Conformal group and fundamental theorem for a class of symmetric spaces

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Abstract. Using a geometric approach, we define and investigate the *conformal group* of a *symmetric space with twist.* In the non-degenerate case we characterize this group by a theorem generalizing the *fundamental theorem of projective geometry.*

0. Introduction

The real projective space $\mathbb{RP}^n = \mathrm{O}(n+1)/(\mathrm{O}(n)\times\mathrm{O}(1))$ and the Sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ are examples of a class of symmetric spaces $M = G/H$ which can also be written as a quotient $M = G^b/Q$ with respect to a "big Lie group" G^b. In the first case, $G^b = \mathbb{P}\,\mathrm{Gl}(n+1,\mathbb{R})$ is the general projective group, characterized by the *fundamental theorem of projective geometry* as the group preserving collinearity (if $n > 1$), and in the second case, $G^b = \mathrm{SO}(n+1,1)$ is the *conformal group*, characterized by a classical result of Liouville (1850) as the group preserving the Riemannian metric of the sphere up to a scalar function (if $n > 2$). Another example is the unitary group $\mathrm{U}(n)$ which is homogeneous under the "big group" $\mathrm{SU}(n,n)$. Here I.E. Segal offered the conjecture that the "big group" can be characterized as the *causal group* of $\mathrm{U}(n)$ ([Se76, p.35]).

In this work we present a unified theory of the "big group". We consider not only symmetric spaces G/H which can globally be written as a quotient G^b/Q, but also spaces having "locally" such a realization; in the semisimple case these are the "open symmetric orbits in symmetric R-spaces" classified by B.O. Makarevič ([Ma73]). In our previous work [Be97] we have shown that algebraically such spaces, called *symmetric spaces with twist*, are equivalent to *Jordan triple systems* (JTS), and we have given an intrinsic geometric characterization which is the starting point for the presentation given here.

Every symmetric space with twist admits a *twisted complexicification*; for example, if $M = \mathbb{RP}^n$, then the twisted complexification is the complex projective space $\mathbb{CP}^n$. In the present work we characterize the twisted complex spaces as *circled spaces*; these are complex manifolds with a real affine connection such that multiplication by i in each tangent space extends to a holomorphic affine (local) diffeomorphism (Chapter 1). By a sort of analytic continuation, we deduce that multiplication by *real* scalars in each tangent space also has a canonical extension to a local diffeomorphism (no longer affine). The corresponding local one-parameter group attached to each point p in the manifold induces a vector field $E = E^p$ which we call an *Euler operator on a symmetric space.* In the example of $M = \mathbb{RP}^n$, using the usual affinization $V = \mathbb{R}^n$ of M, the Euler operator $E = E^0$ is nothing but the usual Euler operator $E(x) = x$ on $\mathbb{R}^n$. In fact, to any Euler operator E^p on a symmetric space we canonically associate a chart, called *Jordan coordinates*, such that E^p is represented by the usual Euler operator (Chapter 2). Now, if $M = G/H$, then the Lie algebra $\mathfrak{g}$ of G together with the Euler operators generate a "big Lie algebra" $\mathfrak{g}^b$ of vector fields on M, called the *(inner) conformal Lie algebra.* It is worth noticing that, because of its relation with the group of non-zero scalars, the concept of an Euler operator on a symmetric space is on the one hand sufficiently close to the original concepts of conformal geometry (which essentially corresponds to the rank-one case; compare e.g. with [Mai98]), and on the other hand it can be seen as the correct generalization of classical conformal geometry to the higher rank case.

The next step is the construction and description of a "big Lie group" belonging to $\mathfrak{g}^b$, called the *conformal group.* This is done on two levels: first (Chapter 3) we give a completely general construction of the conformal group, and second (Chapter 4) we characterize it in the *non-degenerate* case by a theorem generalizing the classical results mentioned above (Th. 4.1). Our general construction follows ideas of M. Koecher ([Koe69a,b]); however, we have chosen a more analytic approach in the spirit of the pseudogroup-concept (cf. [Ko72]) instead of the algebraic framework used by Koecher. This simplifies the theory considerably and is much better adapted to applications to harmonic analysis which we have in mind. Moreover, this approach avoids the use of axiomatic Jordan theory; in fact, it offers new, computation-free proofs for some fundamental algebraic identities in Jordan theory.

The characterization of the conformal group by the "fundamental theorem" Th.4.1 (i) is a further development of our "Liouville theorem for Jordan algebras" ([Be96a]) which we reproduce here for convenience of the reader (Th. 4.1 (ii)). The Liouville theorem deals with a condition on *first* (total) differentials, but in fact it is a "hidden second-differential theorem", i.e. it implies a condition on the second differential which in turn is close to a formulation due to H. Weyl of the fundamental theorem of projective geometry (cf. Ex. 4.8 (d)). Based on this observation, we introduce notions of *T-conformality* and *T-projectivity* and characterize the conformal group (Th.4.1) and the conformal Lie algebra (Th.4.2) by these properties in the non-degenerate case.

Since the 19th century the "conformal group" and its characterization has attracted much mathematical interest. From the more recent work, we would like to mention a result by S. Gindikin and S. Kaneyuki who have proved a theorem generalizing the Liouville-theorem to simple Jordan triple systems not belonging to the projective space ([GiKa98]). The essentially same result has been obtained earlier by A.B. Goncharov ([Go87]); see also the work of R.J. Baston [Ba91] for closely related results on AHS-structures. Special cases of the above mentioned results are theorems of Chow and Dieudonné (cf. Ex. 4.8 (c)) and the determination of *causal groups* (cf. Ex.4.8 (a)). Another interesting problem is the characterization of the conformal group by an invariant generalizing the classical cross-ratio which has been obtained for the classical matrix-spaces by Hua ([Hua45]) and for Jordan algebras by Koecher ([Ko67]). An algebraic approach to the conformal group by using the theory of *Jordan pairs* is due to O. Loos ([Lo79]). In Lie theory, the conformal Lie algebra is introduced by the so-called *Kantor-Koecher-Tits construction* (Section 2.3; cf. [Sa80]). The automorphism groups of symmetric tube domains are special cases of "conformal groups" (cf. [FK94]). Finally, the theory of the conformal group presented here is an essential tool for the further study of the symmetric spaces with twist started in [Be96b], [Be97] and [Be98a]; cf. [Be98b] for an overview.

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1. Circled spaces and Euler operators

1.0. Preliminaries. *(a) Extensions of tangent objects.*

Definition 1.0.1. Let M and M' be real manifolds and $p \in M$, $q \in M'$.

(1) An *extension* of a linear map $a : T_pM \to T_qM'$ is a smooth map $\alpha : M \to M'$ such that

$$\alpha(p) = q, \quad T_p\alpha = a.$$

(2) A *vector field extension of a tangent vector* $v \in T_pM$ is a vector field $\mathbf{v} \in \mathfrak{X}(M)$ such that

$$\mathbf{v}_p = v.$$

(3) A *vector field extension of an endomorphism* $A \in \mathrm{End}(T_pM)$ is a vector field $X \in \mathfrak{X}(M)$ such that $X_p = 0$ and for all $v \in T_pM$ and vector field extensions $\mathbf{v}$ of v,

$$[\mathbf{v}, X]_p = A(v).$$

If the extended object is only defined on a neighbourhood of p, then we speak of *local extensions*.■

The value of $[\mathbf{v}, X]_p$ in part (3) does not depend on the chosen vector field extension of v because the bracket of two vector fields vanishing at p vanishes again at p. If $M \subset V$ is an open domain and $p = 0$, then the condition from part (3) is equivalent to

$$\mathrm{D}\, X(0) = A,$$

where we consider X as a function $M \to V$ after having identified all tangent spaces with V. In this realization it is proved by elementary arguments that X is a (local) vector field extension of $A \in \mathrm{End}(T_pM)$ if and only if the flow φ_t of X is a local extension of $\exp(tA) \in \mathrm{Gl}(T_pM)$.

Example 1.0.2. (The Euler operator.) If $M = V$ is a vector space and $A = \mathrm{id}_V$, then $e^t\,\mathrm{id}_V$ is a (global) extension of e^{tA}. The corresponding vector field extension of A is the vector field $E = E^0$ given by

$$E(p) = p, \tag{1.1}$$

which is called the *Euler operator*. ■

Clearly extensions as defined above are highly non-unique. However, in the presence of additional requirements they may become unique. For instance, extensions of tangent maps to *affine maps* (i.e. maps compatible with affine connections) are always unique if they exist (cf. [Lo69a, p.24]).

(b) Symmetric spaces. We adopt here the group-theoretic definition of a symmetric space: a *symmetric space* is a connected homogeneous space of the form $M = G/H$ where H is an open subgroup of the group of fixed points of some non-trivial involution σ of G. We assume further that G acts effectively on M and consider $\mathfrak{g}$ as a subalgebra of the algebra $\mathfrak{X}(M)$ of vector fields on M. The base point eH of M is denoted by o, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the canonical decomposition of $\mathfrak{g}$ w.r.t. the differential of σ at the origin. The evaluation map

$$\mathrm{ev}_o : \mathfrak{q} \to T_oM, \quad X \mapsto X_o \tag{1.2}$$

is bijective; we denote by

$$l_o : T_oM \to \mathfrak{q} \subset \mathfrak{X}(M), \quad v \mapsto l_o v \tag{1.3}$$

its inverse. Then $l_o v$ is a vector field extension of v in the sense of Def. 1.0.1. Since the base point is arbitrary, we have in fact maps $l_p : T_pM \to \mathfrak{g} \subset \mathfrak{X}(M)$ for all $p \in M$.

We define the *canonical connection* on the tangent bundle of M by the formula

$$(\nabla_X Y)_p := [l_p(X_p), Y]_p \quad (X, Y \in \mathfrak{X}(M)). \tag{1.4}$$

It is easily verified that this defines indeed a connection ∇ and that this connection coincides with the one considered in [KoNo69] and [Lo69a]. It is torsionfree and has covariantly constant curvature: $\nabla R = 0$; a connection with these properties is called *affine locally symmetric* and characterizes symmetric spaces locally. The curvature tensor is given by the formula

$$(R(X,Y)Z)_p = -[[l_p(X_p), l_p(Y_p)], l_p(Z_p)] \quad (X,Y,Z \in \mathfrak{X}(M)). \tag{1.5}$$

It is G-invariant and is therefore equivalent to R_o, i.e. to the *Lie triple system* (LTS) $\mathfrak{q}$ with the Lie triple product

$$[X,Y,Z] := [[X,Y],Z] \quad (X,Y,Z \in \mathfrak{q}). \tag{1.6}$$

Homomorphisms of symmetric spaces are smooth maps which are affine (i.e. compatible with connections). A main result in the theory of symmetric spaces affirms that the category of connected simply connected symmetric spaces is equivalent to the category of (finite-dimensional real) LTS (cf. [Lo69, Th.II.4.12]). This means in particlar that an element of $\mathrm{Gl}(T_oM)$ has an extension to an automorphism of M iff it is a LTS-automorphism, and an element of $\mathrm{End}(T_oM)$ has an extension to an affine vector field on M iff it is a derivation of the Lie triple product R_o.

Proposition 1.0.3. *A manifold M with affine connection ∇ is locally affine symmetric if and only if for all $p \in M$ the endomorphism $-\mathbf{1}_p = -\mathrm{id}_{T_pM}$ has a (local) affine extension.*

Proof. Clearly $-\mathbf{1}_p$ is an automorphism of R_p. Therefore, if (M,∇) is affine symmetric, then $-\mathbf{1}_p$ has an affine extension. Conversely, let s_p be an affine extension of $-\mathbf{1}_p$. Since s_p is affine, it preserves the torsion Tor, the curvature R and all their covariant derivatives, and $-\mathbf{1}_p$, being its derivative at p, preserves these tensors at p. In particular,

$$\mathrm{Tor}_p(u,v) = (-\mathbf{1}_p)\,\mathrm{Tor}_p((-\mathbf{1}_p u, -\mathbf{1}_p v) = -\mathrm{Tor}_p(u,v)$$

for all $u,v \in T_pM$, whence $\mathrm{Tor} = 0$. In the same way we see that $-\mathbf{1}_p$-invariant quadrilinear maps from T_pM to T_pM vanish, whence $\nabla R = 0$. ■

The affine extension s_p of $-\mathbf{1}_p$ defined by the preceding propositon is called the *geodesic symmetry* w.r.t. the point p.

1.1. Circled spaces. We consider a smooth manifold M. Recall that a *tensor field $\mathcal{J}$ of type* $(1,1)$ is a smooth field of endomorphisms $\mathcal{J} = (\mathcal{J}_p)_{p\in M}$, $\mathcal{J}_p \in \mathrm{End}(T_pM)$. If $\mathcal{J}_p^2 = -\mathbf{1}_p$ for all p, then $\mathcal{J}$ is called an *almost complex structure*, and if $\mathcal{J}_p^2 = \mathbf{1}_p$, it is called a *polarization.*

Definition 1.1.1. A *locally (*resp. *globally) circled affine space* or just a *circled space* is a triple $(M,\mathcal{J},\nabla)$, where $\mathcal{J}$ is an almost complex structure and ∇ an affine connection on a real manifold M such that

(1) $\mathcal{J}$ is invariant under ∇ (i.e. $\nabla\mathcal{J} = 0$),
(2) $\mathcal{J}_p$ extends, for all p, to a local (resp. global) affine map j_p. ■

Recall that, by definition, $(\nabla\mathcal{J})(X,Y) = \nabla_X(\mathcal{J}Y) - \mathcal{J}(\nabla_X Y)$, and condition (1) can be written $\nabla_X(\mathcal{J}Y) = \mathcal{J}\nabla_X Y$ for all $X,Y \in \mathfrak{X}(M)$.

Theorem 1.1.2. *The (locally) circled affine spaces are precisely the twisted complex (locally) symmetric spaces, i.e. the (locally) symmetric spaces with invariant almost complex structure $\mathcal{J}$ such that the curvature tensor R satisfies the relation $R(\mathcal{J}X,Y) = -R(X,\mathcal{J}Y)$.*

Proof. Let $(M,\mathcal{J},\nabla)$ be a circled space. Since j_p is an affine extension of $\mathcal{J}_p$, j_p^2 is an affine extension of $\mathcal{J}_p^2 = -\mathbf{1}_p$. According to Prop. 1.0.3, (M,∇) is thus a (locally) affine symmetric space. Now the condition $\nabla\mathcal{J} = 0$ implies that

$$R(X,Y)\,\mathcal{J}Z = \mathcal{J}\,R(X,Y)Z. \tag{1.7}$$

The arguments used in the proof of Prop. 1.0.3 show that $\mathcal{J}_p$ is an automorphism of R_p. In the presence of (1.7) this is equivalent to

$$R_p(\mathcal{J}_p^{-1} v, \mathcal{J}_p^{-1} w, u) = R_p(v, w, u), \tag{1.8}$$

which in turn is equivalent to $R_p(\mathcal{J}_p v, w, u) = -R_p(v, \mathcal{J}_p w, u)$ $(u, v, w \in T_pM)$. This being true for all p, we have shown that $\mathcal{J}$ is a twisted almost complex structure for the symmetric space (M, ∇).

Conversely, given a symmetric space (locally isomorphic to $M = G/H$) with invariant twisted almost complex structure $\mathcal{J}$, let ∇ be the canonical connection of M. The invariance of $\mathcal{J}$ under G_o implies that $\nabla \mathcal{J} = 0$, and it follows that (1.7) holds. By assumption, (1.8) holds, and together with (1.7) this implies that $\mathcal{J}_p$ is for all p an automorphism of R_p. Therefore $\mathcal{J}_p$ has an extension to an affine map of (M, ∇). Thus $(M, \nabla, \mathcal{J})$ is circled. ∎

Since affine extensions are unique, it follows that for a circled space $j_p^2 = s_p$ is the symmetry w.r.t. p.

Proposition 1.1.3. *If $(M, \mathcal{J}, \nabla)$ is a circled space, then $\mathcal{J}_p$ has for all p a unique vector field-extension to an affine vector field J^p of ∇, and the operator $e^{t\,\mathcal{J}_p}$ has, for all $p \in M$ and $t \in \mathbb{R}$, a unique extension to a (local) affine diffeomorphism.*

Proof. We have shown above that the relations $R(\mathcal{J} X, Y) = -R(X, \mathcal{J} Y)$ and $R(X, Y)\,\mathcal{J} Z = \mathcal{J} R(X, Y)Z$ hold. It follows that

$$R(\mathcal{J} X, Y)Z + R(X, \mathcal{J} Y)Z + R(X, Y)\,\mathcal{J} Z = \mathcal{J} R(X, Y)Z, \tag{1.9}$$

i.e. $\mathcal{J}_p$ is a derivation of R_p for all p. As remarked in the preceding section, it follows that $\mathcal{J}^p$ has an affine vector field extension J^p. Moreover, $e^{t\,\mathcal{J}_p}$ is an automorphism of R_p and has thus an affine extension. ∎

1.2. Euler operators on symmetric spaces. We will use the fact that invariant (1,1)-tensor fields $\mathcal{J}$ with the property $\mathcal{J}^2 = -\mathbf{1}$ or $\mathcal{J}^2 = \mathbf{1}$ on a symmetric space $M = G/H$ are *integrable* (see Appendix). This means essentially that the Lie algebra

$$\mathfrak{g}(\mathcal{J}) = \{X \in \mathfrak{X}(M) |\, \forall Y \in \mathfrak{X}(M) : [X, \mathcal{J} Y] = \mathcal{J}[X, Y]\}$$

is stable under $\mathcal{J}$ and has a $\mathcal{J}$-bilinear Lie-bracket (Appendix, Prop. A.2). Since $\mathcal{J}$ is invariant under G, we have $\mathfrak{g} = \mathrm{Lie}(G) \subset \mathfrak{g}(\mathcal{J})$.

Lemma 1.2.1. *If $(M, \mathcal{J}, \nabla)$ is a circled space and J^p denotes, for $p \in M$, the affine vector field extension of $\mathcal{J}_p$, then the vector field*

$$E^p := \mathcal{J}^{-1} J^p \in \mathfrak{g}(\mathcal{J})$$

is a vector field extension of the operator $\mathbf{1}_p = \mathrm{id}_{T_pM}$.

Proof. We remark first that $(E^p)_p = \mathcal{J}_p^{-1}(J^p)_p = 0$. Next, since $J^p \in \mathfrak{g} \subset \mathfrak{g}(\mathcal{J})$ and since $\mathfrak{g}(\mathcal{J})$ is stable under $\mathcal{J}$ (Prop. A.2), $E^p = -\,\mathcal{J} J^p$ belongs to $\mathfrak{g}(\mathcal{J})$. Fix $p \in M$ and choose a vector field extension $\mathbf{v}$ of $v \in T_pM$ such that $\mathbf{v} \in \mathfrak{g}(\mathcal{J})$, e.g. one may take $\mathbf{v} = l_p v$. Then

$$[\mathbf{v}, E^p]_p = [\mathbf{v}, \mathcal{J}^{-1} J^p]_p = \mathcal{J}_p^{-1}[\mathbf{v}, J^p]_p = \mathcal{J}_p^{-1}\,\mathcal{J}_p\, v = v;$$

thus E^p is a vector field extension of $\mathbf{1}_p$. ∎

In the following proposition we use the notation

$$\widehat{\mathfrak{q}} := \mathrm{ad}(E^o)\mathfrak{q} = \{[E^o, X] |\, X \in \mathfrak{q}\} \subset \mathfrak{g}(\mathcal{J}) \tag{1.10}$$

and

$$\mathfrak{q}^b := \mathfrak{q} + \widehat{\mathfrak{q}} \subset \mathfrak{g}(\mathcal{J}). \tag{1.11}$$

Proposition 1.2.2. *In the situation of the preceding lemma, the following holds:*
(E1) $[[\widehat{\mathfrak{q}},\widehat{\mathfrak{q}}],\widehat{\mathfrak{q}}] \subset \widehat{\mathfrak{q}}$ *and* $[[\mathfrak{q}^b,\mathfrak{q}^b],\mathfrak{q}^b] \subset \mathfrak{q}^b$ *(i.e,* $\widehat{\mathfrak{q}}$ *and* $\mathfrak{q}^b$ *are Lie triple-systems).*
(E2) $[[\mathfrak{q},\widehat{\mathfrak{q}}],\mathfrak{q}] \subset \widehat{\mathfrak{q}}$.
(E3) *For all* $X \in \mathfrak{q}^b$, $\mathrm{ad}(E^o)^2 X = X$.
(E4) *For all* $X, Y \in \mathfrak{q}^b$, $\mathrm{ad}(E^o)[X,Y] = 0$.

Proof. Since $\mathcal{J}$ defines on $\mathfrak{g}(\mathcal{J})$ the structure of a complex Lie algebra, the inclusion

$$\iota : \mathfrak{g} \to \mathfrak{g}(\mathcal{J})$$

has a unique $\mathbb{C}$-linear extension to a homomorphism

$$\iota_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g} \to \mathfrak{g}(\mathcal{J}), \quad (X + iY) \mapsto X + \mathcal{J} Y.$$

Note that $\iota_{\mathbb{C}}(i^{-1}J^o) = \mathcal{J}^{-1} J^o = E^o$, and since $\iota_{\mathbb{C}}$ is a homomorphism, the diagram

$$\begin{array}{ccc} \mathfrak{g}_{\mathbb{C}} & \stackrel{\iota_{\mathbb{C}}}{\longrightarrow} & \mathfrak{g}(\mathcal{J}) \\ -\mathrm{ad}(iJ^o)\downarrow & & \downarrow \mathrm{ad}(E^o) \\ \mathfrak{g}_{\mathbb{C}} & \stackrel{\iota_{\mathbb{C}}}{\longrightarrow} & \mathfrak{g}(\mathcal{J}) \end{array}$$

commutes. Using that $[J^o,\mathfrak{q}] = \mathfrak{q}$, this implies $\widehat{\mathfrak{q}} = \mathcal{J}[J^o,\mathfrak{q}] = \mathcal{J}\,\mathfrak{q} = \iota_{\mathbb{C}}(i\mathfrak{q})$ and $\mathfrak{q}^b = \iota_{\mathbb{C}}(\mathfrak{q}_{\mathbb{C}})$. Since $i\mathfrak{q}$ and $\mathfrak{q}_{\mathbb{C}}$ are Lie triple systems, so are their images under $\iota_{\mathbb{C}}$, proving (E1). The relation $[[\mathfrak{q},i\mathfrak{q}],\mathfrak{q}] = i[[\mathfrak{q},\mathfrak{q}],\mathfrak{q}] \subset i\mathfrak{q}$ yields (E2).

By assumption, $\mathrm{ad}(J^o)$ is an invariant twisted complex structure on $\mathfrak{q}$; therefore $\mathrm{ad}(J^o) : \mathfrak{q}_{\mathbb{C}} \to \mathfrak{q}_{\mathbb{C}}$, being its $\mathbb{C}$-linear continuation, is an invariant twisted complex structure on $\mathfrak{q}_{\mathbb{C}}$. Since $(i\,\mathrm{ad}(J^o))^2 = i^2(\mathrm{ad}(J^o))^2$ is the identity on $\mathfrak{q}_{\mathbb{C}}$, it follows that $\mathrm{ad}(E^o)^2$ is the identity on $\mathfrak{q}^b$, whence (E3). Since, for all $X \in \mathfrak{h}_{\mathbb{C}}$, $[iJ^o,X] = i[J^0,X] = 0$, it follows that $[E^o,Y] = 0$ for all $Y \in \iota_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}})$, in particular for all $[U,V]$, $U,V \in \mathfrak{q}^b = \iota_{\mathbb{C}}(\mathfrak{q}_{\mathbb{C}})$; whence (E4). ■

Note that (E3) and (E4) imply that $[\mathrm{ad}(E^o)X,Y] + [X,\mathrm{ad}(E^o)Y] = \mathrm{ad}(E^o)[X,Y] = 0$, i.e.

$$\text{(E5)} \qquad \forall X,Y,Z \in \mathfrak{q}^b : \quad [[\mathrm{ad}(E^o)X,Y],Z] = -[[X,\mathrm{ad}(E^o)Y],Z]$$

holds. Using the terminology introduced in [Be97], (E3), (E4) and (E5) say that $\mathrm{ad}(E^o)$ is a twisted invariant polarization on the LTS $\mathfrak{q}^b$, and (E1) and (E2) say that $\mathfrak{q}$ has the essential properties of a para-real form of this polarized LTS. In fact, if the homomorphism $\iota_{\mathbb{C}}$ from the preceding proof is injective, then $\mathfrak{q}^b$ is isomorphic to the polarized LTS $\mathfrak{q}_{\mathbb{C}}$; we return to this point below (Prop. 2.4.1).

Definition 1.2.3. An *Euler operator on a symmetric space* $M = G/H$ is given by a vector field extension $E := E^o$ of $\mathbf{1}_o := \mathbf{1}_{T_oM}$ having properties (E1) – (E4). Property (E4) allows to define a vector field extension E^p of $\mathbf{1}_p$ for all $p \in M$ in a G-invariant way which is uniquely determined by E^o; then $(E^p)_{p\in M}$ will be called a *distribution of Euler operators on* M.

The Lie algebra $\mathfrak{g}^b := [\mathfrak{q}^b,\mathfrak{q}^b] + \mathfrak{q}^b$ of vector fields generated by $\mathfrak{q}^b$ is called the associated *inner conformal Lie algebra.* ■

We will see below that, in a suitable chart, the vector field E^o indeed coincides with the usual Euler operator (cf. Ex. 1.0.2). Rephrasing Prop. 1.2.2, we can say that an Euler operator is canonically associated to a circled space, as a sort of real version of the circled structure. It is therefore not surprising that this structure is inherited by any real form of a circled space. Recall that a *real form* is given by a conjugation, i.e. by an involutive automorphism τ with $\tau_* \mathcal{J} = -\mathcal{J}$.

Proposition 1.2.4. *If τ is a conjugation of the circled space $(M, \mathcal{J}, \nabla)$ such that $\tau(o) = o$, then the vector field E^o is invariant under τ_*, and it defines an Euler operator on the real form M^τ of M.*

Proof. We have $\tau_*(J^o) = -J^o$ since $\tau_*(J^o)$ is an affine vector field extending $\tau_*(\mathcal{J}_o) = -\mathcal{J}_o$. Using this, we get

$$\tau_*(E^o) = \tau_*(\mathcal{J}^{-1} J^o) = -\mathcal{J}^{-1} \tau_*(J^o) = E^o.$$

Thus E^o can be restricted to M^τ as a (locally defined) vector field, and $\mathfrak{q}^\tau + \mathrm{ad}(E^o)\mathfrak{q}^\tau$ is the real form $(\mathfrak{q}^b)^\tau$ of $\mathfrak{q}^b$; the properties (E1) – (E4) continue to hold for all $X, Y, Z \in (\mathfrak{q}^b)^\tau$. ■

Recall from [Be97] that a *symmetric space with twist* is by definition a real form of a twisted complex symmetric space, i.e. of a circled space. Thus Prop. 1.2.4 assigns canonically an Euler operator to any symmetric space with twist. A "twist" is by definition a *Jordan-extension* T of the curvature tensor R, i.e. an invariant tensor field such that T_p defines a Jordan triple system (JTS) on the tangent space T_pM such that the relation

$$T(X,Y,Z) - T(Y,X,Z) = -R(X,Y)Z \quad (X,Y,Z \in \mathfrak{X}(M)) \tag{1.12}$$

holds (cf. [Be97]).

Proposition 1.2.5. *Let $(E^p)_{p\in M}$ be a distribution of Euler operators on a symmetric space $M = G/H$ and recall from* Section 1.0 *the vector field extension $l_p : T_pM \to \mathfrak{g}$.*

(i) *The formula*

$$T(X,Y,Z)_p := [[l_p(X_p), \frac{l_p(Y_p) + \mathrm{ad}(E^p) l_p(Y_p)}{2}], l_p Z_p]_p$$

defines a Jordan-extension of the curvature tensor R of M.

(ii) *If M is a symmetric space with twist and $(E^p)_{p\in M}$ is the associated distribution of Euler operators, then T is equal to the structure tensor of M.*

Proof. (i) From the definition of T it is clear that T is a tensor field (i.e. it is function-linear in all three arguments) and that it is G-invariant. Thus we have to show that T_o is a Jordan-extension of R_o. If we identify T_oM with $\mathfrak{q}$ and let $X, Y, Z \in \mathfrak{q}$, then

$$T_o(X,Y,Z) = \frac{1}{2}([[X,Y],Z] - \mathrm{ad}(E^o)[[X, \mathrm{ad}(E^o)Y], Z]). \tag{1.13}$$

As we have remarked after the proof of Prop. 1.2.2, $\mathrm{ad}(E^o)$ is an *invariant twisted polarization* on the LTS $\mathfrak{q}^b$ in the sense of [Be97, Def. 1.4.1]. Therefore the formula

$$\widetilde{T}(X,Y,Z) := \frac{1}{2}([[X,Y],Z] - \mathrm{ad}(E^o)[[X, \mathrm{ad}(E^o)Y], Z])$$

defines a Jordan-extension of the LTS $\mathfrak{q}^b$, ([Be97, Th.1.4.2]). Because of (E1), (E2) and (E3), $\mathfrak{q}$ is stable under $\widetilde{T}$, and the restriction of $\widetilde{T}$ to a triple product on $\mathfrak{q}$ is a Jordan-extension of $\mathfrak{q}$. But then formula (1.13) shows that this means precisely that T_o is a Jordan-extension of $\mathfrak{q}$.

(ii) Let us assume first that M is a circled space and (E^p) its associated field of Euler operators. Then by definition (cf. [Be97, Section 1.3]) the structure tensor of M, evaluated at the base point, is for $X, Y, Z \in \mathfrak{q}$ given by

$$\begin{aligned}\frac{1}{2}(R(X,Y)Z - \mathcal{J} R(X, \mathcal{J}^{-1} Y)Z)_o &= \frac{1}{2}([[X,Y],Z] - \mathrm{ad}(J_o)[[X, \mathrm{ad}(J^o)^{-1}Y], Z])_o \\ &= \frac{1}{2}([[X,Y],Z] - \mathcal{J}^{-1}\mathrm{ad}(E^o)[[X, \mathcal{J}\,\mathrm{ad}(E^o)^{-1}Y], Z])_o \\ &= \frac{1}{2}([[X,Y],Z] - \mathrm{ad}(E^o)[[X, \mathrm{ad}(E^o)Y], Z])_o;\end{aligned}$$

according to Eqn. (1.13), this is the tensor T_o defined in the claim. Since both Euler operators and structure tensor can be restricted to real forms, it follows that the structure tensor and T coincide at the base point. ■

2. The conformal Lie algebra

We assume in this chapter an Euler operator E^o given on a symmetric space $M = G/H$. In this chapter we are interested in local properties; strictly speaking, we should assume M to be a *germ* of a symmetric space (cf. [Be97]).

2.1. Local Borel-imbedding.

Proposition 2.1.1. *The LTS* $\widehat{\mathfrak{q}} = \mathrm{ad}(E^o)\mathfrak{q}$ *is isomorphic to the dual of* $\mathfrak{q}$ *(i.e. its Lie triple bracket is isomorphic to the negative of the one in* $\mathfrak{q}$*). Consequently, the c-dual symmetric space of* M *has locally a canonical realization with base point* o *on the underlying manifold of* M.

Proof. From (E3) and (E4) we get for all $X, Y, Z \in \mathfrak{q}$,

$$[[\mathrm{ad}(E^o)X, \mathrm{ad}(E^o)Y], \mathrm{ad}(E^o)Z] = \mathrm{ad}(E^0)[[X, -\mathrm{ad}(E^o)^2 Y], Z] = -\mathrm{ad}(E^o)[[X,Y],Z];$$

therefore $\mathrm{ad}(E^o)$ is an isomorphism from $\mathfrak{q}$ onto the LTS obtained from $\mathfrak{q}$ by taking the negative of the usual triple Lie bracket. The second statement is otained by integrating locally at o the vector fields $X \in \widehat{\mathfrak{q}}$. ■

If $(M, \mathcal{J}, \nabla)$ is a circled space, the c-dual symmetric space of (M, ∇) admits, according to the preceding proposition, a local imbedding into M. This is a local generalization of the *Borel-imbedding* of a Hermitian symmetric space; it carries over to real forms.

2.2. Jordan coordinates and local Harish-Chandra imbedding.

Lemma 2.2.1. *The inner conformal Lie algebra* $\mathfrak{g}^b$ *(cf.* Def. 1.3.4*) is* $\mathbb{Z}/(3)$*-graded. More precisely,* $\mathfrak{g}^b$ *is stable under the derivation* $\mathrm{ad}(E^o)$ *and decomposes as a direct sum*

$$\mathfrak{g}^b = \mathfrak{m}^{-1} \oplus \mathfrak{m}^0 \oplus \mathfrak{m}^1$$

of the eigenspaces $\mathfrak{m}^k$ *for the eigenvalues* $k = 1, 0, -1$ *of* $\mathrm{ad}(E^o)$*, and* $[\mathfrak{m}^k, \mathfrak{m}^l] \subset \mathfrak{m}^{k+l}$*. In particular,* $\mathfrak{m}^{\pm 1}$ *are abelian and* $[[\mathfrak{m}^1, \mathfrak{m}^{-1}], \mathfrak{m}^{\pm 1}] \subset \mathfrak{m}^{\pm 1}$*. Moreover,*

$$\mathfrak{q}^b = \mathfrak{m}^{-1} \oplus \mathfrak{m}^1, \quad [\mathfrak{q}^b, \mathfrak{q}^b] = \mathfrak{m}^0.$$

Proof. By definition, $\mathfrak{g}^b = [\mathfrak{q}^b, \mathfrak{q}^b] + \mathfrak{q}^b$; this is a Lie algebra since $\mathfrak{q}^b$ is a Lie triple system. Further, $\mathfrak{q}^b$ is stable under $\mathrm{ad}(E^o)$ whose square is, by (E3), the identity there; therefore $\mathfrak{q}^b \subset \mathfrak{m}^1 \oplus \mathfrak{m}^{-1}$. On the other hand, by (E4), $[\mathfrak{q}^b, \mathfrak{q}^b] \subset \mathfrak{m}^0$, and we actually have equalities. This establishes the decomposition of $\mathfrak{g}^b$ into eigenspaces.

Now, since $\mathrm{ad}(E^o)$ is a derivation, for all $X \in \mathfrak{m}^k$, $Y \in \mathfrak{m}^l$: $\mathrm{ad}(E^o)[X,Y] = [\mathrm{ad}(E^o)X, Y] + [X, \mathrm{ad}(E^o)Y] = [kX, Y] + [X, lY] = (k+l)[X,Y]$, proving the statement about the grading. ■

We will use the notation $\mathfrak{m}^{\pm} := \mathfrak{m}^{\pm 1}$, $\mathfrak{h}^b := \mathfrak{m}^0$. Note that $H_o = [H, E^o]_o = 0$ for all $H \in \mathfrak{h}^b$, and also $X_o = 0$ for all $X \in \mathfrak{m}^+$ since $X_o = [E^o, X]_o = -[X, E^o]_o = -X_o$.

Lemma 2.2.2. *Every tangent vector* $v \in T_oM$ *has a unique vector field extension by an element* $\mathbf{v} \in \mathfrak{m}^-$*. It is given by the formula*

$$\mathbf{v} = \frac{1}{2}(l_o v - [E^o, l_o v]),$$

where l_o *is the inverse of the bijective evaluation map* $\mathfrak{q} \to T_oM$.

Proof. From the definition of $\mathfrak{q}^b$ (Eqn. (1.11)) it is clear that $\mathfrak{m}^- = \{\mathbf{v} | \, v \in T_oM\}$. We evaluate at the base point:

$$\mathbf{v}_o = \frac{1}{2}(l_o v + [l_o v, E^o])_o = \frac{1}{2}(v + \mathbf{1}_o(l_o v)_o) = v.$$

It follows that the evalution map $\mathfrak{m}^- \to T_oM$ is bijective. ■

Definition 2.2.3. If E^o is an Euler operator on a symmetric space M, we define a chart, called *Jordan coordinates*, as follows: to a vector v lying in a suitable neighbourhood U of 0 in $V := T_oM$ we assign the value of the integral curve $\varphi_t(o)$ of $\mathbf{v}$ at $t = 1$.

Having fixed the base point o, we will consider Jordan coordinates as an identification of an open neighborhood U of $o \in M$ with an open domain U in $V = T_oM$, i.e. we use the same notation v for $v \in U$ and for the corresponding point in M. Vector fields on U will be identified with smooth functions $U \to V$; thus the vector fields $\mathbf{v} \in \mathfrak{m}^-$ are identified with the constant vector fields. ■

The notion of Jordan coordinates can be seen as a *local Harish-Chandra imbedding*: in fact, integrating the vector fields $X \in \mathfrak{q}$ yields a local realization of M as an open domain in the vector space $V \cong \mathfrak{m}^-$. When M is a Hermitian symmetric space, this realization is known to be global and is precisely the well-known Harish-Chandra imbedding.

2.3. Geometric version of the Kantor-Koecher-Tits construction.

Theorem 2.3.1. *Let E^o be an Euler operator on a symmetric space M and $\mathfrak{g}^b$ the associated inner conformal Lie algebra. Then $\mathfrak{g}^b$ is represented in Jordan coordinates as a Lie algebra of quadratic vector fields, i.e. by polynomial maps $V \to V$ of degree at most two, such that the grading*

$$\mathfrak{g}^b = \mathfrak{m}^- \oplus \mathfrak{h}^b \oplus \mathfrak{m}^+$$

(cf. Lemma 2.2.2*) coincides with the grading given by the degree. More precisely, if we define for $v \in V$ a polynomial $\mathbf{p}_v$ by*

$$\mathbf{p}_v(x) := \frac{1}{2} T_o(x, v, x),$$

where T_o is the trilinear map on $V = T_oM$ defined in Prop. 1.2.5*, then*

(1) $\mathfrak{m}^- = \{\mathbf{v} | \, v \in V\}$,
(2) $\mathfrak{m}^+ = \{\mathbf{p}_v | \, v \in V\}$,
(3) $\mathfrak{q} = \{\mathbf{v} - \mathbf{p}_v | \, v \in V\}$,
(4) $\widehat{\mathfrak{q}} = \{\mathbf{v} + \mathbf{p}_v | \, v \in V\}$,
(5) $\mathfrak{h}^b = [\mathfrak{m}^+, \mathfrak{m}^-] = \mathrm{Span}\{T(w, v) | \, v, w \in V\}$*, where* $T(u, v)p := T(u, v, p)$,
(6) *E^o is identified with the usual Euler operator on V, i.e. $E^o(x) = x$.*

Proof. (1) is true by definition of the Jordan coordinates. In order to prove the other relations, recall the formula

$$[X, Y](z) = \mathrm{D}\, Y(z) \cdot X(z) - \mathrm{D}\, X(z) \cdot Y(z) \tag{2.1}$$

for the Lie bracket of two vector fields X, Y, considered as smooth functions on V. This formula implies that for any Y and $v \in V$,

$$\mathrm{ad}(\mathbf{v})^k Y(p) = \mathrm{D}^k\, Y(p) \cdot (\otimes^k v),$$

where $\mathrm{D}^k\, Y : V \to \mathrm{Hom}(S^k V, V)$ is the ordinary k-th total differential of $Y : V \to V$. Now, it follows from Lemma 2.1.2 that $[\mathfrak{m}^-, [\mathfrak{m}^-, [\mathfrak{m}^-, \mathfrak{g}^b]]] = 0$, and therefore, for all $Y \in \mathfrak{g}^b$ and p in the domain of the Jordan coordinates,

$$\mathrm{D}^3\, Y(p) = 0.$$

Integration shows that Y is quadratic as claimed.

Let us prove (6): E^o is a linear vector field since $[\mathbf{v}, [\mathbf{v}, E^o]] = -[\mathbf{v}, \mathbf{v}] = 0$ and $(E^o)_o = 0$. Since $[\mathbf{v}, E^o] = \mathbf{v}_o = v$, $E^o = \mathrm{id}_V$.

Now, the solutions of the Euler differential equation

$$[E^o, X](p) = \mathrm{D}\, X(p) \cdot p - X(p) = k\, X(p)$$

for $k \in \mathbb{N}_0$ are precisely the homogeneous polynomials of degree $k+1$; therefore $\mathfrak{h}^b$ is represented by linear vector fields and $\mathfrak{m}^+$ by homogenous quadratic ones.

It remains to calculate the second differential of the vector field $l_o w \in \mathfrak{q}$ for $w \in V$. We use that for all $w \in \mathfrak{q}$, $l_o w = \mathbf{w} + \widetilde{\mathbf{w}}$ with $\widetilde{\mathbf{w}} := \frac{l_o w + [E^o, l_o w]}{2}$ is the decomposition of $l_o w$ in a constant term plus a homogeneous quadratic polynomial. Using the commutativity of $\mathfrak{m}^+$ and $\mathfrak{m}^-$, we get $[\mathbf{v}, l_o w] = [\mathbf{v}, \widetilde{\mathbf{w}}] = [l_o v, \widetilde{\mathbf{w}}]$ and

$$\begin{aligned}(\mathrm{D}^2(l_o w))(0) \cdot v \otimes v &= [\mathbf{v}, [\mathbf{v}, l_o w]]_o \\ &= [\mathbf{v}, [l_o v, \widetilde{\mathbf{w}}]]_o \\ &= [l_o v, [l_o v, \widetilde{\mathbf{w}}]]_o \\ &= T_o(v, w, v)\end{aligned}$$

whith T_o as in Prop. 1.2.5. Since $\mathrm{D}^2\, \mathbf{w} = 0$, we have $\mathrm{D}^2(l_o w) = \mathrm{D}^2(\widetilde{\mathbf{w}})$; the vector field $\widetilde{\mathbf{w}}$ being homogeneous quadratic, it is given by the formula

$$\widetilde{\mathbf{w}}(x) = \frac{1}{2} D^2(l_o w)(0)(x, x) = -\frac{1}{2} T(x, w, x) = -\mathbf{p}_w(x).$$

This proves (2) and gives the formula

$$(l_o w)(x) = w - \frac{1}{2} T(x, w, x),$$

proving (3). Next, $[E^o, \mathbf{v} - \mathbf{p}_v] = -\mathbf{v} - \mathbf{p}_v$, proving (4).

We prove (5): Clearly, $[\mathfrak{q}^b, \mathfrak{q}^b] = [\mathfrak{m}^+, \mathfrak{m}^-]$, and we have explicitly

$$[\mathbf{p}_v, \mathbf{w}](p) = -(\mathrm{D}\, \mathbf{p}_v)(p) w = -T(w, v, p) = -T(w, v) p; \tag{2.2}$$

this implies the claim. ∎

Theorem 2.3.2. *There is a canonical bijection between Jordan-extensions of the curvature tensor and Euler operators on a symmetric space M.*

Proof. In Prop. 1.2.5 we have associated to an Euler operator E^o a Jordan-extension T of R. Conversely, given a Jordan-extension T of R, we let E^o be the Euler operator induced from the twisted complexification given by T.

These two constructions are indeed inverse to each other: Starting with a Jordan-extension T, we get again T from the associated Euler operator E^o (Prop. 1.2.5 (ii)). Starting with an Euler operator E^o, let T be the associated Jordan-extension. Then it is easily verified that in Jordan coordinates associated to E^o the Euler operator derived from T is again the usual Euler operator on V and thus coincides with E^o by part (6) of the preceding theorem. ∎

2.4. Faithful Jordan triple systems.

Proposition 2.4.1. *Let E^o be an Euler operator on a symmetric space M corresponding to a Jordan-extension T. Then the following are equivalent:*

(i) $\mathfrak{q} \cap \widehat{\mathfrak{q}} = 0$

(ii) *The map $V \to \mathrm{Hom}(V \otimes V, V)$, $v \mapsto T(\cdot, v, \cdot)$ is injective.*

If these properties hold, then the formula

$$\Theta(X+Y) := X - Y, \quad X \in \mathfrak{q}, Y \in \widehat{\mathfrak{q}}$$

defines an involutive automorphism of the LTS $\mathfrak{q}^b$ *such that* $\Theta \circ \operatorname{ad}(E^o) = -\operatorname{ad}(E^o) \circ \Theta$. *This automorphism extends to an involution of the inner conformal Lie algebra* $\mathfrak{g}^b$ *given in the notation of* Th. 2.3.1 *by the formula*

$$\Theta(\mathbf{v} + T(a,b) + \mathbf{p}_w) = -\mathbf{w} - T(b,a) - \mathbf{p}_v.$$

Proof. The equivalence of (i) and (ii) follows immediately from a comparison of formulas (3) and (4) of Th. 2.3.1.

Now we assume that $E^o = -\mathcal{J} J^o$ is the Euler operator of a circled space, and as in the proof of Prop. 1.2.2 we consider the inclusion $\iota : \mathfrak{q} \to \mathfrak{g}(\mathcal{J})$ and its complexification

$$\iota_{\mathbb{C}} : \mathfrak{q}_{\mathbb{C}} = \mathfrak{q} \oplus i\mathfrak{q} \to \mathfrak{g}(\mathcal{J}), \quad X + iY \mapsto X + \mathcal{J} Y.$$

Its kernel is

$$\ker \iota_{\mathbb{C}} = \{X + i\mathcal{J} X \,|\, X \in \mathfrak{q}, \mathcal{J} X \in \mathfrak{q}\} = \{Z \in \mathfrak{q}_{\mathbb{C}} \,|\, i\mathcal{J} Z \in \mathfrak{q}_{\mathbb{C}}\}.$$

Thus

$$\ker \iota_{\mathbb{C}} = (\mathfrak{q} \cap \widehat{\mathfrak{q}})_{\mathbb{C}},$$

and we see that (i) holds iff $\iota_{\mathbb{C}}$ is injective. If this is the case, Θ is nothing but complex conjugation of $\mathfrak{q}_{\mathbb{C}}$ w.r.t. $\mathfrak{q}$, which has clearly the properties stated. Moreover, automorphisms of LTS extend always to automorphisms of the standard imbedding; thus Θ extends to an automorphism of $\mathfrak{g}^b$, and the explicit formula follows from Th. 1.3.1 along with the formula $[\mathbf{p}_v, \mathbf{w}] = -T(w,v)$.

Passing to real forms of circled spaces, the claim now follows in the general case of a symmetric space with twist. ■

Definition 2.4.2. A Jordan triple system T is called *faithful* if Condition (ii) from the preceding proposition holds. ■

We will see later that faithfulness is a rather "weak" condition which is satisfied in all interesting cases.

2.5. The structure group and its Lie algebra. Recall that, if X is a vector field on V (considered as a smooth function $V \to V$) and φ a (locally defined) diffeomorphism of V, then the natural forward transport of X is given by

$$(\varphi_* X)(p) = (D\varphi^{-1}(p))^{-1} \cdot X(\varphi^{-1}(p)). \tag{2.3}$$

Lemma 2.5.1. *A locally defined diffeomorphism g of M satisfies the conditions*

$$g(o) = o, \quad g_*(\mathfrak{m}^-) = \mathfrak{m}^-, \quad g_*(\mathfrak{m}^+) = \mathfrak{m}^+ \tag{2.4}$$

(i.e. the locally defined vector field $g_* X$ for $X \in \mathfrak{m}^\pm$ coincides on some domain $U \subset V$ with an element of $\mathfrak{m}^\pm$) if and only if it is represented in the canonical chart by an invertible linear map $g : V \to V$ such that there exists an element $g^\sharp \in \mathrm{Gl}(V)$ with

$$\forall u,v,w \in V: \quad gT(u,v,w) = T(gu, g^\sharp v, gw). \tag{2.5}$$

Proof. Let g satisfy (2.4). Then, for all $v \in V$, $(g_* \mathbf{v})(p) = (\mathrm{D}\, g^{-1}(p))^{-1} v$ has to be a constant vector field, i.e., $\mathrm{D}\, g^{-1}$ is a constant function. Integrating, we obtain that g^{-1} is an affine map of V; it preserves the base point and is hence linear. Conversely, the invertible linear maps preserve $\mathfrak{m}^-$.

Using the description of $\mathfrak{m}^+$ given by Th.2.3.1 (2), we see that the condition $g_*(\mathfrak{m}^+) = \mathfrak{m}^+$ means that, for every $v \in V$, there exists $w \in V$ such that $g_*(\mathbf{p}_v) = \mathbf{p}_w$. If we denote by V_0 the kernel of the linear map $\mathbf{p} : V \to \mathfrak{m}^+, \quad v \mapsto \mathbf{p}_v$, and require w to be in a fixed complementary subspace V_1 of V_0, then the choice of w becomes unique. Now we define $g^\sharp$ to be arbitrary invertible on V_0 and let $g^\sharp(v) := w$ for $v \in V_1$. Then clearly the condition

$$g_*(\mathbf{p}_v) = \mathbf{p}_{g^\sharp v} \tag{2.6}$$

holds for all $v \in V$, and since $\mathbf{p}$ is linear, so is $g^\sharp$. Finally, since

$$g_*(\mathbf{p}_v)(p) = g(\mathbf{p}_v)(g^{-1}(x)) = \frac{1}{2} g T(g^{-1}x, v, g^{-1}x),$$

and T is symmetric in the outer variables, (2.6) is equivalent to (2.5). Moreover, all arguments can be reversed, and we thus see that (2.5) implies that $g_*(\mathfrak{m}^+) = \mathfrak{m}^+$. ∎

Definition 2.5.2. The group

$$\mathrm{Str}(T) = \{ g \in \mathrm{Gl}(V) |\, \exists g^\sharp \in \mathrm{Gl}(V) : \forall u,v,w \in V : \, gT(u,v,w) = T(gu, g^\sharp v, gw) \}$$

described in the previous lemma is called the *structure group* of the JTS T, resp. of the Euler operator E^o which is equivalent to T. Its closure in $\mathrm{End}(V)$ is called the *structure monoid.* ∎

Proposition 2.5.3. *If T is faithful and Θ denotes the involution of $\mathfrak{g}^b$ described in* Prop. 2.4.1*, then for all $g \in \mathrm{Str}(T)$ the element $g^\sharp$ is unique, it belongs again to* $\mathrm{Str}(T)$*, and $\sharp : g \mapsto g^\sharp$ is an involutive automorphism of* $\mathrm{Str}(T)$*, determined by the condition*

$$(g^\sharp)_* = \Theta \circ g_* \circ \Theta.$$

Proof. Everything follows by comparing Eqn. (2.6) with

$$g_* \mathbf{p}_v = g_*(\Theta(-\mathbf{v})) = \Theta \circ (\Theta \circ g_* \circ \Theta)(-\mathbf{v}) = \mathbf{p}_{(\Theta \circ g_* \circ \Theta)(\mathbf{v})}$$

and noticing that $g \mapsto g_*$ is injective. ∎

Lemma 2.5.4. *The structure group is a closed subgroup of* $\mathrm{Gl}(V)$ *whose Lie algebra is*

$$\begin{aligned} \mathfrak{str}(T) &= \{ X \in \mathfrak{gl}(V) |\, [X, \mathfrak{m}^+] \subset \mathfrak{m}^+ \} \\ &= \{ X \in \mathfrak{gl}(V) |\, \exists X^\sharp \in \mathfrak{gl}(V) : \forall u,v,w \in V : \\ &\qquad XT(u,v,w) = T(Xu,v,w) + T(u, X^\sharp v, w) + T(u,v,Xw) \}. \end{aligned}$$

It contains the algebra $\mathrm{Der}(T)$ *of derivations of T as a subalgebra. The endomorphisms $T(u,v)$ ($u,v \in V$) belong to $\mathfrak{str}(T)$, and one may choose $T(u,v)^\sharp = -T(v,u)$.*

Proof. Clearly $\mathrm{Str}(T)$ is an algebraic, hence closed subgroup of $\mathrm{Gl}(V)$. Differentiating, resp. integrating, we see that for $X \in \mathfrak{gl}(V)$ the conditions: $\forall t : (\exp tX)_* \mathfrak{m}^+ \subset \mathfrak{m}^+$ and $[X, \mathfrak{m}^+] \subset \mathfrak{m}^+$ are equivalent, proving the first equation. If we define for $X \in \mathfrak{str}(T)$ the element $X^\sharp$ similarly as in the proof of Lemma 2.5.1 by $[X, \mathbf{p}_v] = \mathbf{p}_{[X^\sharp, \mathbf{v}]}$, we get the second equality (note that this condition does not define $X^\sharp$ uniquely). The derivations of T arise precisely if $X = X^\sharp$.

Finally, it is clear that $\mathfrak{h}^b = [\mathfrak{m}^+, \mathfrak{m}^-]$ satisfies the first condition. In Prop. 1.2.5 we have proved that T is a Jordan triple product, and the defining identity (JT2) of a Jordan triple product (cf. [Be97] or [Sa80]) says precisely that the condition of the second description holds with $T(u,v)^\sharp = -T(v,u)$. ∎

Definition 2.5.5. The Lie algebra $\mathfrak{str}(T)$ is called the *structure algebra of* T, and its subalgebra $\mathfrak{h}^b$ spanned by the $T(u,v)$, $u,v \in V$ is called the *inner structure algebra.* ∎

A JTS T is called *non-degenerate* if the *trace-form* $(x,y) \mapsto \operatorname{Tr}(T(x,y))$ is non-degenerate (cf. [Sa80]).

Proposition 2.5.6. *Let* T *be a non-degenerate JTS. Then*

(i) T *is faithful.*

(ii) *For all* $g \in \operatorname{Str}(T)$, $g^\sharp = (g^*)^{-1}$, *and for all* $X \in \mathfrak{str}(T)$, $X^\sharp = -X^*$, *where the adjoint is taken w.r.t. the trace-form.*

Proof. (i) By definition, T is non-degenerate iff the map

$$^* : V \to V^*, \quad v \mapsto v^* := (x \mapsto \operatorname{Tr} T(x,v))$$

is injective. But * is a composition of

$$A : V \mapsto W \subset \operatorname{Hom}(S^2V, V), \quad v \mapsto T(\cdot, v, \cdot)$$

and

$$\kappa : \operatorname{Hom}(S^2V, V) \to V^*, \quad B \mapsto (x \mapsto \operatorname{Tr}(B(x,\cdot))).$$

Therefore A has to be injective, i.e. T is faithful.

(ii) By definition of $\sharp$, A has the equivariance property $g \cdot A(v) = A(g^\sharp v)$, and κ is natural, i.e $\kappa(g \cdot B) = (g^*)^{-1}\kappa(B)$. When we identify V and V^* via *, this proves the claim. Similarly for $X^\sharp$. ∎

2.6. The conformal Lie algebra: general properties.

Definition 2.6.1. The Lie algebra

$$\mathfrak{co}(T) := \mathfrak{m}^- \oplus \mathfrak{str}(T) \oplus \mathfrak{m}^+$$

is called the *conformal Lie algebra of* T. Recall that its subalgebra $\mathfrak{g}^b = \mathfrak{m}^- \oplus \mathfrak{h}^b \oplus \mathfrak{m}^+$ is called the *inner conformal Lie algebra of* T. ∎

Definition 2.6.2. A (possibly only locally defined) vector field X on V is called T*-conformal* if it is of class $\mathcal{C}^3$ and, for all $p \in V$ where X is defined, the first differential $\mathrm{D}\, X(p)$ belongs to $\mathfrak{str}(T)$. It is called *T-projective* if in addition, for all p where X is defined, the second differential $\mathrm{D}^2\, X(p) : V \otimes V \to V$ belongs to the space

$$W := \{T(\cdot, v, \cdot) |\, v \in V\} \subset \operatorname{Hom}(V \otimes V, V).$$ ∎

Proposition 2.6.3. *The Lie algebra* $\mathfrak{co}(T)$ *is a Lie algebra of* T*-conformal and* T*-projective vector fields.*

Proof. Both properties are trivial for the vector fields belonging to $\mathfrak{m}^-$ or $\mathfrak{str}(V)$. Let us assume that $X = \mathbf{p}_v \in \mathfrak{m}^+$. Then $\mathrm{D}\, X(p) = T(p,v) \in \mathfrak{str}(T)$ and $\mathrm{D}^2\, X(p) = T(\cdot, v, \cdot) \in W$. ∎

We will prove in Chapter 4 that in the non-degenerate case the converse also holds.

Theorem 2.6.4. *Every derivation of the conformal Lie algebra is an inner derivation, and the adjoint representation*

$$\operatorname{ad} : \mathfrak{co}(T) \to \operatorname{Der}(\mathfrak{co}(T)), \quad X \mapsto \operatorname{ad}(X)$$

is an isomorphism onto.

Proof. [Koe69b, Satz 3.1] ∎

3. Conformal group and conformal completion

3.1. Definition of the conformal group. The conformal group is a Lie group with Lie algebra $\mathfrak{co}(T)$, coming in a certain realization, namely as a *group of birational maps of the vector space* V. Since birational maps are not defined everywhere, we start with *local diffeomorphisms* φ of V, i.e. diffeomorphisms defined on some domain $U \subset V$.

Theorem 3.1.1. *Let $\varphi : U \to \varphi(U) \subset V$ be a local ($\mathcal{C}^1$-regular) diffeomorphism of V such that*

$$\varphi_*(\mathfrak{co}(T)) = \mathfrak{co}(T),$$

i.e. if for all $X \in \mathfrak{co}(T)$, the vector field $\varphi_*(X)$ coincides on $\varphi(U)$ with an element of $\mathfrak{co}(T)$. Then φ has a unique extension to a birational map $\widetilde{\varphi}$ of V such that $\widetilde{\varphi}_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$. The birational maps thus obtained form a group.

Proof. Recall formula (2.1) for the push-forward of a vector field. Since $(\varphi^{-1})_* = (\varphi_*)^{-1} =: \varphi^*$, the conditions $\varphi_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$ and $\varphi^*(\mathfrak{co}(T)) = \mathfrak{co}(T)$ are equivalent.

Let $X = \mathbf{v}$ with $v \in V$ (constant vector field). By assumption, $\varphi^*\mathbf{v}$ coincides on U with an element of $\mathfrak{co}(T)$. Since $\mathfrak{co}(T)$ is a Lie algebra of (quadratic) polynomial vector fields (Th. 2.3.1), the map $U \to V$, $p \mapsto (\varphi^*\mathbf{v})(p) = (\mathrm{D}\,\varphi(p))^{-1}v$ is (quadratic) polynomial for all $v \in V$, and thus

$$d_\varphi : U \to \mathrm{Gl}(V) \subset \mathrm{End}(V), \quad p \mapsto d_\varphi(p) := (\mathrm{D}\,\varphi(p))^{-1}$$

is (quadratic) polynomial. Now consider the Euler operator $X = E^o \in \mathfrak{co}(T)$. The same arguments as before show that

$$n_\varphi : U \to V, \quad p \mapsto (\varphi^* E^o)(p) = (\mathrm{D}\,\varphi(p))^{-1} \cdot \varphi(p)$$

is (quadratic) polynomial. Therefore

$$\varphi : U \to V, \quad p \mapsto \varphi(p) = d_\varphi(p)^{-1} \cdot n_\varphi(p)$$

is rational. We continue n_φ and d_φ to polynomials on V, and define

$$\widetilde{\varphi}(x) := d_\varphi(x)^{-1} \cdot n_\varphi(x)$$

for all x with $\det d_\varphi(x) \neq 0$ (since $\det d_\varphi$ is a non-zero polynomial, $\widetilde{\varphi}$ is defined almost everywhere). Then $\widetilde{\varphi}$ is a birational map, its inverse given by $\widetilde{\varphi^{-1}}$. Let us prove that $\widetilde{\varphi}_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$. For all $X \in \mathfrak{co}(T)$,

$$((\widetilde{\varphi})^* X)(p) = (\mathrm{D}\,\widetilde{\varphi}(p))^{-1} X(\widetilde{\varphi}(p)) = d_\varphi(p) X(\widetilde{\varphi}(p))$$

is rational in p and coincides for $p \in U$ with the polynomial $Y := \varphi^* X \in \mathfrak{co}(T)$, whence $((\widetilde{\varphi})^* X)(p) = \varphi^* X(p)$ everywhere. In other words, $(\widetilde{\varphi})^*(\mathfrak{co}(T)) \subset \mathfrak{co}(T)$. The same being true for φ^{-1}, we actually have equality.

If φ and ψ with $\varphi_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$ and $\varphi_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$ are given, then (although φ and ψ may be not composable) $\widetilde{\varphi}$ and $\widetilde{\psi}$ are always composable on a dense open set, and

$$(\widetilde{\varphi} \circ \widetilde{\psi})_*(\mathfrak{co}(T)) = \widetilde{\varphi}_*(\widetilde{\psi}_*(\mathfrak{co}(T))) = \mathfrak{co}(T).$$

This, and the remark made above that $(\widetilde{\varphi})^{-1} = \widetilde{\varphi^{-1}}$, prove the last claim. ∎

Definition 3.1.2. The group of birational maps of V defined in the preceding theorem is denoted by $\mathrm{Co}(T)$ and is called the *conformal group of* T. For any $g \in \mathrm{Co}(T)$, the quadratic polynomial

$$d_g : V \to \mathrm{End}(V), \quad x \mapsto (\mathrm{D}\, g(x))^{-1}$$

is called its *denominator*, and the quadratic polynomial

$$n_g : V \to V, \quad x \mapsto (\mathrm{D}\, g(x))^{-1} \cdot g(x) = d_g(x) \cdot g(x)$$

is called its *numerator* (cf. the preceding proof). ■

The following result is due to Koecher ([Koe69b, Satz 2.2]). In order to introduce some relevant notation, we include its proof.

Theorem 3.1.3. *The representation*

$$* : \mathrm{Co}(T) \to \mathrm{Aut}(\mathfrak{co}(T)), \quad g \mapsto g_*$$

is injective, and its image $\mathrm{Co}_*(T)$ *is an open subgroup of* $\mathrm{Aut}(\mathfrak{co}(T))$*; in particular,* $\mathrm{Co}(T)$ *has the structure of a Lie group with Lie algebra* $\mathfrak{co}(T)$.

Proof. We have seen above that $g(p) = d_g(p)^{-1} n_g(p)$ with $d_g(p)v = (g^*\mathbf{v})(p)$ and $n_g(p) = (g^*E)(p)$. This defines an inverse of $*$ from $\mathrm{Co}_*(T)$ onto $\mathrm{Co}(T)$; thus $*$ is injective.

In order to prove that $\mathrm{Co}_*(T)$ is open in $\mathrm{Aut}(\mathfrak{co}(T))$ we are going to describe some special elements in $\mathrm{Co}(T)$. For $X \in \mathfrak{co}(T)$ let φ_t be the local flow of X. If $X \in \mathfrak{m}^-$ or $X \in \mathfrak{str}(T)$, then the flows are global on V; we write $\exp(X) := \varphi_1$. Explicitly, we have $\exp(\mathbf{v}) = t_v$ (translation by v), and for $X \in \mathfrak{str}(T)$, $\exp(X)$ is given by the usual exponential of a matrix; according to Lemma 2.5.4 it belongs to the structure group $\mathrm{Str}(T)$.

Now consider $X \in \mathfrak{m}^+$. Since X vanishes of order 2 at 0, there is a neighbourhood U of 0 in V such that $\varphi_t(x)$ is defined for $x \in U$ and $t = 1$ (cf. [Be96a, annexe (A2)]). We let

$$\exp X : U \to V, \quad x \mapsto \varphi_1(x).$$

Since $X \in \mathfrak{co}(T)$, it follows that $(\varphi_t)_*$ preserves $\mathfrak{co}(T)$ in the sense of Th. 3.1.1; in particular $\exp(X) \in \mathrm{Co}(T)$. For simplicity of notation, we identify $\exp(X)$ with its rational continuation guaranteed by Th. 3.1.1. Then the relation

$$\forall Y \in \mathfrak{co}(T) : \quad (\exp X)_* Y = e^{-\operatorname{ad}(X)} Y$$

holds (the sign-change comes in because group actions and Lie algebras of vector fields are defined on levels which behave contravariantly to each other). Summing up, $e^{\operatorname{ad}(X)}$ belongs to $\mathrm{Co}_*(T)$ for all $X \in \mathfrak{m}^{\pm}$ and $X \in \mathfrak{str}(T)$; thus the identity component of the group $\mathrm{Int}(\mathfrak{co}(T))$ of inner automorphisms, generated by $e^{\operatorname{ad}(\mathfrak{co}(T))}$, belongs to $\mathrm{Co}_*(T)$. But this is equal to the identity component of $\mathrm{Aut}(\mathfrak{co}(T))$ because $\operatorname{ad}(\mathfrak{co}(T)) = \mathrm{Der}(\mathfrak{co}(T))$ (Th. 2.6.4). Thus $\mathrm{Co}_*(T)$ is open in $\mathrm{Aut}(\mathfrak{co}(T))$. ■

3.2. Conformality. We will precise now in which sense the conformal group is indeed "conformal". The question whether all conformal maps belong to the group $\mathrm{Co}(T)$ leads to the *Liouville theorem* which will be discussed in the next chapter.

Definition 3.2.1. Let V be a vector space and $G \subset \mathrm{Gl}(V)$ a closed subgroup. We say that a locally defined diffeomorphism g of V is *G-conformal*, if, for all x where g is defined,

$$\mathrm{D}\, g(x) \in G.$$

■

Theorem 3.2.2. *All elements of* $\mathrm{Co}(T)$ *are* $\mathrm{Str}(T)$*-conformal.*

Proof. Given $x \in V$ with $\det d_g(x) \neq 0$, we let $y := g(x)$ and $g' := t_{-y} \circ g \circ t_x$. Then $g' \in \mathrm{Co}(T)$, $g'(0) = 0$ and $\mathrm{D}\, g'(0) = \mathrm{D}\, g(x)$. Replacing g by g', we may assume that $x = 0$ and $g(0) = 0$.

We claim that then $g_*(\mathfrak{m}^+) \subset \mathfrak{m}^+$. In order to prove this, note that $\mathfrak{m}^+$ is the subalgebra of elements of $\mathfrak{co}(T)$ vanishing of order 2 at the origin. Differentiating the expression $(g_*X)(p) = \mathrm{D}\, g(g^{-1}(x)) \cdot X(g^{-1}(p))$, we obtain

$$\begin{aligned} \mathrm{D}(g_*X)(p) = &\left(\mathrm{D}^2\, g(g^{-1}(p)) \cdot X(g^{-1}(p))\right) \circ \mathrm{D}\, g^{-1}(p) \\ &+ \mathrm{D}\, g(g^{-1}(p)) \circ \mathrm{D}\, X(g^{-1}(p)) \circ \mathrm{D}\, g^{-1}(p). \end{aligned} \tag{3.1}$$

If $g(0) = 0$, $X(0) = 0$ and $\mathrm{D}\, X(0) = 0$, then the right-hand side vanishes at the origin. Also, $(g_*X)(0) = 0$, and thus $g_*(\mathfrak{m}^+) \subset \mathfrak{m}^+$.

Next we want to show that for all $X \in \mathfrak{m}^+$, $g_*X = (\mathrm{D}\, g(0))_*X$. In order to prove this, we differentiate the expression (3.1) and evaluate at 0. Then, under our assumptions on g and X, only one term remains, namely

$$\mathrm{D}^2(g_*X)(0) = \mathrm{D}\, g(0) \circ \mathrm{D}^2\, X(0) \circ \mathrm{D}\, g(0)^{-1} \otimes \mathrm{D}\, g(0)^{-1}.$$

If we replace in this expression g by $\mathrm{D}\, g(0)$, the right-hand side does not change; therefore

$$\mathrm{D}^2(g_*X)(0) = \mathrm{D}^2((\mathrm{D}\, g(0))_*X)(0).$$

Since both g_*X and $(\mathrm{D}\, g(0))_*X$ are are homogeneous quadratic vector fields, it follows that $g_*X = (\mathrm{D}\, g(0))_*X$.

We have shown that $\mathfrak{m}^+ = g_*(\mathfrak{m}^+) = (\mathrm{D}\, g(0))_*(\mathfrak{m}^+)$. According to Lemma 2.5.1 and Def. 2.5.2, $\mathrm{D}\, g(0)$ belongs thus to the structure group. ■

Corollary 3.2.3. *For all* $g \in \mathrm{Co}(T)$*, the image of the polynomial* d_g *is contained in the structure monoid.*

Proof. This is an immediate consequence of the definition of the structure monoid as the closure of the structure group in $\mathrm{End}(V)$. ■

3.3. Projectivity. Conformality was defined by a condition on the *first* differential. Similarly, projectivity will be defined by an additional condition on the *second* differential. The question whether all transformations which are projective in this sense belong to $\mathrm{Co}(T)$ leads to the *fundamental theorem* (Th. 4.1 (i)).

Definition 3.3.1. A locally defined diffeomorphism g of V is called *T-projective* if it is $\mathrm{Str}(T)$-conformal and in addition for all x where g is defined,

$$(\mathrm{D}\, g(x))^{-1} \circ \mathrm{D}^2\, g(x) \in W,$$

where $W = \{T(\cdot, v, \cdot) | \, v \in V\} \subset \mathrm{Hom}(V \otimes V, V)$ is the subspace introduced in Def. 2.6.2. ■

We give a more conceptual definition of this condition: it is equivalent to requiring that

$$(g^*\nabla - \nabla)_x \in W, \tag{3.2}$$

where ∇ is the canonical flat connection of V given by $(\nabla_X Y)_p = \mathrm{D}\, Y(p) \cdot X(p)$ and $g^*\nabla$ is the push-forward of a connection by a diffeomorphism g. In other words, $g^*\nabla - \nabla$ is a section of the subbundle $\mathbf{W}$ of $\mathrm{Hom}(S^2(TV), V)$ with constant fiber $\mathbf{W}_p = W$. Conformality means just that g preserves $\mathbf{W}$. Therefore, if both g and h are T-projective, then

$$(gh)^*\nabla - \nabla = h^*(g^*\nabla - \nabla) + g^*\nabla - \nabla$$

is again a section of $\mathbf{W}$. This means that the composition of two T-projective maps, if it is defined, is again T-projective. In other words, locally defined T-projective diffeomorphisms form a *pseudogroup of diffeomorphisms* (cf. [Ko72] for the formal definition).

Theorem 3.3.2. *All elements of* $\mathrm{Co}(T)$ *are* T*-projective.*

Proof. We know already that all elements of $\mathrm{Co}(T)$ are $\mathrm{Str}(T)$-conformal (Th. 3.2.2). In order to prove T-projectivity, we reduce as in the proof of Th. 3.2.2 to the case $x = 0$ and $g(x) = x$; composing with $\mathrm{D}\, g(0)^{-1}$, we may further reduce to the case $\mathrm{D}\, g(0) = \mathrm{id}_V$.

We specialize Eqn. (3.1) to the Euler operator $X = E$:

$$\mathrm{D}(g_*E)(p) = (\mathrm{D}^2\, g(g^{-1}(p)) \cdot g^{-1}(p)) \circ \mathrm{D}\, g^{-1}(p) + \mathrm{id}_V .$$

Thus, under our assumptions on g, $\mathrm{D}(g_*E)(0) = \mathrm{id}_V$, and differentiating further and evaluating at 0, we find that

$$\mathrm{D}^2(g_*E)(0) = (\mathrm{D}^2\, g)(0).$$

Since $g_*E \in \mathfrak{co}(T)$ by definition of $\mathrm{Co}(T)$, it follows from Prop. 2.6.3 that $(\mathrm{D}^2\, g)(0) \in W$. ∎

3.4. Fine structure of the conformal group. We return to the general case.

Proposition 3.4.1. *An element* $g \in \mathrm{Co}(T)$ *is uniquely determined by its 2-jet at one point* $p \in V$ *with* $d_g(p) \neq 0$*.*

Proof. Let $g_1, g_2 \in \mathrm{Co}(T)$ have the same 2-jet at a point $p \in V$. Then $g := g_1^{-1} g_2$ has the 2-jet of the identity at p. Replacing g by $t_{-g(p)} \circ g \circ t_p$, we may further assume that $p = 0$. We thus have to show that the only element $g \in \mathrm{Co}(T)$ with $g(0) = 0$, $\mathrm{D}\, g(0) = \mathrm{id}_V$ and $\mathrm{D}^2\, g(0) = 0$ is the identity. The arguments proving Th. 3.3.2 show that, under these conditions on g, $g_*E = E$. Then, for all $v \in V$,

$$[E, g_*\mathbf{v}] = [g_*E, g_*\mathbf{v}] = g_*[E, \mathbf{v}] = -g_*\mathbf{v}.$$

However, the only solutions of the differential equation $[E, X] = -X$ are the constant vector fields, therefore $(g_*\mathbf{v})(p) = (\mathrm{D}\, g^{-1}(p))^{-1} v$ is constant; this means that $\mathrm{D}\, g^{-1} : V \to \mathrm{End}(V)$ is constant and thus g^{-1} is linear. Since $\mathrm{D}\, g(0) = \mathrm{id}_V$ by assumption, this implies that $g = \mathrm{id}_V$. ∎

Proposition 3.4.2. *Let* $\mathrm{Co}'(T) := \{g \in \mathrm{Co}(T) | \, d_g(0) \neq 0\}$*. Then the map*

$$\kappa : \mathrm{Co}'(T) \to V \times \mathrm{Str}(T) \times W, \quad g \mapsto (g(0), \mathrm{D}\, g(0), \mathrm{D}\, g(0)^{-1} \circ \mathrm{D}^2\, g(0))$$

is bijective.

Proof. The map is well-defined since g is regular at 0 by definition of $\mathrm{Co}'(T)$, and $\mathrm{D}\,g(0)$ and $\mathrm{D}\,g(0)^{-1}\,\mathrm{D}^2\,g(0)$ lie in the correct spaces according to Th. 3.2.2 and 3.3.2. Since these data describe precisely the 2-jet of g, the preceding proposition now implies that κ is injective.

In order to prove that κ is surjective, we have to calculate the 2-jet of $\exp X$ for $X \in \mathfrak{m}^+$ (defined in the proof of Th. 3.1.3). It is easily seen that $(\exp X)(0) = 0$ and $(D \exp X)(0) = \mathrm{id}_V$ (cf. [Be96a, Appendix A2]). Therefore, as we have seen in the proof of Th.3.3.2,

$$\mathrm{D}^2(\exp X)(0) = \mathrm{D}^2((\exp X)_* E)(0),$$

and we can calculate

$$\begin{aligned} \mathrm{D}^2(\exp X)(0) &= \mathrm{D}^2(e^{-\operatorname{ad}(X)}E)(0) \\ &= \mathrm{D}^2(E - [X,E])(0) \\ &= \mathrm{D}^2(X)(0) \end{aligned}$$

since $[X,E] = -X$ for all $X \in \mathfrak{m}^+$ und thus $\operatorname{ad}(X)^k E = 0$ for all $k > 1$.

Now, given $v,w \in V$, $g \in \mathrm{Str}(T)$, we let $g := t_v \circ g \circ \exp(\mathbf{p}_w) \in \mathrm{Co}(T)$ and prove that $\kappa(g) = (v, g, 2T(\cdot, w, \cdot))$: In fact, $g(0) = v$, $\mathrm{D}\,g(0) = g$ and $\mathrm{D}\,g(0)^{-1} \circ \mathrm{D}^2\,g(0) = (\mathrm{D}^2 \exp(\mathbf{p}_w))(0) = \mathrm{D}^2(\mathbf{p}_w)(0) = 2T(\cdot, w, \cdot)$. ∎

We collect the preceding results. By

$$\mathrm{Co}(T)_0 := \{g \in \mathrm{Co}'(T) |\, d_g(0) \neq 0, g(0) = 0\}$$

we denote the "stabilizer of the base point", and by

$$\mathrm{Co}(T)_{00} := \{g \in \mathrm{Co}(T)_0 |\, Dg(0) = \mathrm{id}_V\}$$

the kernel of the first isotropy representation $g \mapsto \mathrm{D}\,g(0)$ of $\mathrm{Co}(T)_0$.

Theorem 3.4.3. *$\mathrm{Co}(T)$ is a Lie group with Lie algebra $\mathfrak{co}(T)$. It is generated by $\exp(\mathfrak{m}^-)$, $\exp(\mathfrak{m}^+)$ and $\mathrm{Str}(T)$. More precisely:*

(1) $\mathrm{Co}(T)_{00} = \exp(\mathfrak{m}^+)$.

(2) $\mathrm{Co}(T)_0 = \mathrm{Str}(T) \exp(\mathfrak{m}^+)$*; this is a semidirect product.*

(3) $\mathrm{Co}'(T) = \exp(\mathfrak{m}^-)\,\mathrm{Str}(T) \exp(\mathfrak{m}^+)$*, and this decomposition is unique, i.e. every element φ of the open dense set $\mathrm{Co}'(T) \subset \mathrm{Co}(T)$ has a unique decomposition*

$$g = t_{g(0)}\ \mathrm{D}\,g(0)\ \exp(\mathbf{p}_w), \qquad t_{g(0)} \in \exp(\mathfrak{m}^-), \mathrm{D}\,g(0) \in \mathrm{Str}(V),\ \mathbf{p}_w \in \mathfrak{m}^+.$$

(4) $\mathrm{Co}(T) = \exp(\mathfrak{m}^-)\,\mathrm{Str}(T) \exp(\mathfrak{m}^+) \exp(\mathfrak{m}^-)$ *(the corresponding decomposition is not unique).*

(5) *If T is faithful, then $\mathrm{Co}(T)$ carries an involution Θ such that $\Theta(\mathrm{Co}'(T))^{-1} = \mathrm{Co}'(T)$ and for all $v,w \in V$, $h \in \mathrm{Str}(T)$,*

$$\Theta(t_v h \exp(\mathbf{p}_w))^{-1} = t_{-w} (h^\sharp)^{-1} \exp(\mathbf{p}_{-v}). \tag{3.3}$$

The decomposition from part (3) *can be written*

$$g = t_{g(0)}\ \mathrm{D}\,g(0)\,\Theta(t_{-\Theta(g)^{-1}(0)}), \tag{3.4}$$

and the relation

$$\mathrm{D}(\Theta(g^{-1}))(0) = \Theta(\mathrm{D}\,g(0))^{-1} \tag{3.5}$$

holds.

(6) *If T is associated to a Jordan algebra with Jordan inverse $j(x) = x^{-1}$ via $T(u,v,w) = 2(u(vw) - v(uw) + (uv)w)$, then* $\mathrm{Co}(T)$ *is generated by the translations, elements of* $\mathrm{Str}(T)$ *and j. In this case, T is faithful and Θ is given by conjugation with j.*

Proof. Claims (1), (2) and (3) are a direct consequence of Prop. 3.4.2.

(4): If $g \in \mathrm{Co}(T)$, we choose a point p with $d_g(p) \neq 0$; we let $g' := g \circ t_p$; then $d_{g'}(0) \neq 0$, and we can apply (3).

(5): The involutive automorphism Θ of $\mathfrak{co}(T)$ described in Prop. 2.4.1 induces by conjugation an involution of the group $\mathrm{Aut}(\mathfrak{co}(T))$ which is again denoted by Θ. We have to show that the subgroup $\mathrm{Co}_*(T) \subset \mathrm{Aut}(\mathfrak{co}(T))$ is stable under Θ. But this is easily checked on the generators of $\mathrm{Co}(T)$ exhibited in Part (4):

$$\Theta(t_v) = \Theta(\exp(-\mathbf{v})) = \exp(\Theta(-\mathbf{v})) = \exp(\mathbf{p}_v)$$

and $\Theta(g) = g^\sharp$ for $g \in \mathrm{Str}(T)$ (cf. Prop. 2.5.3). This also proves Equation (3.3) and that $\Theta(\mathrm{Co}'(T))^{-1} = \mathrm{Co}'(T)$.

If $g = vhn$ abreviates the decomposition (3) of $g \in \mathrm{Co}'(T)$, then $\Theta(g)^{-1} = \Theta(n)^{-1}\Theta(h)^{-1}\Theta(v)^{-1}$ is the corresponding decomposition of $\Theta(g)^{-1}$, and the unicity statement in (3) implies that $\Theta(n)^{-1} = t_{\Theta(g)^{-1}(0)}$ and $\Theta(h)^{-1} = \mathrm{D}(\Theta(g)^{-1}(0)$. This yields formulas (3.4) and (3.5).

(6): The Jordan inverse j is a birational map of V with differential $\mathrm{D}\, j(x) = -\frac{1}{2}P(x)^{-1}$ where $P(x) = T(x,\cdot,x)$ (cf. [FK94, Ch.II]). Thus, for $v \in V$, $jt_{tv}j$ is a one-parameter group of birational maps of V. It is generated by the vector field

$$X(p) = \frac{d}{dt}|_{t=0} j(tv + jp) = \mathrm{D}\, j(j(p)) \cdot v = -\frac{1}{2}P(x)v = -\mathbf{p}_v(p).$$

It follows that $\mathfrak{m}^+ = j_*(\mathfrak{m}^-)$ and therefore $j_*(\mathfrak{co}(T)) = \mathfrak{co}(T)$, and thus $j \in \mathrm{Co}(T)$. Comparing with Prop. 2.4.1, we see that T is faithful and $\Theta = j_*$. ■

Note that our proof, in contrast to related results of Loos ([Lo79]) and Koecher ([Koe69a,b]), does not use other algebraic identities than the ones defining a JTS. Therefore this approach can be used to derive alternative proofs of many identies for Jordan pairs and -triple systems (cf. the identities JP1 – JP35 in [Lo77]). Since methods and results are similar as in the Jordan algebra case treated in [Be98a] (item (6) of the preceding theorem), we content ourselves here with defining the principal objects appearing in Jordan-theoretic formulas.

Definition 3.4.4. The polynomial

$$B : V \times V \to \mathrm{End}(V), \quad (x,y) \mapsto B(x,y) := \mathrm{id}_V - T(x,y) + \frac{1}{4}P(x)P(y),$$

where $P(z) = T(z,\cdot,z)$, is called the *Bergman polynomial of the JTS T*. ■

Proposition 3.4.5.

(i) *For all $x, y \in V$,*

$$d_{\exp(\mathbf{p}_y)}(x) = B(x,y).$$

(ii) *For all $x, y \in V$,*

$$n_{\exp(\mathbf{p}_y)}(x) = x - \frac{1}{2}P(x)y.$$

(iii) *If* $\mathrm{Det}\, B(x,y) \neq 0$*, then* $B(x,y) \in \mathrm{Str}(T)$ *and*

$$\exp(\mathbf{p}_y)(x) = B(x,y)^{-1}(x - \frac{1}{2}P(x)y).$$

Proof. One uses the definition of the nominators and denominators in a similar way as in the proof of [Be98a, Prop. 1.2.1]. ■

Note that, if T is faithful, Part (iii) together with $\Theta(t_y)^{-1} = \exp(\mathbf{p}_y)$, yields the explicit formula

$$(\Theta(t_y))^{-1}(x) = B(x,y)^{-1} \cdot (x - \frac{1}{2}P(x)y). \tag{3.6}$$

In [Lo77] and [Lo79] this expression is called the *quasi-inverse* and is denoted by x^y.

3.5. Conformal completion and structure bundle. We denote by $P := \mathrm{Str}(T)\exp(\mathfrak{m}^+)$ the stabilizer $\mathrm{Co}(T)_0$ and by $\mathfrak{p} := \mathfrak{str}(T) \oplus \mathfrak{m}^+$ its Lie algebra. Then P is closed in $\mathrm{Co}'(T)$ and in $\mathrm{Co}(T)$.

Definition 3.5.1. The space $V^c := \mathrm{Co}(T)/P$ as well as the map

$$V \to V^c, \quad v \mapsto t_v P$$

are called the *conformal completion of* V (w.r.t T). If V^c is compact, then it is also called the *conformal compactification of* V. ■

Proposition 3.5.2. *The conformal completion is an imbedding with open dense image.*

Proof. The map $V \to V^c$ is injective since $P \cap t_V = \{\mathrm{id}_V\}$. Its image is open dense since it is the image of the open dense set $\mathrm{Co}(T)' \subset \mathrm{Co}(T)$ under the surjective submersion $\pi : \mathrm{Co}(T) \to V^c$, $g \mapsto gP$. ■

We identity V with the corresponding open dense domain of V^c. It is clear that $\mathfrak{co}(T)$ is realized as an algebra of vector fields on V^c and that $\mathrm{Co}(T)$ is precisely the group normalizing this algebra. It suffices to assume that diffeomorphisms normalizing $\mathfrak{co}(T)$ are only locally defined in order to conclude a global continuation onto V^c; in particular, all elements of $\mathfrak{co}(T)$ are complete vector fields on V^c, and in this sense V^c is "complete". It is known that, if T is non-degenerate, then P is a parabolic subgroup of the semisimple group $\mathrm{Co}(T)$ and hence V^c is compact.

Proposition 3.5.3. *The subbundle* $\mathbf{W}$ *of* $\mathrm{Hom}(S^2(TV), TV)$ *with constant fiber* $\mathbf{W}_p = W$ *(where* W *is as in* Def. 2.6.2*) has a unique extension to a* $\mathrm{Co}(T)$*-invariant subbundle (again denoted by* $\mathbf{W}$*) of the bundle* $\mathrm{Hom}(S^2(T(V^c)), T(V^c))$.

Proof. We only have to check that $W = \mathbf{W}_0$ is invariant under the natural action of the stabilizer P. But this is clear since, for all $p \in P$, $\mathrm{D}\,p(0)$ belongs to $\mathrm{Str}(T)$ which is precisely the linear group preserving W. ■

The arguments of the preceding proof show that, more generally, a locally defined diffeomorphism preserves the structure bundle if and only if it is $\mathrm{Str}(T)$-conformal.

Definition 3.7.4. The subbundle $\mathbf{W}$ of $\mathrm{Hom}(S^2T(V^c), T(V^c))$ is called the *structure bundle associated to* T. ■

4. Liouville theorem and fundamental theorem

Let us recall from Definitions 2.6.2, 3.2.1 and 3.3.1 the notions of *conformality* and *projectivity*: given a JTS T on a vector space V, we introduce the following conditions on a vector field X, resp. on a locally defined diffeomorphism g: for all $p \in V$, where the expression is defined,

$$\begin{array}{ll} \text{(C1)} & \mathrm{D}\,X(p) \in \mathfrak{str}(T), \\ \text{(P1)} & \mathrm{D}^2\,X(p) \in W, \\ \text{(C2)} & \mathrm{D}\,g(p) \in \mathrm{Str}(T), \\ \text{(P2)} & (\mathrm{D}\,g(p))^{-1} \circ \mathrm{D}^2\,g(p) \in W. \end{array} \tag{4.1}$$

Then conditions (C1) and (C2) define T-conformality and (P1) and (P2) define T-projectivity.

Theorem 4.1. *Let T be a non-degenerate JTS containing no ideal isomorphic to $\mathbb{R}$ or $\mathbb{C}$.*

(i) (The Fundamental Theorem) *Any local T-projective diffeomorphism (of class $\mathcal{C}^4$) is birational, and its birational continuation belongs to* $\mathrm{Co}(T)$. *In other words,* $\mathrm{Co}(T)$ *is exactly the group of T-projective transformations.*

(ii) (The Liouville-Theorem) *If T is associated to a semisimple Jordan algebra via $T(x,y,z) = \frac{1}{2}(x(yz) - y(xz) + (xy)z)$, then every local* $\mathrm{Str}(T)$*-conformal diffeomorphism is birational, and its birational continuation belongs to the group* $\mathrm{Co}(T)$, *i.e. it is a composition of the translations t_v, $v \in V$, of elements of* $\mathrm{Str}(V)$ *and of the Jordan inverse $j(x) = x^{-1}$.*

Proof. We have already seen that elements of $\mathrm{Co}(T)$ satisfy the conditions (C2) and (P2) (Th. 3.2.2 and Th. 3.3.2). The proof of the converse occupies the remainder of this chapter. It is done in both cases by proving first an algebraic version on the level of vector fields, which we state next; the corresponding statement on the group-level is then deduced.

Theorem 4.2. *Let assumptions be as in the preceding theorem.*

(i) (Infinitesimal version of the Fundamental Theorem) *Any locally defined T-projective vector field (of class $\mathcal{C}^3$) is polynomial and belongs to* $\mathfrak{co}(T)$. *In other words,* $\mathfrak{co}(T)$ *is precisely the space of T-conformal vector fields.*

(ii) (Infinitesimal version of the Liouville Theorem) *If T is associated to a semisimple Jordan algebra, then* $\mathfrak{co}(T)$ *is exactly the space of T-conformal vector fields.*

Proof. We have already seen that elements of $\mathfrak{co}(T)$ satisfy the conditions (C1) and (P1) of Eqn. (4.1) (Prop. 2.6.3). Conversely, let X be a vector field defined on some domain U and satisfying (C1) and (P1). We abbreviate $L := \mathrm{Str}(T) \subset \mathrm{Gl}(V)$ and $\mathfrak{l} := \mathfrak{str}(T) \subset \mathfrak{gl}(V)$. Then by (C1), $\mathrm{D}\,X$ is a map

$$\mathrm{D}\,X : U \to \mathfrak{l} \subset \mathrm{Hom}(V,V), \quad p \mapsto \mathrm{D}\,X(p);$$

therefore the second differential $\mathrm{D}^2 X = \mathrm{D}(\mathrm{D}\,X)$ takes values in $\mathrm{Hom}(V,\mathfrak{l})$. On the other hand, $\mathrm{D}^2 X(p)$ is a *symmetric* bilinear map $V \times V \to V$. Thus $\mathrm{D}^2 X$ can be considered as a map

$$\begin{aligned}\mathrm{D}^2 X : U \to \mathrm{Hom}_s(V,\mathfrak{l}) &:= \mathrm{Hom}(V,\mathfrak{l}) \cap \mathrm{Hom}(S^2V,V) \\ &= \{T : V \to \mathfrak{l} \,|\, \forall u,v \in V : T(u)v = T(v)u\}.\end{aligned}$$

Deriving further, we see that $\mathrm{D}^k X$ takes values in

$$\mathrm{Hom}_s(S^{k-1}V,\mathfrak{l}) := \mathrm{Hom}(S^kV,V) \cap \mathrm{Hom}(S^{k-1}V,\mathfrak{l}).$$

Note that this is a L- and a $\mathfrak{l}$-submodule of $\mathrm{Hom}(S^kV,V)$ (sometimes denoted by $\mathfrak{l}^{(k-1)}$, cf. [Ko72]). Similarly, condition (P1) implies that $\mathrm{D}^3 X$ takes values in

$$\begin{aligned}\mathrm{Hom}_s(V,W) &:= \mathrm{Hom}(V,W) \cap \mathrm{Hom}(S^3V,V) \\ &= \{\alpha : S^3V \to V \,|\, \forall x \in V : \alpha(x,\cdot,\cdot) \in W\}.\end{aligned}$$

Note that

$$W \subset \mathrm{Hom}_s(V,\mathfrak{l})$$

since $T(u,x) \in \mathfrak{str}(T) = \mathfrak{l}$ and $T(u,x)v = T(v,x)u$ for all $u,v,x \in V$. Thus condition (P1) is a refinement of condition (C1).

Lemma 4.3.

(i) *If* $\mathrm{Hom}_s(V,W)=0$*, then the Lie algebra* $\mathfrak{co}(T)$ *is precisely the space of* T*-projective vector fields.*

(ii) *If* $\mathrm{Hom}_s(V,\mathfrak{g})=W$ *and* $\mathrm{Hom}_s(S^2V,\mathfrak{g})=0$*, then the Lie algebra* $\mathfrak{co}(T)$ *is precisely the space of* $\mathfrak{l}$*-conformal vector fields.*

Proof. (i) If $\mathrm{Hom}_s(V,W)=0$ and X is T-conformal, then $\mathrm{D}^3 X=0$ and we can integrate. We then find that X is in fact quadratic, given by

$$X(x)=X(0)+\mathrm{D}\,X(0)\cdot x+\frac{1}{2}\mathrm{D}^2 X(0)\cdot(x,x).$$

Since $\mathrm{D}(0)\in\mathfrak{str}(T)$ and $\mathrm{D}^2 X(0)\in W$ by conformality, X is of the form given in Th. 2.3.1, i.e. $X\in\mathfrak{co}(T)$.

(ii) If $\mathrm{Hom}_s(S^2V,\mathfrak{l})=0$ and X is $\mathfrak{l}$-conformal, then $\mathrm{D}^3 X=0$, and we can conclude as above, using that $\mathrm{D}^2 X\in\mathrm{Hom}_s(V,\mathfrak{l})=W$. ■

Proposition 4.4. *If* T *is a non-degenerate JTS containing no ideal isomorphic to* $\mathbb{R}$ *or* $\mathbb{C}$*, then* $\mathrm{Hom}_s(V,W)=0$.

Proof. By Prop. 2.5.6, the map $A:V\to W$, $v\mapsto T(\cdot,v,\cdot)$ is bijective. Therefore

$$\mathrm{Hom}_s(V,W)\to\mathrm{End}(V),\quad \alpha\mapsto\beta:=A^{-1}\circ\alpha$$

is injective, i.e., we define β by the relation $\alpha(u)\cdot v\otimes w=T(v,\beta u,w)$. By the assumption on α, this expression is totally symmetric in u,v,w. In particular, the relation

$$T(v,\beta u)=T(u,\beta v) \tag{4.2}$$

holds for all $u,v\in V$. When applying $\sharp$ to both sides, the relation $T(a,b)^\sharp=-T(b,a)$ (cf. Lemma 2.5.4) yields $T(\beta u,v)=T(\beta v,u)$. Using the notaton $R(a,b)=T(b,a)-T(a,b)$, we get $R(v,\beta u)=R(u,\beta v)$, i.e.

$$R(v,\beta u)=-R(\beta v,u) \tag{4.3}$$

holds for all $u,v\in V$. We claim that this implies the relation

$$R(v,u)\circ\beta=-\beta\circ R(v,u) \tag{4.4}$$

for all $u,v\in V$. In order to prove this, we note that for all $x,y\in V$, $(y|\beta x)=\mathrm{tr}\,T(y,\beta x)=\mathrm{tr}\,T(\beta y,x)=(\beta y|x)$, i.e. β is self-adjoint. Using this and the relation

$$(R(x,y)u|v)=(R(u,v)x|y),$$

which holds for all $x,y,u,v\in V$ (cf. [Hel62, p.68]), we get from (4.3)

$$\begin{aligned}(R(u,v)\beta x|y)&=(R(\beta x,y)u|v)\\&=-(R(x,\beta y)u|v)\\&=-(R(u,v)x|\beta y)\\&=-(\beta R(u,v)x|y).\end{aligned}$$

Since the trace-form is non-degenerate, (4.4) follows by comparing the first and the last expression. From (4.4) we deduce that for all $a,b,u,v\in V$,

$$[R(a,b),R(u,v)]\circ\beta=\beta\circ[R(a,b),R(u,v)]. \tag{4.5}$$

Thus equations (4.4) and (4.5) together imply that for all $H \in [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$,

$$H \circ \beta = \beta \circ H = 0. \tag{4.6}$$

It follows that $\beta(V) \subset V$ is a submodule on which $[\mathfrak{h}, \mathfrak{h}]$ acts trivially.

Let us assume now that T is simple and not isomorphic to the one-dimensional real or complex JTS. Then $R \neq 0$ and $[\mathfrak{h}, \mathfrak{h}] \neq 0$. If T is simple, then by a result of E. Neher ([N85, Th. 1.11]) either

(A) R is a simple LTS, or

(B) R is is the direct sum of two simple non-abelian LTS, or

(C) R is the direct sum of a simple LTS and a non-dimensional center. This case arises precisely if T comes from a simple Jordan algebra (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with product $(x, y) \mapsto xy$ via the formula $T(u, v, w) = 2(u(vw) - v(uw) + (uv)w)$, and the trivial submodule is the space $\mathbb{K}e$ of multiples of the unit element e.

Moreover, if R is simple, then either V is an irreducible $\mathfrak{h}$-module or it is the direct sum of two irreducible $[\mathfrak{h}, \mathfrak{h}]$-modules which are dual to each other ([Koh65, Th. 3]). Thus we end up with the following two cases:

(a) V is the direct sum of several irreducible $[\mathfrak{h}, \mathfrak{h}]$-modules of dimension bigger than one;

(b) V is the direct sum of a trivial one-dimensional and an irreducible non-trivial $[\mathfrak{h}, \mathfrak{h}]$-module.

In the first case the only submodule on which $[\mathfrak{h}, \mathfrak{h}]$ acts trivially is zero, whence $\beta = 0$. In the second case β has to be a multiple of the projection onto the trivial one-dimensional module. Thus β is given by $\beta(x) = \lambda(x|e)e$ with $\lambda \in \mathbb{K}$, and

$$T(u, \beta v) = \lambda(v|e) T(u, e) = 2\lambda(v|e) L(u),$$

where $L(u)x = ux$. If $\lambda \neq 0$, then condition (4.2) with $v = e$ implies that $L(u) = (u|e)L(e) = (u|e)\,\mathrm{id}_V$ for all $u \in V$. Since $\mathfrak{h} = [L(V), L(V)]$, this implies that $\mathfrak{h} = 0$; this case corresponds to the simple Jordan algebra $V = \mathbb{K}$ and is excluded. Therefore $\lambda = 0$ and $\beta = 0$.

Thus the proposition is proved in the case that T is simple. If T is a non-degenerate JTS on V, we decompose $V = \oplus_i V_i$ into simple ideals w.r.t. T; then $W = \oplus_i W_i$ where the W_i correspond to the V_i. From the definition of $\mathrm{Hom}_s(V, W)$ one easily gets that

$$\mathrm{Hom}_s(V, W) = \oplus_i \mathrm{Hom}_s(V_i, W_i),$$

and in this way the general statement is reduced to the case of a simple JTS (cf. also [Be96a, Lemme 1.3.3]). ∎

The preceding proposition together with Lemma 4.3 (i) proves part (i) of Th. 4.2.

Proposition 4.5. *If V is a semisimple Jordan algebra and $\mathfrak{l} = \mathfrak{str}(V)$ is its structure algebra, then* $\mathrm{Hom}_s(V, \mathfrak{l}) = W$.

Proof. [Be96a, Prop. 1.2.1] ∎

Note that, for $S \in \mathrm{Hom}_s(S^2V, \mathfrak{l})$ and $v \in V$ fixed, $S(v, \cdot)$ belongs to $\mathrm{Hom}_s(V, \mathfrak{l})$. Therefore the preceding two propositions together imply that, if V is a semisimple Jordan algebra having no ideal isomorphic to $\mathbb{R}$ or $\mathbb{C}$, then $\mathrm{Hom}_s(S^2V, \mathfrak{l}) \subset \mathrm{Hom}_s(V, W) = 0$. Now part (ii) of Theorem 4.2 follows from Prop. 4.3 (ii), and Theorem 4.2 is completely proved. In order to deduce Theorem 4.1 from Theorem 4.2, we need the following proposition.

Proposition 4.6. *Let g be a locally defined diffeomorphism of V and X a vector field on V.*
(i) *If g satisfies* (C2) *and X satisfies* (C1)*), then g_*X satisfies* (C1)*).*
(ii) *If X satisfies* (C1) *and* (P1) *and g satisfies* (C2) *and* (P2)*, then g_*X satisfies* (P1).

Proof. (i) We denote by $\Gamma(\mathbf{W})$ the space of smooth sections over V of the structure bundle $\mathbf{W}$; these are just the smooth functions $V \to W$. Thus $\Gamma(\mathbf{W})$ is a subspace of the space of smooth functions $V \to \mathrm{Hom}(S^2V, V)$, and condition (C2) means that g preserves this subspace. Similarly, (C1) is equivalent to the condition

$$X \cdot \Gamma(\mathbf{W}) \subset \Gamma(\mathbf{W}).$$

Thus, if g satisfies (C2) and X satisfies (C1), then

$$(g_*X) \cdot \Gamma(\mathbf{W}) = g \cdot (X \cdot (g^{-1} \cdot \Gamma(\mathbf{W}))) \subset \Gamma(\mathbf{W}).$$

(ii) Let ∇ be the canonical flat connection of V. Recall that (P2) is equivalent to $g \cdot \nabla - \nabla \in \Gamma(\mathbf{W})$. Similarly, (P1) is equivalent to

$$X \cdot \nabla \in \Gamma(\mathbf{W}).$$

Using this and part (i), we get

$$(g_*X) \cdot \nabla = g \cdot (X \cdot (g^{-1} \cdot \nabla)) \in g \cdot (X \cdot (\nabla + \Gamma(\mathbf{W}))) \subset g \cdot \Gamma(\mathbf{W}) \subset \Gamma(\mathbf{W});$$

thus g_*X satisfies (P1), and (ii) is proved. ∎

Now we prove part (i) of Theorem 4.1. By Theorem 4.2 (i), $\mathfrak{co}(T)$ is precisely the space of vector fields (of class $\mathcal{C}^3$) satisfying (C1) and (P1). If g is of class $\mathcal{C}^4$ and satisfies (C2) and (P2), then part (ii) of the preceding proposition implies that g_* preserves $\mathfrak{co}(T)$. Thus by Th. 3.1.1, g is actually birational and belongs to the conformal group $\mathrm{Co}(T)$ (Def. 3.1.2). In a similar way, Th. 4.2 (ii) and part (i) of the preceding proposition imply Th. 4.1 (ii). ∎

Remark 4.7. The proof of Th. 4.1 (i) shows that a "fundamental theorem" holds whenever $\mathrm{Hom}_s(V, W) = 0$. ∎

Example 4.8. (a) If V is a Euclidean Jordan algebra (cf. [FK94]), then Th. 4.1 (ii) contains the determination of the *causal group* of V, cf. [Be96a, Th.2.3.1 (iii)], and of some causal symmetric spaces such as the de-Sitter and the anti de-Sitter model of general relativity and of the unitary group $\mathrm{U}(n)$, thus giving an affirmative answer to the conjecture of I.E. Segal mentioned in the introduction (cf. [Be96b]).
(b) If V is the Jordan algebra associated to a positive definite quadratic form, then Th. 4.1 (ii) is the original theorem of Liouville.
(c) If $V = M(n, n; \mathbb{R})$, $\mathrm{Sym}(n, \mathbb{R})$ or $\mathrm{Herm}(n, \mathbb{C})$, then Th. 4.1 (ii) is essentially equivalent to results of Chow and Dieudonné (cf. [D63] and [Be96b, Th.1.8.1]).
(d) If $V = M(1, n; \mathbb{R})$, then Th. 4.1 (i) is equivalent to the fundamental theorem of projective geometry. Here $V = \mathbb{R}^n$ with the JTS given by

$$T(u, v, w) := uv^t w + wv^t u, \tag{4.7}$$

and $\mathrm{Str}(T) = \mathrm{Gl}(n, \mathbb{R})$. Oberserve that (C1) and (C2) are empty in this case. Condition (P2) is now a special case of a well-known Theorem of H. Weyl stating that two torsionfree affine connections have same (unparametrized) geodesics iff their difference is a section of the bundle with fibre W associated to the JTS (4.7). ∎

Appendix: Integrability of almost complex structures

It is well-known that invariant almost complex structures on symmetric spaces are integrable (cf. [KoNo69]). For convenience of the reader, we give short proofs of the basic results. Recall that an almost complex structure (resp. polarization) is a tensor field $\mathcal{J}$ of type (1,1) such that $\mathcal{J}_p^2 = -\mathbf{1}_p$ (resp. $\mathcal{J}_p^2 = \mathbf{1}_p$) for all $p \in M$. It is called *integrable* if its *torsion tensor*

$$N(X,Y) := [\mathcal{J} X, \mathcal{J} Y] + \mathcal{J}^2[X,Y] - \mathcal{J}[X, \mathcal{J} Y] - \mathcal{J}[\mathcal{J} X, Y]$$

vanishes.

Proposition A.1. *Let $\mathcal{J}$ be a G-invariant almost complex structure or polarization on a symmetric space $M = G/H$. Then $\mathcal{J}$ is integrable.*

Proof. We use the canonical connection ∇ of M (cf. Section 1.0). Since $[X,Y] = \nabla_X Y - \nabla_Y X$ for all $X, Y \in \mathfrak{X}(M)$, we get

$$N(X,Y) = \nabla_{\mathcal{J} X} \mathcal{J} Y - \nabla_{\mathcal{J} Y} \mathcal{J} X + \mathcal{J}^2(\nabla_X Y - \nabla_Y X) - \mathcal{J}(\nabla_X \mathcal{J} Y - \nabla_{\mathcal{J} Y} X - \nabla_{\mathcal{J} X} Y + \nabla_Y \mathcal{J} X).$$

The condition $G \cdot \mathcal{J} = \mathcal{J}$ implies $\nabla \mathcal{J} = 0$; this means that $\nabla_Z \mathcal{J} S = \mathcal{J} \nabla_Z S$ for all $Z, S \in \mathfrak{X}(M)$. Using this property, all terms in the above expression of $N(X,Y)$ cancel out. ■

We consider the invariance group $G(\mathcal{J})$ of $\mathcal{J}$; this is the group of diffeomorphisms g of M such that for all $Y \in \mathfrak{X}(M)$, $g_*(\mathcal{J} Y) = \mathcal{J} g_* Y$ holds. We call $G(\mathcal{J})$ also the *group of almost (para-) holomorphic transformations of* $(M, \mathcal{J})$. The corresponding infinitesimal object is the Lie algebra

$$\mathfrak{g}(\mathcal{J}) := \{X \in \mathfrak{X}(M) | \, \forall Y \in \mathfrak{X}(M) : [X, \mathcal{J} Y] = \mathcal{J}[X,Y]\}.$$

We remark that for all $X, Y \in \mathfrak{g}(\mathcal{J})$,

$$N(X,Y) = [\mathcal{J} X, \mathcal{J} Y] - \mathcal{J}^2[X,Y],$$

and therefore, if $\mathcal{J}$ is integrable, then we have for all $X, Y \in \mathfrak{g}(\mathcal{J})$,

$$[\mathcal{J} X, \mathcal{J} Y] = \mathcal{J}^2[X,Y]. \tag{A.1}$$

Proposition A.2. *Let $\mathcal{J}$ be an integrable almost complex structure or a polarization on a manifold M and assume that $\mathfrak{g}(\mathcal{J})$ contains for any point $p \in M$ a local basis of $\mathfrak{X}(M)$ around p. Then $\mathfrak{g}(\mathcal{J})$ is stable under the map $X \mapsto \mathcal{J} X$. In particular, if $\mathcal{J}^2 = -\,\mathrm{id}_{\mathfrak{X}(M)}$, then $\mathfrak{g}(\mathcal{J})$ is a complex Lie algebra with complex structure $\mathcal{J}$.*

Proof. Let $X \in \mathfrak{g}(\mathcal{J})$ and $Y \in \mathfrak{X}(M)$. On a neighbourhood of a point $p \in M$ we can write $Y = \sum_{i=1}^m f_i Y_i$ with smooth functions f_i and $Y_i \in \mathfrak{g}(\mathcal{J})$. Without loss of generality we may assume that $m = 1$ and write $Y = f\widetilde{Y}$. Then using (A.1), locally,

$$\begin{aligned} [\mathcal{J} X, \mathcal{J} Y] &= [\mathcal{J} X, f \mathcal{J} \widetilde{Y}] = df(\mathcal{J} X) \cdot \mathcal{J} \widetilde{Y} + f[\mathcal{J} X, \mathcal{J} \widetilde{Y}] \\ &= \mathcal{J}(df(\mathcal{J} X) \cdot \widetilde{Y} + [\mathcal{J} X, \widetilde{Y}]) \\ &= \mathcal{J}[\mathcal{J} X, Y]. \end{aligned}$$

Using a partition of unity, we get the same relation for general $Y \in \mathfrak{X}(M)$, whence $\mathcal{J} X \in \mathfrak{g}(\mathcal{J})$, proving the claim. ■

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