreted in terms of *buildings* in the sense of J. Tits.
- (4) It would be interesting to define “generalized projective geometries over skew-fields \mathbb{K} ”. However, the commutativity of \mathbb{K} is essential in the approach [Be01b], and it is not at all clear how a theory for a non-commutative field or ring \mathbb{K} could look like. I conjecture that it should correspond somehow to *quaternionic symmetric spaces*, which in turn correspond to some aspects of *non-commutative Jordan structures* – see [Be01d] for a short account.

References

(This list contains just the references quoted in the talk; see [Be01b] for further references related to the approach presented here and [Be00], [BeHi99] for a more extensive bibliography related to Jordan structures and symmetric spaces.)

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