Geometric Bergman and Hardy spaces

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0. Introduction

If Ω is an open convex cone in a real vector space V, then one defines a *Hardy space* of holomorphic functions on the *tube domain* $T_\Omega := V + i\Omega \subset V_{\mathbb{C}}$ by

$$H^2(T_\Omega) := \{f \in \mathcal{O}(T_\Omega) |\, ||f||^2 := \mathrm{sup}_{y\in\Omega} \int_V |f(x+iy)|^2 \, dx < \infty\}, \tag{0.1}$$

where dx is a fixed Lebesgue measure on V. It is well-known that if Ω does not contain affine lines, then the Hardy space is a non-trivial Hilbert space such that the point evaluations $f \mapsto f(z)$ are continuous. Thus there exists a vector $K_z \in H^2(T_\Omega)$ such that $f(z) = \langle f, K_z\rangle$ for all $f \in H^2(T_\Omega)$; the function $K(z,w) = K_w(z)$ is called the *reproducing kernel* of $H^2(T_\Omega)$ (cf. e.g. [FK94, Section IX.4]). Another classical Hardy space of holomorphic functions can be defined for a bounded domain $D \subset \mathbb{C}^n$ which is starshaped around zero: given a measure μ on the Shilov boundary Σ of D one defines

$$H^2(D,\mu) := \{f \in \mathcal{O}(D) |\, ||f||^2 := \mathrm{sup}_{0<r<1} \int_\Sigma |f(rx)|^2 \, d\mu(x) < \infty\}. \tag{0.2}$$

In both cases the integration is carried out over a region of the boundary of a complex domain, after translating the function by elements of a certain semigroup of (strict) *compressions* of the domain (i.e. a semigroup of diffeomorphisms carrying the closure of the domain into its interior).

The simplest example arises when D is the unit disc; then $\Sigma = S^1$ is the unit circle, and the most natural choice for μ is the normalized rotation invariant measure on S^1. As is well-known, the *Cayley transform* $C(z) = i(1+z)(1-z)^{-1}$ carries the unit disc onto the upper half-plane which is just the tube domain $\mathbb{R} + i\mathbb{R}^+$. There is an important class of bounded domains, called *bounded symmetric domains of tube type*, for which there exists via a generalized Cayley transform an unbounded realization as a tube domain T_Ω for a *symmetric cone* Ω (cf. [FK94, Ch. X]). In this case, there exist a number of classical and also more recent results on the Hardy spaces defined above which, however, have so far not been explained in a satisfactory way. In particular, we are interested in the following four facts for which one wishes to have new and geometric proofs:

Fact 1: The unitary action of a "big" group. If D is the unit disc and μ the rotation invariant probability measure on the circle, then the group $G = \mathrm{SU}(1,1)$ operates unitarily on $H^2(D,\mu)$ via

$$(g.f)(z) = (\mathrm{Det}\, \mathrm{d}g^{-1}(z))^{1/2} f(g^{-1}z) \tag{0.3}$$

(the square root can be defined in a consistent way since $\mathrm{SU}(1,1)$ is a double cover of the group $\mathrm{Aut}(D) = \mathbb{P}\,\mathrm{SU}(1,1)$). By the same formula the group $\mathrm{Sl}(2,\mathbb{R})$ acts on the Hardy space

$H^2(T_\Omega)$, and, more generally, a double cover of the automorphism group of a tube type domain acts unitarily on the corresponding Hardy space. However, this action is not obvious from the geometric data because the measure μ, resp. dx on the boundary is far from being invariant under $\mathrm{SU}(1,1)$, resp. under $\mathrm{Sl}(2,\mathbb{R})$.

Fact 2: The isomorphism of classical Hardy spaces. The Cayley transform C induces an isomorphism of Hardy spaces

$$H^2(D) \to H^2(T_\Omega), \quad f \mapsto (z \mapsto (\mathrm{Det}\, \mathrm{d}C^{-1}(z))^{1/2} f(C^{-1}z)). \tag{0.4}$$

This is not at all obvious from the definitions: firstly, the spaces $V+iy$ over which one integrates in the definition of $H^2(T_\Omega)$ are transformed via C into *horocycles* of D which are not the same as the "concentric circles" $r\Sigma$ used in the definition of $H^2(D)$, and secondly the measures used on the Shilov boundary (resp. on its open dense part $C(V)$) are not the same. In fact, no geometric reason for this isomorphism is given in the literature; it is only deduced from the explicit knowledge of the reproducing kernels.

Fact 3: Hardy spaces are "square roots" of Bergman spaces. The reproducing kernel of $H^2(D)$, called the *Cauchy-kernel* and denoted by $S(z,w)$, turns out to be a square root of the *Bergman-kernel* $K(z,w)$ of D: there is a constant $c \neq 0$ such that

$$cS(z,w)^2 = K(z,w). \tag{0.5}$$

The Bergman kernel is by definition the reproducing kernel of the *Bergman space*

$$B^2(D) = \{f \in \mathcal{O}(D) | \int_D |f(z)|^2\, dz < \infty\}, \tag{0.6}$$

where dz is Lebesgue measure on $V_\mathbb{C}$, restricted to D.

Similarly, the reproducing kernel of $H^2(T_\Omega)$ is the square root of the Bergman kernel of T_Ω (this information allows to prove that (0.4) is an isomorphism). Put in another way, the Hardy space turns out to belong to a parameter in the *analytic continuation of a family of weighted Bergman spaces* (cf. [FK94, Ch.XIII]). Once more this seems to be rather an accident and no geometric interpretation of this fact is given.

Fact 4: Imbedding of classical Hardy spaces into non-classical Hardy spaces. Motivated by the so-called "Gelfand-Gindikin program", the problem of imbedding "classical" or "commutative" Hardy spaces into "non-classical" or "non-commutative" Hardy spaces has attracted much interest during the last years (cf. [BH98b], [BO98], [Cha98], [KØ96], [KØ97], [OØ98]). One wants to understand a class of Hardy spaces which are of interest in group theory but whose kernels are very complicated by comparing them with the more classical and better known Hardy spaces on bounded symmetric domains of tube type. When working on this problem ([BH98b]), we remarked that usually the Facts 1 – 3 mentioned above are taken for granted although they remain mysterious in the usual framework of Hardy spaces. Once this observation was made, it became clear that a geometric explanation of Facts 1 – 3 is precisely the groundwork needed to understand what is really going on in the problem of imbedding one class of Hardy spaces into another.

"Geometric Bergman- and Hardy spaces". The definition of "geometric Bergman spaces" is well-known: the Bergman space of a complex manifold M is the space of holomorphic n-forms ω on M such that the real $2n$-form

$$i^{n^2} \omega \otimes \overline{\omega}$$

is integrable over M. From this definition it is immediately clear that the group $\mathrm{Aut}(M)$ of holomorphic automorphisms of M acts unitarily on the geometric Bergman space. If we trivialize the geometric Bergman space w.r.t. a nowhere vanishing holomorphic n-form, we obtain a function space in which the group $\mathrm{Aut}(M)$ acts via a multiplier representation similar to (0.3). This should be compared to Fact 1.

Fact 3 indicates what kind of bundle we have to take in order to realize the Hardy space as a space of sections: the bundle must be a "square root" of the bundle defining Bergman spaces, i.e. a holomorphic line bundle $\mathbf{L}$ such that $\mathbf{L}\otimes\mathbf{L}$ is isomorphic to the canonical bundle $\mathbf{K}_M=\Lambda^n(T^*M)$ of M. Such bundles are called (holomorphic) *half-form bundles*, cf. [GS77]. For the definition of Hardy spaces of sections one needs much more structure than for the definition of Bergman spaces: besides a half-form bundle over a domain we need a certain "boundary" of this domain and a semi-group over which the supremum in the definition of the Hardy-norm will be taken. This geometric information will be called "Hardy-space data" (Def. 1.3.3). Just as the definition of geometric Bergman spaces does not require a measure, the definition of geometric Hardy spaces will not require a measure on the boundary. This allows a "big group" to operate unitarily on the geometric Hardy space, explaining Fact 1. Also Fact 3 is explained in a natural way: whenever the "big group" acts transitively on the domain (as is the case for bounded symmetric domains), both the reproducing kernels of the geometric Bergman- and Hardy space define invariant sections of $\mathbf{K}$, resp. of $\mathbf{L}$, and since $\mathbf{L}\otimes\mathbf{L}\cong\mathbf{K}$, there must be a constant c with $cS\otimes S=K$. However, the main problem is now to show that $c\neq 0$: it is more difficult to prove that a geometric Hardy space is not reduced to zero than to prove this for Hardy spaces of functions. In the case of a bounded symmetric domain D we prove that $c\neq 0$ (Thm. 2.2.3) using a result of J.-L. Clerc ([Cl98]) saying that elements of the *compression semigroup* $S(D)$ are also *contractions* of the Bergman metric of D. Once we know that the geometric Hardy space of D is not reduced to zero, we can explain Fact 2: The Hardy spaces $H^2(D)$ and $H^2(T_\Omega)$ are essentially defined by taking suprema over certain *subsemigroups* of $S(D)$; this supremum is smaller than the one over $S(D)$ used in the definition of the geometric Hardy space, and therefore the geometric Hardy space can be realized as a subspace of $H^2(D)$ and $H^2(T_\Omega)$. But this subspace contains enough elements in order to prove that we have in fact equality (Thm. 2.3.1 and Thm. 2.3.3).

This paper is meant to provide some quite general background information to a wide range of non-expert readers who are interested in the geometric analysis of Hardy spaces. Therefore we do not go too far into the details related to Fact 4. We just collect some general remarks on Bergman- and Hardy spaces related to homogeneous spaces (Section 3) which, however, already lead directly to some of the most subtle problems which have implicitly shown up in the literature mentioned above (Fact 4): namely, before comparing Hardy spaces, one needs to compare holomorphic half-form bundles together with "equivariant" group- or semigroup actions. Examples show that half-form bundles which are equivalent as vector bundles may very well carry several essentially different actions of a given group. This observation naturally leads to the problem of describing and classifying such objects; it will be taken up elsewhere.

1. Definition of geometric Bergman- and Hardy spaces

1.1. Bergman spaces of holomorphic sections. Before defining geometric Hardy-spaces, it is useful to recall quickly the definition and basic properties of geometric Bergman spaces (cf. [KN69, p.163]). For any complex manifold M of dimension n we denote by $\mathbf{K}_M$ the canonical bundle $\bigwedge^n T^*M$ whose holomorphic sections are the holomorphic n-forms. The exterior product of a holomorphic n-form ω_1 with an antiholomorphic n-form $\overline{\omega}_2$ gives, up to a factor i^{n^2}, a real $2n$-form. Since M, viewed as a real manifold, is automatically orientable this form can be integrated over M and the resulting number, if finite, is the inner product $(\omega_1 \mid \omega_2)$. The *Bergman space* is then the space

$$\mathcal{B}^2(M) := \{\omega \in \mathcal{O}(M, \mathbf{K}_M) | \, i^{n^2} \int_M \omega \wedge \overline{\omega} < \infty\} \tag{1.1.1}$$

of holomorphic n-forms ω for which the inner product $(\omega \mid \omega)$ is finite. It is known that this is a Hilbert space admitting a reproducing kernel which is a holomorphic $2n$-form on $M \times \overline{M}$ (cf. Section 2). From the definition of the Bergman space it is immediately clear that the group $\mathrm{Aut}(M)$ of holomorphic diffeomorphisms acts unitarily on $\mathcal{B}^2(M)$ by $(g, \omega) \mapsto (g^{-1})^* \omega$, where g^* is the usual pull-back of forms.

Remark 1.1.1. (Trivialization of line bundles.) A line bundle $\mathbf{V}$ over a complex manifold N is isomorphic to the trivial line bundle if and only if it admits a holomorphic nowhere vanishing section ν. In fact, the constant function 1 is a nowhere vanishing section of the trivial bundle, and conversely, given such a section ν, we define a bundle map

$$N \times \mathbb{C} \to \mathbf{V}, \quad (p, z) \mapsto z\nu_p$$

whose inverse is given by $\mathbf{V}_p \ni v \mapsto (p, z)$ with z defined by $v = z\nu_z$. The corresponding isomorphism

$$\mathcal{O}(N, \mathbf{V}) \to \mathcal{O}(N)$$

whose inverse is given by $f \mapsto f\nu$ will be called the *trivialization map associated to* ν. Clearly we can make similar remarks for real line bundles over real manifolds.

Applying this to the Bergman space, given a nowhere vanishing holomorphic n-form ν, we obtain an isomorphism of Hilbert spaces

$$\mathcal{B}^2(M) \to B^2(M, \nu) := \{f \in \mathcal{O}(M) | \int_M |f(z)|^2 i^{n^2} (\nu \wedge \overline{\nu})(z) < \infty\} \tag{1.1.2}$$

with inverse given by $f \mapsto f\nu$. The function space $B^2(M, \nu)$ is called a *trivialization* of $\mathcal{B}^2(M)$.

Note that the unitary action of $\mathrm{Aut}(M)$ in the trivialized picture is no longer canonical, but depends on ν: for every $g \in \mathrm{Aut}(M)$ there exists a function $j(g) = j_\nu(g)$ such that

$$g^*\nu = j(g) \cdot \nu.$$

Thus $g^*(f\nu) = g^*f \cdot g^*\nu = g^*f \cdot j(g) \cdot \nu$, implying that the action of g is transferred to

$$(g.f)(z) = j(g^{-1}, z) f(g^{-1}.z),$$

where $j(g, \cdot) := j(g)$. ■

Remark 1.1.2. (Bundle-valued Bergman spaces.) If M is a complex n-dimensional manifold and $\mathbf{H}$ a holomorphic vector bundle over M, then by definition a *holomorphic* n*-form with values in* $\mathbf{H}$ is a holomorphic section of the bundle

$$\mathbf{K}_M \otimes \mathbf{H} = \mathrm{Hom}(\bigwedge^n TM, \mathbf{H}); \tag{1.1.3}$$

the space of such forms is denoted by $\Omega^n(M, \mathbf{H})$. In order to define Bergman spaces of holomorphic n-forms with values in $\mathbf{H}$, we need the additional assumption that $\mathbf{H}$ is a *Hermitian* vector bundle. Then to $v, v' \in \mathbf{H}_z$ and $\alpha, \alpha' \in \bigwedge^n T^*_z M$, we associate a scalar valued (n,n)-form

$$\langle \alpha \otimes v, \alpha' \otimes v' \rangle := i^{n^2} (v \mid v')_{\mathbf{H}_z} (\alpha \wedge \overline{\alpha'}). \tag{1.1.4}$$

In this way we obtain a sesquilinear map

$$\langle\cdot,\cdot\rangle\colon \Omega^n(M,\mathbf{H})\times\Omega^n(M,\mathbf{H})\to\mathcal{E}^{(n,n)}(M_{\mathbb{R}}), \tag{1.1.5}$$

where $\mathcal{E}^{(n,n)}(M_{\mathbb{R}})$ denotes the differential forms of type (n,n) on M considered as an almost complex real manifold $M_{\mathbb{R}}$. With these definitions

$$\mathcal{B}^2(M,\mathbf{H}) := \{\omega\in\Omega^n(M,\mathbf{H}) | \int_{M_{\mathbb{R}}}\langle\omega,\omega\rangle<\infty\}, \tag{1.1.6}$$

is called the *Bergman space* of *square integrable sections* (cf. [Ko68, p.639]). It turns out that $\mathcal{B}^2(M,\mathbf{H})$ is a Hilbert space with respect to the inner product

$$(\omega\mid\omega')_{\mathcal{B}} := \int_{M_{\mathbb{R}}}\langle\omega,\omega'\rangle. \tag{1.1.7}$$

If $\mathbf{H}$ is a trivial vector bundle with typical fiber $\mathbf{H}_o$, then the Hermitian metric is of the form $(v\mid v')_{\mathbf{H}_z} = (v\mid P(z)v')_{\mathbf{H}_o}$ with a Hermitian positive definite operator $P(z)$. If we assume moreover that ν is a nowhere vanishing n-form on M, then the map $f\mapsto\nu\otimes f$ defines an isomorphism of

$$B^2(M,\mathbf{H}_o,P) := \{f\in\mathcal{O}(M,\mathbf{H}_o) | \, i^{n^2}\int_{M_{\mathbb{R}}}||P(z)f(z)||^2\,(\nu\wedge\overline{\nu})(z)<\infty\} \tag{1.1.8}$$

onto $\mathcal{B}^2(M,\mathbf{H})$. In case $\mathbf{H}$ is a line bundle, $z\mapsto P(z)$ is a scalar function with positive values, and Eqn. (1.1.8) coincides with the usual definition of *weighted Bergman spaces* with weight function P. ∎

If g is a holomorphic diffeomorphism of the complex manifold M and ω a holomorphic n-form, then we have the equation

$$\int_M (g^*\omega)\otimes\overline{(g^*\omega)} = \int_{g(M)}\omega\otimes\overline{\omega}$$

which means that g acts unitarily on the classical Bergman space. The statement is immediately generalized to the case of general bundle-valued Bergman spaces if we assume that g acts isometrically on the Hermitian bundle used in the construction.

A situation that will be relevant later on arises as follows: we assume that $\mathcal{U}$ is a domain in a complex manifold M, $\mathbf{H}$ a Hermitian vector bundle over $\mathcal{M}$ and Γ is a semigroup of biholomorphic maps on M which preserves $\mathcal{U}$ and acts on the bundle $\mathbf{H}$ with isometric fiber maps.

Proposition 1.1.3. *The Bergman space $\mathcal{B}^2(\mathcal{U})$ is stable under the natural pull-back $\omega\mapsto s^*\omega$ for $s\in\Gamma$, and this action is contractive.*

Proof. In the scalar case, we have

$$\int_{\mathcal{U}}(s^*\omega)\otimes\overline{(s^*\omega)} = \int_{s(\mathcal{U})}\omega\otimes\overline{\omega}\le\int_{\mathcal{U}}\omega\otimes\overline{\omega}$$

since s is a compression of $\mathcal{U}$. The argument carries over to the bundle valued case with the obvious changes. ∎

1.2. Spaces of square integrable half-forms.

Definition 1.2.1. A *half-form bundle* on a real n-dimensional manifold N is a complex line bundle $\mathbf{L}$ over N such that the square $\mathbf{L}^2 = \mathbf{L} \otimes \mathbf{L}$ is isomorphic to the complexified line bundle $(\bigwedge^n T^*N)_{\mathbb{C}}$. In other words, the transition functions $h_{\alpha\beta}: U'_\alpha \cap U'_\beta \to \mathbb{C}^\times$ of $\mathbf{L}$ satisfy

$$h^2_{\alpha\beta}(z) = \mathrm{Det}_{\mathbb{R}} \left(\mathrm{d}(\psi_\beta \circ \psi_\alpha^{-1})(\psi_\alpha(z))\right). \tag{1.2.1}$$

if $(U'_\alpha, \psi_\alpha)_{\alpha \in A}$ is an atlas for N. A section of a half-form bundle is called a *half-form.* ■

Example 1.2.2. Assume that N is orientable; thus there is a nowhere vanishing n-form ν on N, and (w.r.t. an oriented atlas) the bundle $(\bigwedge^n T^*N)_{\mathbb{C}}$ is isomorphic to the trivial line bundle which we denote by $\mathbf{1}$ (cf. Remark 1.1.1). Since $\mathbf{1} \otimes \mathbf{1} \cong \mathbf{1}$, it follows that $\mathbf{1}$ is also a half-form bundle on N. We cannot conclude, however, that in this situation all half-form bundles are trivial. For example, if N is the circle, the complexification of the Moebius band (considered as a real non-trivial line bundle) is a non-trivial half-form bundle. ■

Remark 1.2.3. Not every manifold admits half-form bundles and if they exist they are not always uniquely determined.

To make this precise recall that the isomorphism classes of line bundles form an abelian group, called the Picard group, under tensoring (the trivial bundle being the identity). This group is isomorphic to $H^1(M, \mathcal{A}^\times)$ where $\mathcal{A}$ is the sheaf of invertible differentiable maps (C^∞, respectively holomorphic, depending on whether M is real or complex). Thus M admits a half-form bundle iff the isomorphism class of $\mathbf{K}_M$ is a square in the Picard group. Consider the exact sequence

$$\{1\} \longrightarrow \{\pm 1\} \longrightarrow \mathcal{A}^\times \xrightarrow{z \mapsto z^2} \mathcal{A}^\times \longrightarrow \{1\}$$

of sheaves of abelian groups. According to [Go73, p.174] we obtain a long exact sequence of group homomorphisms in cohomology

$$\ldots \to H^1(M, \{\pm 1\}) \xrightarrow{\iota} H^1(M, \mathcal{A}^\times) \xrightarrow{z \mapsto z^2} H^1(M, \mathcal{A}^\times) \xrightarrow{\delta} H^2(M, \{\pm 1\}) \to \ldots$$

Thus the canonical bundle is a square iff $\delta(\mathbf{K}_M)$ is the identity in $H^2(M, \{\pm 1\})$. Moreover, if $\mathbf{K}_M$ is a square, then the set of square roots is parametrized by the image of ι in $H^1(M, \{\pm 1\})$.

Inparticular we find: If $H^j(M, \{\pm 1\})$ is trivial for $j = 1, 2$, then M admits a unique half-form bundle. ■

For two half-forms $\omega^{(1)}, \omega^{(2)}$ the tensor product $\omega^{(1)} \otimes \overline{\omega^{(2)}}$ is a *density*, i.e. it is a section of the *density bundle* $|\bigwedge^n|T^*N$ which is the complex line bundle defined by the transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{C}^\times$ given as

$$g_{\alpha\beta}(z) = \left|\mathrm{Det}_{\mathbb{R}} \left(\mathrm{d}(\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(z))\right)\right|,$$

where $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is an atlas for N. The fiber of $|\bigwedge^n|T^*N$ at $x \in N$ can be viewed as the set of maps $\rho: (T_xN)^n \to \mathbb{C}$ such that

$$\rho(A\eta_1, \ldots, A\eta_n) = |\,\mathrm{Det}(A)|\rho(\eta_1, \ldots, \eta_n)$$

for all $A \in \mathrm{End}_{\mathbb{R}}(T_xN)$ (cf. [GS77, p.53]). We call a density ν *real*, if $\nu(x): (T_xN)^n \to \mathbb{C}$ only takes real values, and *positive*, if $\nu(x): (T_xN)^n \to \mathbb{C}$ take only non-negative values for all $x \in N$.

Thus, if $\mathbf{HF}_N$ is a half-form bundle on N,

$$(\omega^{(1)}, \omega^{(2)}) \mapsto (\omega^{(1)} \mid \omega^{(2)})_{\mathbf{HF}_N} := \int_N \omega^{(1)} \otimes \overline{\omega^{(2)}}$$

defines an inner product on the space of continuous sections of $\mathbf{HF}_N$ with compact support whose completion we denote by $L^2(N, \mathbf{HF}_N)$.

Example 1.2.4. (Trivialization) Assume that $\mathbf{HF}_N$ has a nowhere vanishing section ω_o, then $\nu := \omega_o \otimes \overline{\omega_o}$ is a nowhere vanishing positive density on N defining a measure dn on N, and

$$L^2(N, dn) \to L^2(N, \mathbf{HF}_N), \quad f \mapsto f\omega_o$$

is an isomorphism of Hilbert spaces (cf. Remark 1.2.1). Densities can be pulled back under smooth maps via

$$\varphi^*\nu(x)(\eta_1, \ldots, \eta_n) = \nu\big(\varphi(x)\big)(\mathrm{d}\varphi(x)\eta_1, \ldots, \mathrm{d}\varphi(x)\eta_n).$$

Thus, for any diffeomorphism g of N there is a strictly positive function $j(g)$ such that

$$g^*\nu = j(g) \cdot \nu.$$

Then

$$(g.f)(x) = j(g^{-1})^{1/2} \cdot (f \circ g^{-1})$$

defines a unitary action of the group G of diffeomorphisms of N in $L^2(N, \mathbf{HF}_N)$. It corresponds to an action $\omega \mapsto g^*\omega$ of G on the space of sections of $\mathbf{HF}_N$ such that $g^*\omega \otimes \overline{g^*\omega} = g^*(\omega \otimes \overline{\omega})$.

In the special case where $N = V$ is a vector space, the bundle $\bigwedge^n T^*V$ clearly is trivial, and thus the trivial bundle is isomorphic to a half-form bundle. The forms ω_o and ν can both be chosen translation-invariant; we will use also the notation $(dx)^{1/2}$ for ν. Then

$$L^2(V, dx) \to L^2(V, \mathbf{HF}_V), \quad f \mapsto f(dx)^{1/2}$$

is the trivialization map. We have $j(g)(x) = |\operatorname{Det}(\mathrm{d}g(x))|$, and thus the action of G is given in the trivialized picture by

$$(g.f)(x) = |\mathrm{d}g^{-1}(x)|^{-1/2} f(g^{-1}.x).$$

In this case the unitarity of the action can be seen as a direct consequence of the transformation formula for integrals. ■

Remark 1.2.5. In general, given a half-form bundle $\mathbf{HF}$, there is no natural pullback action of the group of all diffeomorphisms on $\mathbf{HF}$ which is compatible with the natural action on the bundle $\bigwedge^n T^*N$ in the sense that $g^*(\omega \otimes \rho) = g^*\omega \otimes g^*\rho$. However, one is not very far from this situation. One can prove:

(1) A single diffeomeorphism of M can always be lifted to a diffeomorphism of the half-form bundle which is compatible with the pullback of holomorphic forms.

(2) The lifting can be done simultaneously for all elements of a connected topological semigroup in such a way that one gets a representation of a double cover of the semigroup. ■

1.3. Hardy spaces of holomorphic half-forms.

Definition 1.3.1. Let M be a complex manifold. A holomorphic line bundle $\mathbf{L} \to M$ is called a *holomorphic half-form bundle* if $\mathbf{L} \otimes \mathbf{L}$ is isomorphic to the canonical line bundle $\mathbf{K}_M = \bigwedge^n T^*M$. A *holomorphic half-form* is a holomorphic section of a holomorphic half-form bundle. ■

In order to define Hardy spaces, we have to integrate holomorphic half-forms over real forms of M. For this we need the following proposition.

Proposition 1.3.2. *Let M be a complex manifold and* $\mathbf{HF}$ *a holomorphic half-form bundle on M. If $N \subseteq M$ is a real form (i.e. a totally real real-analytic submanifold with $\dim_{\mathbb{R}} N = \dim_{\mathbb{C}} M$), then $\mathbf{HF}|_N$ is a half-form bundle on the real manifold N, and $(\mathbf{HF} \otimes \overline{\mathbf{HF}})|_N$ is a density bundle on the real manifold N.*

Proof. We can find an atlas $(U_\alpha, \varphi_\alpha)$ for M such that $\varphi_\alpha(N \cap U_\alpha) \subseteq \mathbb{R}^n$ for each of the coordinate functions $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$. In fact, such charts are obtained by parametrizing a piece of N by a real-analytic map $f : U \to N$, where U is a neighborhood of 0 in $\mathbb{R}^n$, and then complexifying f to get a holomorphic map from a neighborhood of 0 in $\mathbb{C}^n$ onto a piece of M. (In a local chart on M all we do is write f as a power series in real variables $(x_1, ..., x_n)$ and then make x complex; the power series will converge in some neighborhood of the origin.) Total reality of N ensures that the extended f is locally biholomorphic, and its inverse is then a local chart on M that is compatible with N.

With respect to this atlas, $(U_\alpha \cap N, \varphi_{\alpha,N})_{\alpha \in A}$ with $\varphi_{\alpha,N} : N \cap U_\alpha \to \mathbb{R}^n$ is an atlas for N and

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha) \cap \mathbb{R}^n \to \varphi_\alpha(U_\alpha) \cap \mathbb{R}^n.$$

In particular, for $x \in N \cap U_\alpha \cap U_\beta$ we find

$$\mathrm{Det}_{\mathbb{R}}\left(\mathrm{d}(\varphi_{\alpha,N} \circ \varphi_{\beta,N}^{-1})(\varphi_\beta(x))\right) = \mathrm{Det}_{\mathbb{C}}\left(\mathrm{d}(\varphi_\alpha \circ \varphi_\beta^{-1})(\varphi_\beta(x))\right) = h_{\beta\alpha}(x)^2$$

for $x \in N \cap U_\alpha \cap U_\beta$. This also implies

$$h_{\alpha\beta}(x)\overline{h_{\alpha\beta}(x)} = \left|\mathrm{Det}_{\mathbb{R}}\left(\mathrm{d}(\varphi_{\alpha,N} \circ \varphi_{\beta,N}^{-1})(\varphi_\beta(x))\right)\right|$$

and hence the claim. ■

Note that, if ω is a holomorphic half-form and N as in the proposition, the restriction $\omega|_N$ is a half-form on the real manifold N and thus we can integrate $\omega_N \otimes \overline{\omega}_N$ over N. Hardy-spaces are defined by a finiteness-condition on such integrals. However, in order to obtain Hilbert-spaces, one needs the more specific geometric situation where N plays the role of a boundary of the domain on which the holomorphic half-forms live.

Definition 1.3.3. A quadruple $(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ is called *Hardy space data* if

(1) $\mathcal{U}$ is a domain in a complex manifold M,

(2) $\mathbf{HF}$ is a holomorphic half-form bundle over M,

(3) N is a real form of M such that N is contained in the boundary $\partial\mathcal{U}$ of $\mathcal{U}$,

(4) Γ is a semigroup of compressions of $\mathcal{U}$ acting by local automorphisms of $\mathbf{HF}$. More precisely, $s \in \Gamma$ acts by a local holomorphic diffeomorphism

$$s : \mathcal{D}_s \mapsto s(\mathcal{D}_s),$$

where $\mathcal{D}_s \subset M$ is a domain containing $\mathcal{U}$. The assumption that s is a compression of $\mathcal{U}$ means that $s(\mathcal{U}) \subset \mathcal{U}$. Moreover, we assume to each s a bundle isomorphism

$$s^* : \mathbf{HF}|_{s(\mathcal{D}_s)} \to \mathbf{HF}|_{\mathcal{D}_s}$$

is given (thus we assume that $p \circ s^* = s \circ p$, where p is the projection onto the base space of $\mathbf{HF}$) such that $s_2^* \circ s_1^* = (s_1 s_2)^*$ and

$$s^*(v_y \otimes w_y) = s^* v_y \otimes s^* w_y$$

for all $v_y, w_y \in \mathbf{HF}_y$. ■

It is clear that the manifold M and the domains $\mathcal{D}_s$ can be replaced by smaller neighborhoods of $\mathcal{U}$; thus we omit M in our notation, and we are in fact only interested in equivalence classes when considering two elements s, s' as equivalent if they coincide on $\mathcal{U}$.

We denote by Γ^o the semigroup ideal

$$\Gamma^o = \{s \in \Gamma | \, s(\overline{\mathcal{U}}) \subset \mathcal{U}^o\}.$$

Then under the above assumptions $s.N$ is a real form of $\mathcal{U}$ for all $s \in \Gamma^o$. Therefore we are in the situation of Proposition 1.3.2, implying that for any holomorphic section ω of $\mathbf{HF}$ we have a density $\omega \otimes \overline{\omega}$ on $s.N$.

Definition 1.3.4. Given Hardy space data $(\mathcal{U}, N, \mathbf{HF}, \Gamma)$, we set for any holomorphic section ω of $\mathbf{HF}$

$$\|\omega\|^2_{\mathcal{H}} := \sup_{s \in \Gamma^o} \int_{s.N} \omega|_{s.N} \otimes \overline{\omega|_{s.N}}$$

and define a *Hardy space*

$$\mathcal{H}^2(\mathcal{U}) := \mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma) := \{\omega \in \mathcal{O}(\mathcal{U}, \mathbf{HF}) \mid \|\omega\|^2_{\mathcal{H}} < \infty\}. \qquad \blacksquare$$

One can also define general bundle-valued Hardy spaces in the same spirit as we passed from the Bergman space of scalar valued forms to bundle-valued forms. However, in order to let notation not become too heavy, we stick to the case of scalar valued half-forms. At the moment we do not even know whether the Hardy space is a vector space.

Proposition 1.3.5.

(i) *The Hardy space $\mathcal{H}^2(\mathcal{U})$ together with $\| \cdot \|^2_{\mathcal{H}}$ is a normed complex vector space.*

(ii) *The semigroup Γ acts naturally by contractions on the Hardy space.*

(iii) *If Γ is a monoid, i.e. a multiplicative semigroup with a neutral element $\mathbf{1}$, and the element $\mathbf{1}$ acts as the identity on $\mathcal{U}$ and $\mathbf{HF}$, then the group $G = \Gamma \cap \Gamma^{-1}$ of units acts naturally by isometries on the Hardy space and hence also on its norm completion.*

Proof. (i) Using property (4) of the Hardy space data, we calculate

$$\int_{s.N} \omega|_{s.N} \otimes \overline{\omega}|_{s.N} = \int_{s.N} (\omega \otimes \overline{\omega})|_{s.N} = \int_N s^*\omega \otimes s^*\overline{\omega} = \|(s^*\omega|_N)\|^2_{L^2(N, \mathbf{HF}_N)},$$

thus

$$\|\omega\|^2_{\mathcal{H}} = \sup_{s \in \Gamma^o} \|(s^*\omega)|_N\|_{L^2(N, \mathbf{HF}_N)}.$$

Using the triangle inequality in $L^2(N, \mathbf{HF}_N)$, it follows that, if $\|\omega\|_{\mathcal{H}} < \infty, \|\nu\|_{\mathcal{H}} < \infty$, we have also $\|\omega + \nu\|_{\mathcal{H}} < \infty$. The claim follows.

(ii) Using the calculation from part (i), we have

$$\begin{aligned}
\|s^*\omega\|^2_{\mathcal{H}} &= \sup_{g \in \Gamma^o} \|(g^*s^*\omega)|_N\|_{L^2(N, \mathbf{HF}_N)} = \sup_{g \in s\Gamma^o} \|(g^*\omega)|_N\|_{L^2(N, \mathbf{HF}_N)} \\
&\leq \sup_{g \in \Gamma^o} \|(g^*\omega)|_N\|_{L^2(N, \mathbf{HF}_N)} \\
&\leq \|\omega\|^2_{\mathcal{H}}.
\end{aligned}$$

(iii) This is immediate with (ii). $\blacksquare$

Note that it is by no means clear at this point that the Hardy space $\mathcal{H}^2(\mathcal{U})$ is an inner product space. In other words, we do not have the means to show that the norm $\|\cdot\|^2_{\mathcal{H}}$ satisfies the parallelogram identity

$$2\|\omega\|^2_{\mathcal{H}} + 2\|\omega'\|^2_{\mathcal{H}} = \|\omega+\omega'\|^2_{\mathcal{H}} + \|\omega-\omega'\|^2_{\mathcal{H}}.$$

1.4. The boundary value map. From the theory of Hardy spaces of functions one expects an isometric boundary value map $b : \mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma) \to L^2(N, \mathbf{HF}_N)$. We formulate an assumption on the geometry of $\mathcal{U}$, N and Γ under which this is indeed the case:

Assumption 1.4.1. (Polar decomposition.) In addition to the data given by Def. 1.3.3, assume that (1) G preserves N, (2) Γ is a locally compact semigroup for which the group $G = \Gamma \cap \Gamma^{-1}$ is a Lie group such that Γ admits a *polar decomposition*

$$\Gamma = G \exp(iC)$$

with a $\mathrm{Ad}(G)$-invariant regular (i.e. convex, open and pointed) cone C in the Lie algebra $\mathfrak{g}$ of G. In more detail, this means: each element $s \in \Gamma$ can be uniquely written as $s = gp$, where g belongs to G and p belongs to a one-parameter subsemigroup $t \mapsto \gamma(t)$ of Γ for which the vector field $X(x) := i\frac{d}{dt}|_{t=0}\gamma(t)x$ is of the form $x \mapsto \frac{d}{dt}|_{t=0}\exp(tY).x$ for Y in C. Here multiplication by i on the tangent space of M is given by the almost complex structure of M. ■

Theorem 1.4.2. *Let $(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ be Hardy space data satisfying* Assumption 1.4.1. *Then there is an isometric boundary value map given by*

$$b : \mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma) \to L^2(N, \mathbf{HF}_N), \quad \omega \mapsto \lim_{t\to 0}((\exp(itX)^*\omega)|_N)$$

which is independent of the element $X \in \mathrm{int}C$.

Proof. This can be proved in the same way as the corresponding statement for Hardy-spaces of functions on Ol'shanskiĭ-semigroups, cf. e.g. [HN93, p. 277 – 279]. Let us briefly recall the main points: We write $\Gamma = G\exp(iC)$ and let $X \in \mathrm{int}(iC)$. For a fixed $\omega \in \mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ we consider the map

$$F : \{z \in \mathbb{C} |\ \mathrm{Re}\, z > 0\} \to L^2(N, \mathbf{HF}_N), \quad z \mapsto (\exp(zX)^*\omega)|_N.$$

In this situation, a Paley-Wiener type lemma due to G. Ol'shanskiĭ (cf. [HN93, Lemma 9.11]) implies that $b_X(\omega) := \lim_{z\to 0} F(z)$ exists in $L^2(N, \mathbf{HF}_N)$. Now it is proved by standard arguments that b_X is an isometry which does not depend on X (cf. [HN93, p. 278/279]). ■

We note the following consequences of Theorem 1.4.2 for later use

Corollary 1.4.3. *Let $(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ be Hardy space data satisfying* Assumption 1.4.1. *Then we have:*

(i) *The Hardy space $\mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ is an inner product space.*
(ii) *The space of sections of* **HF** *which are holomorphic in some neighborhood of $\mathcal{U}$ is dense in $\mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$.* ■

Recall the basic concepts of the holomorphic representation theory for involutive semigroups from [Ne99] : An *involutive semigroup* is a pair $(\Gamma, \sharp)$, where Γ is a semigroup and $\sharp : \Gamma \to \Gamma$ is an involutive anti-automorphism. A *Hermitian semigroup-representation* of a semigroup Γ with involution $\sharp$ on a pre-Hilbert space $\mathcal{H}^0$ is a semigroup homomorphism $\pi : \Gamma \to B_0(\mathcal{H}^0)$ preserving the involutions, i.e. $\pi(s^\sharp) = \pi(s)^*$. Here $B_0(\mathcal{H}^0)$ is the vector space of linear operators

$A\colon \mathcal{H}^0 \to \mathcal{H}^0$ for which a formal adjoint exists. This is an involutive semigroup, so the above definition makes sense. π is called *bounded* if $\pi(s)$ is a bounded operator for all $s \in \Gamma$. If Γ is a topological semigroup with a continuous involution, then a bounded representation π on a Hilbert space $\mathcal{H}$ is called *continuous* if $\pi\colon \Gamma \to B(\mathcal{H})$ is continuous w.r.t. the weak operator topology on the algebra $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$.

If Γ in addition is a complex manifold and $\sharp$ is anti-holomorphic, then $(\Gamma, \sharp)$ is called a *complex involutive semigroup* and a bounded representation $\pi\colon \Gamma \to B(\mathcal{H})$ is called *holomorphic* if it is holomorphic as a map when $B(\mathcal{H})$ is endowed with its natural Banach space structure.

Remark 1.4.4. Note that the existence of a polar decomposition for Γ automatically shows that we have an involutive self-map $s \mapsto s^\sharp$ of Γ via $(g \exp X)^\sharp = (\exp X) g^{-1}$. In fact, we calculate

$$(s^\sharp)^\sharp = \exp(\mathrm{Ad}(g)X) g = g(\exp X) g^{-1} g = g \exp X.$$

But the equality $(s_1 s_2)^\sharp = s_2^\sharp s_1^\sharp$ cannot automatically be deduced from this definition unless Γ is abelian. Below we will encounter various examples of semigroups with polar decomposition such that the associated involutive selfmap is actually an involution defined as follows: there is a "complex conjugation" τ of M w.r.t. the real form N, and then the equation $s^\sharp = \tau \circ s^{-1} \circ \tau$ holds on some neighborhood of $\mathcal{U}$. ∎

Theorem 1.4.5. *Suppose that* Assumption 1.4.1 *holds. In addition we assume that* $(\Gamma, \sharp)$ *is a complex involutive semigroup such that*

$$(g \exp X)^\sharp = (\exp X) g^{-1} = g^{-1} \exp(\mathrm{Ad}(g)X).$$

Then the representation of Γ *from* Proposition 1.3.5 *on the completion of* $\mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ *is a holomorphic Hermitian representation.*

Proof. We set $\pi(s)\omega = s^*\omega$ for $\omega \in \mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ and use Proposition 1.3.5 to extend $\pi(s)$ to a contraction on the completion $\mathcal{H}$ of $\mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$. In this way we obtain, again by Proposition 1.3.5, a bounded representation $\pi\colon \Gamma \to B(\mathcal{H})$. Once more from Proposition 1.3.5 we see that the restriction of π to $G = \Gamma \cap \Gamma^{-1}$ is unitary representation.

Using charts and a suitable partition of unity we can view the integrals which give the norms of the half-forms involved as ordinary integrals of functions. Since Γ acts by contractions we can use Lebesgue's theorem of dominated convergence to see that π is holomorphic (cf. [HN93, Lemma 9.7]).

It remains to be shown that $\pi(s^\sharp) = \pi(s)^*$. The unitarity of $\pi|_G$ shows that it suffices to show that $\pi(\exp X)$ is self adjoint for $X \in iC$. Consider the derived representation $d\pi$ of $\mathfrak{g}$ on $\mathcal{H}$. Since $iX \in \mathfrak{g}$ we know that $d\pi(X)$ is a self adjoint operator on $\mathcal{H}$ which is the infinitesimal generator of the unitary one-parameter group $\pi\big(\exp(itX)\big)$. Let P be the spectral measure of $d\pi(X)$ and set $A = P(]-\infty, 0])d\pi(X)$.

Now for $\omega \in \mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ consider the function $z \mapsto F_\omega(z) = \pi\big(\exp(zX)\big)\omega = (\exp zX)^*\omega$ which is defined on $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}\, z \geq 0\}$. Then $F_\omega(z + it) = \pi(\exp itX) F_\omega(z)$ and Ol'shanskiĭ's Paley-Wiener Lemma ([HN93, Lemma 9.11]) shows that there exists a $\xi \in \mathcal{H}$ such that $F_\omega(z) = e^{zA}\xi$ for $z \in \mathbb{C}_+$. Since $F_\omega(0) = \omega$ we see that $\xi = \omega$ which implies

$$\pi\big(\exp(zX)\big)\omega = e^{zA}\omega.$$

By continuity we now find $\pi\big(\exp(zX)\big) = e^{zA}$. Since A is self adjoint the claim follows from the case $z = 1$. ∎

Remark 1.4.6. There is an analog of Theorem 1.4.5 for Bergman spaces: Suppose that $(\Gamma, \sharp)$ satisfies the assumptions of Th. 1.4.5. Then the representation of Γ on $\mathcal{B}^2(\mathcal{U})$ from Proposition 1.1.3 is a holomorphic Hermitian representation.

In fact, we can obtain this result copying the proof of Theorem 1.4.5 with $\mathcal{B}^2(\mathcal{U})$ instead of $\mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$. It even gets a little simpler since $\mathcal{B}^2(\mathcal{U})$ already is complete. ■

1.5. Completeness of Hardy spaces. In general the normed vector space from Def. 1.3.4 will not be complete. We introduce a sufficient condition on the Hardy space data which will allow us to prove completeness in various cases.

Definition 1.5.1. Hardy space data $(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ are called *complete* if Assumption 1.4.1 is satisfied and the inclusion map

$$\mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma) \hookrightarrow \mathcal{O}(\mathcal{U}, \mathbf{HF})$$

is continuous, where $\mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ carries the norm topology and $\mathcal{O}(\mathcal{U}, \mathbf{HF})$ the topology of compact open convergence. ■

In the following proposition we use the concept of a reproducing kernel for a Hilbert space of sections as explained in the appendix.

Proposition 1.5.2. *Let $(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ be complete Hardy space data. Then $\mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ is a Hilbert space which admits a reproducing kernel.*

Proof. In view of Theorem 1.4.2 this can be proved just as the corresponding result [HN93, Theorem 9.31, in particular p. 280]. ■

In general it is not easy to check the completeness of Hardy space data. In fact, we don't have a general criterion that covers all the known cases of complete data.

Proposition 1.5.3. *Consider the Hardy space data $(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ and assume*

(a) *They satisfy Assumption 1.4.1 (polar decomposition).*

(b) *The map $\Gamma^o \times N \to \mathcal{U}, (s, n) \mapsto s.n$ is a submersion.*

Then $(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ is a set of complete Hardy space data.

Proof. Since the question is of local nature we may assume that the bundle $\mathbf{HF}$ admits a trivialization on some neighborhood of $\mathcal{U}$. So let us assume that ν is a global nowhere vanishing section of $\mathbf{HF}$ on some neighborhood of $\mathcal{U}$. Identifying holomorphic sections ω of $\mathbf{HF}$ with holomorphic functions f via $\omega = f\nu$, the semigroup Γ acts by

$$(s^* f)(z) = j(s, z) f(s.z)$$

with $j(s, z)$ defined by $s^*\nu = j(s, \cdot)\nu$ (cf. Remark 1.1.1). Let $d\mu$ be the measure defined by the density $\nu \otimes \overline{\nu}$ on N. Then the trivialized picture of the Hardy space $\mathcal{H}^2$ is

$$H^2 = \{f \in \mathcal{O}(\mathcal{U}) |\ \sup_{s \in \Gamma^o} \int_N |f(s.u)|^2 |j(s, u)|^2 \, d\mu(u) < \infty\}.$$

Now we follow the proof for the case of Hardy spaces on Ol'shanskiĭ semigroups (cf. [HN93, p.279 – 280] or [Ne98, Lemma 1.3]): the main point is to prove that for any compact subset $K \subset \mathcal{U}$ we can find a constant c_K depending only on K such that

$$\sup_{z \in K} |f(z)| \leq c_K ||f||_{H^2}$$

for all $f \in H^2$. For the proof of this estimate Assumption (b) is essential, since it permits to introduce local coordinates adapted to the problem and thus to show that f satisfies a local Bergman-type condition. The absolute value of the cocycle factor $j(s, z)$ can locally be estimated from above and from below and does not affect this way of reasoning. ■

We note here that Assumption (b) of Proposition 1.5.3 is not necessary. In fact, it is not satisfied for the data leading to the classical Hardy spaces $H^2(D)$ with D a bounded symmetric domain of complex dimension n greater than one (cf. Section 2.3): In that case Γ is one-dimensional, therefore ΓN is $n+1$-dimensional and hence not open in the real $2n$-dimensional D.

Remark 1.5.4. If one strengthens the assumptions on Γ by saying that Γ is a complex Ol'shanskiĭ semigroup (cf. [Ne98]), then the proof of [Ne98, Prop. 2.4] can be adapted to yield the following result: If the Hardy space data $(\mathcal{U}, N, \mathbf{HF}, \Gamma)$ are complete, then the subspace $b(\mathcal{H}^2(\mathcal{U}, N, \mathbf{HF}, \Gamma)) \subset L^2(N, \mathbf{HF}_N)$ is the largest subspace $F \subset L^2(N, \mathbf{HF}_N)$ such that all the self-adjoint operators iX, $X \in C$, are negative on F. ■

2. Bergman- and Hardy spaces on tube type domains

2.1. Bergman spaces on bounded symmetric domains. In the following we assume that $D = G/K$ is a Hermitian symmetric space. Then D has a canonical realization as a circled bounded symmetric domain in a complex vector space $V_{\mathbb{C}} = \mathbb{C}^n$ (cf. [Lo77] or [Sa80]). The vector space $V_{\mathbb{C}}$ in turn is realized as an open dense subset in the compact dual $(V_{\mathbb{C}})^c$ of D. The imbedding $D \subset V_{\mathbb{C}} \subset (V_{\mathbb{C}})^c$ is called the *Borel-imbedding*. The space $(V_{\mathbb{C}})^c$ is a complex manifold and can be written as the quotient $(V_{\mathbb{C}})^c = G_{\mathbb{C}}/Q$ of complex Lie groups. Then Q is a maximal parabolic subgroup of $G_{\mathbb{C}}$ containing the translation group $t_{V_{\mathbb{C}}}$.

There exist unbounded realizations of the Hermitian symmetric space G/K as *Siegel domains of the second kind*. In case G/K is *of tube type*, this expression reduces to a Siegel domain of the first kind which is by definition a tube domain $T_\Omega = V + i\Omega$ over a homogeneous self-dual cone Ω in a Euclidean vector space V. This space carries the structure of a Euclidean Jordan algebra with unit element e, and the Cayley transform C relating these two realizations via $C(D) = T_\Omega$ can be written in terms of this algebra as $C(z) = i(e+z)(e-z)^{-1}$ (cf. [FK94, p. 190]).

One defines a Bergman space of holomorphic functions on the bounded symmetric domain D by

$$B^2(D) := \{f \in \mathcal{O}(D) | \int_D |f(z)|^2 dz < \infty\}.$$

If we denote by dz also the translation invariant holomorphic top-degree form on $V_{\mathbb{C}}$, restricted to D, then

$$B^2(D) \to \mathcal{B}^2(D), \quad f \mapsto f(dz) \tag{2.1.1}$$

is an isomorphism of Hilbert spaces. One deduces that $\mathcal{B}^2(D)$ is non-trivial since $B^2(D)$ contains all holomorphic polynomials. It follows that the Bergman-kernel K is non-zero. The following explicit formula for the Bergman-kernel is well-known. In order to show that it can be proved by *geometric* methods we indicate a short proof. One defines a polynomial on $V_{\mathbb{C}}$ by

$$B(x,y) := \mathrm{id}_{V_{\mathbb{C}}} - 2x \,\square\, y + P(x)P(y),$$

where $(x \,\square\, y)z = T(x,y,z)$ and $P(x)z = T(x,z,x)$ are defined via the Jordan triple product $T(x,y,z)$ associated to the bounded symmetric domain D (cf. [Lo77] or [Sa80] where the notation $\{xyz\}$ for $T(x,y,z)$ is used).

Theorem 2.1.1. *The reproducing kernel of $B^2(D)$ is (up to a non-zero constant factor) given by*

$$k(z,w) = \operatorname{Det} B(z,\overline{w})^{-1},$$

and the reproducing kernel of $\mathcal{B}^2(D)$ by $K(z,w) = k(z,w)dz \boxtimes \overline{dw}$.

Proof. The group $\operatorname{Aut}(D)$ acts unitarily on the geometric Bergman space. Therefore its reproducing kernel is G-invariant (Thm. A.2.1). Since G acts transitively on D, the kernel $K(z,z)$ is uniquely determined by its value at a base point, and by holomorphy $K(z,w)$ is then also determined. In other words, the reproducing kernel is determined up to a factor by its invariance property. In the trivialized picture the invariance translates into the covariance property

$$k(g.z, g.z) = \operatorname{Det}(\mathrm{d}g(z))^{-2} k(z,z).$$

It can be proved by geometric methods that the function $z \mapsto \operatorname{Det} B(z,\overline{z})^{-1}$ has precisely this covariance property (cf. [Lo77]; in [Be98, Section 1.4] a more direct proof is given which in fact does not need the trivialization). Since the Bergman-kernel is not zero, we conclude that there is a scalar $\lambda \neq 0$ with $k(z,w) = \lambda \operatorname{Det}(B(z,\overline{w}))^{-1}$. ■

Moreover, since the group G acts transitively on D by holomorphic diffeomorphisms, a theorem of S. Kobayashi (cf. [Ko68] or [BH98a, Thm. 2.5]) implies that $\mathcal{B}^2(D)$ is an irreducible unitary G-module. Next one defines a family of *weighted Bergman spaces* of holomorphic functions by

$$B^2_m(D) := \{f \in \mathcal{O}(D) | \int_D |f(z)|^2 \operatorname{Det} B(z,\overline{z})^{m-1}\, dz < \infty\}$$

(cf. [FK94, Ch.XIII]). This is the trivialized picture of the bundle-valued Bergman space $\mathcal{B}^2(D, \mathbf{K}_D^{m-1})$ with values in the line bundle $\mathbf{K}_D^{m-1}$ with the Hermitian metric given by the $(m-1)$th power of the Bergman kernel function. Since this metric is G-invariant, we have again a unitary and irreducible G-representation on this space.

2.2. The geometric Hardy space of a tube-type domain. We are going to define Hardy-space data associated to a bounded symmetric domain D of tube type. On $M = V_{\mathbb{C}}$ we choose the natural trivial half-form bundle $\mathbf{HF}$ and denote its translation-invariant section by $(dz)^{1/2}$. The space N will be given by the *Shilov-boundary* which is described in terms of the complex Jordan algebra $V_{\mathbb{C}}$ by

$$\Sigma = \{z \in V_{\mathbb{C}} | \, z^{-1} = \overline{z}\};$$

this shows that Σ is indeed a real form of $V_{\mathbb{C}}$ (if D is not of tube type, then the Shilov boundary is not a real form of $V_{\mathbb{C}}$). As semigroup Γ we take the double covering $\widetilde{S}_2(D)$ of the *compression semigroup*

$$S(D) = \{s \in G_{\mathbb{C}} | \; s(D) \subset D\}.$$

(For the definition of this double cover and the double cover $\widetilde{G}_2$ of G cf. [BH98b, Remark 2.2.3] or [KØ97].)

Lemma 2.2.1. *The data $(D, \Sigma, \mathbf{HF}, \widetilde{S}_2(D))$ define Hardy space data on D.*

Proof. Using the trivialization $f \mapsto f(dz)^{1/2}$, the group $\widetilde{G}_2$ acts on the space of holomorphic sections of $\mathbf{HF}|_D$ via

$$(g.f)(z) = ((\operatorname{Det} \mathrm{d}g^{-1})(z))^{1/2} f(g^{-1}.z). \tag{2.2.1}$$

The definition of the group $\widetilde{G}_2$ assures that this is indeed a representation. The map $z \mapsto (\operatorname{Det} \mathrm{d}g^{-1}(z))^{-1}$ is a holomorphic polynomial on $V_{\mathbb{C}}$ which vanishes precisely at the points

$z \in V_{\mathbb{C}}$ with $g^{-1}(z) \notin V_{\mathbb{C}}$ (cf. [Be98] or [Lo77]). In particular, it does not vanish on the open neighborhood $U_1 = g^{-1}(V_{\mathbb{C}}) \cap V_{\mathbb{C}}$ of D. Choose a connected simply connected open neighborhood U of D inside U_1. Then the holomorphic function $z \mapsto (\mathrm{Det}\, \mathrm{d}g^{-1}(z))^{1/2}$ has a unique extension to a holomorphic function on U, and therefore (2.2.1) defines an action of g by a local automorphism of **HF** in the sense of Def. 1.3.3 (4). All arguments go through for $\widetilde{G}_2$ replaced by the semigroup $\widetilde{S}_2(D)$. Thus assumption (4) of the Hardy space data (Def. 1.3.3) is verified. ■

Definition 2.2.2. The geometric Hardy space given by the Hardy space data from Lemma 2.2.1 is denoted by $\mathcal{H}^2(D, \widetilde{S}_2(D))$ or just by $\mathcal{H}^2(D)$. ■

Theorem 2.2.3.

(i) *The Hardy space $\mathcal{H}^2(D)$ is complete, has a reproducing kernel S and admits a boundary value map.*

(ii) *There is a constant $\lambda \in \mathbb{C}$, $\lambda \neq 0$, such that*

$$S \otimes S = \lambda K,$$

where K is the Bergman kernel of D. In particular, $\mathcal{H}^2(D)$ is not reduced to zero.

Proof. (i) We verify the assumptions of Thm. 1.4.2 and Prop. 1.5.3: it is well-known that $G(D)$ preserves Σ, and according to a result of G. Ol'shanskiĭ, the semigroup $S(D)$ admits a polar decomposition $S(D) = G \exp(iC)$ where $C = C_{max}$ is in fact a maximal invariant regular $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$. Thus Assumption 1.4.1 is verified. In order to prove completeness of the Hardy-space data, it remains to show that $\Gamma^o \times \Sigma \to D$ is a submersion. But since any element of Γ^o maps Σ into D and G, which is contained in Γ, acts transitively on D, this is clear.

(ii) We use the same arguments as in the proof of Thm. 2.1.1: the group $\widetilde{G}_2$ acts unitarily both on $\mathcal{H}^2$ and on $\mathcal{B}^2$; thus according to Thm. 2.2.1 the kernels S and K are $\widetilde{G}_2$-invariant. The invariance of S implies that also $S \otimes S$ is an invariant kernel. It is a section of $(\mathbf{HF} \otimes \mathbf{HF}) \boxtimes (\mathbf{HF} \otimes \mathbf{HF}) = \mathbf{K}_{\mathrm{D}} \boxtimes \mathbf{K}_D$, and is uniquely determined by its restriction to the diagonal. This restriction is an invariant section of $\mathbf{K}_D$. Since $\widetilde{G}_2$ acts transitively on D, there is up to a factor $\lambda \in \mathbb{C}$ just one invariant section of the line bundle $\mathbf{K}_D$: $S \otimes S = \lambda K$.

It remains to prove that $\lambda \neq 0$, i.e. that $\mathcal{H}^2(D)$ is not reduced to zero. Here we have to use some specific information on the geometric situation. We proceed in steps:

1. We trivialize $\mathcal{H}^2(D)$ via $(dz)^{1/2}$: If f is a holomorphic function on D, we let

$$||f||^2 := ||f||^2_{H^2(D)} := \sup_{s \in \widetilde{S}_2(D)^o} \int_\Sigma |f(s.u)|^2 |\,\mathrm{Det}\, \mathrm{d}s(u)|\, d\sigma(u),$$

where $d\sigma$ is the (normalized) K-invariant measure on Σ. Then we define a Hardy space of holomorphic functions

$$H^2(D, \widetilde{S}_2(D)) := \{f \in \mathcal{O}(D) \,|\, ||f||^2 < \infty\}. \tag{2.2.2}$$

We claim that

$$H^2(D, \widetilde{S}_2(D)) \to \mathcal{H}^2(D), \quad f \mapsto f\,(dz)^{1/2}$$

is a Hilbert space isomorphism. In fact, the group K is a subgroup of the unitary group of $V_{\mathbb{C}}$. Therefore the density $|dz| = (dz)^{1/2} \otimes \overline{(dz)^{1/2}}$ is K-invariant, and so is the restriction of this density to Σ. It thus defines a K-invariant measure on Σ. Such a measure being unique up to a constant factor, it must be proportional to $d\sigma$. Now formula (2.2.1) shows that the definitions of both Hardy spaces correspond to each other under the trivialization.

2. Let f be a function which is holomorphic on some neighborhood of D. We will prove that then

$$||f||_{H^2(D)} \leq \int_\Sigma |f(u)|^2 \, d\sigma(u) = ||f||_{L^2(\Sigma)} \tag{2.2.3}$$

(actually, we will have equality). Since f is continuous on the compact space Σ, this integral is finite, and it follows that $f \in H^2(D, \widetilde{S}_2(D))$. This proves that $H^2(D, \widetilde{S}_2(D))$ contains e.g. all holomorphic polyomials and therefore is not reduced to zero.

Since $S(D) = G\exp(iC)$ and G preserves the Hardy-space norm, it is enough to take the supremum in the definition of the Hardy space over $\exp(iC)$. But then, letting for any $X \in iC$,

$$F_t := ||(\exp(tX)_*)(\omega)||^2_{L^2(\Sigma,\mathbf{HF})} = \int_\Sigma |f(u)|^2 |\operatorname{Det} \mathrm{d}(\exp(tX))(u)| \, d\sigma(u)$$

(where $\omega = f(dz)^{1/2}$), it is enough to prove that F_t is bounded from above by $||f||_{L^2(\Sigma)}$ for all $t \in \mathbb{R}^+$.

3. The function $t \mapsto F_t$, being a composition of a holomorphic map and a norm-function, is subharmonic on a half-plane. It is constant in imaginary direction, and therefore its restriction to $\mathbb{R}^+$ is convex. We will prove that

$$\lim_{t\to\infty} F_t = 0. \tag{2.2.4}$$

Together with the convexity this implies that $t \mapsto F_t$ is decreasing on $\mathbb{R}^+$. Then its supremum is obtained by taking the limit for $t \to 0$, proving that this supremum is $||f||_{L^2(\Sigma)}$.

4. Since $\exp(X)$ is a strict compression of D, i.e. $\exp(X).\overline{D} \subset D$, there is by compactness a real number $r < 1$ with $\exp(X).\overline{D} \subset rD$. Let $d(x,y)$ be the distance of x and y w.r.t. the Bergman metric on D. Then on $r\overline{D}$ the Bergman metric and the Euclidean metric of $V_\mathbb{C}$ are equivalent in the sense that there exist constants $c_0 < 1$ and $c_1 > 1$ such that for all $x, y \in r\overline{D}$ the inequality

$$c_0 d(x,y) < |x-y| < c_1 d(x,y) \tag{2.2.5}$$

holds. This follows from the fact that the Bergman metric is given by

$$h_z(u,v) = \langle B(z,\overline{z})^{-1}u, v\rangle,$$

where $\langle u, v\rangle$ is the Euclidean scalar product on $V_\mathbb{C}$ (cf. [Lo77, Th. 2.10]).

5. By a result of J.-L. Clerc ([Cl98]; note that its proof requires only standard facts on the geometry of D), the map $\exp(X)$ is a strict contraction of the Bergman distance on D, i.e. there exists a constant $k < 1$ with $d(\exp(X).x, \exp(X).y) \leq kd(x,y)$ for all $x, y \in D$. Taking powers of $\exp(X)$, we can find $s > 0$ and $a < 1$ such that for all $x, y \in D$

$$d(\exp(sX).x, \exp(sX).y) \leq a\frac{c_0}{c_1} d(x,y). \tag{2.2.6}$$

6. For $g = \exp(sX)$ we estimate for all $x, y \in r\overline{D}$, using (2.2.5) and (2.2.6):

$$|g.x - g.y| < c_1 d(g.x, g.y) < ac_0 d(x,y) < a|x-y|,$$

i.e. g is a strict contraction for the Euclidean metric on $r\overline{D}$. (Using Banach's fixed point theorem, we can now conclude that g has a fixed point in $r\overline{D}$, but we don't need that at this point.) It follows that for all $z \in r\overline{D}$ and $N \in \mathbb{N}$

$$|\operatorname{Det} \mathrm{d}g^N(z)| \leq a^{nN}$$

where $n := \dim V$.

7. Using the chain rule, we have for all $N \in \mathbb{N}$ and $u \in \Sigma$,

$$\mathrm{d}(\exp(NsX)\exp(X))(u) = (\mathrm{d}\exp(NsX))(\exp(X).u) \circ \mathrm{d}(\exp(X))(u).$$

Now we take determinants and put $M := \sup_{u\in\Sigma} |\operatorname{Det}(\mathrm{d}\exp(X))(u)|$. Note that $\exp(X).u \in rD$. We get

$$|\operatorname{Det}(\mathrm{d}(\exp(NsX)\exp(X))(u)| < Ma^{nN}.$$

From this we get

$$F_{Ns+1} = \int_\Sigma |f(u)|^2 |\operatorname{Det}\mathrm{d}(\exp(Ns+1)X)(u)| d\sigma(u) \leq Ma^{nN} \int_\Sigma |f(u)|^2 d\sigma(u).$$

Since $||f||_{L^2(\Sigma)} < \infty$ by boundedness of f on $\overline{D}$, this tends to zero as N tends to infinity. Together with the convexity of $t \mapsto F_t$ this implies that $\lim_{t\to\infty} F_t = 0$, and as explained in Steps 2 and 3 the claim follows. ■

Remark 2.2.4.

(i) The proof of Theorem 2.2.3 shows that the dynamical system $\exp(\mathbb{R}^+X)$ behaves very much like the dynamical system $\mathbb{R}^+\mathrm{id}_{V_\mathbb{C}}$: it has precisely one fixed point in D which is an attractor for all points in some neighborhood of D. This makes the proof work. The behavior of one-parameter semigroups of compressions which are not strict is in general much more complicated.

(ii) The proof works for any $f \in \mathcal{O}(D)$ such that f has almost everywhere a pointwise limit on Σ defining a square integrable function there. It would be interesting to know whether this condition already describes the Hardy space (i.e. is the boundary value map pointwise almost everywhere?) ■

2.3. Classical versus geometric Hardy spaces. Theorem 2.2.3 gives a satisfactory explanation of Facts 1 and 3 (Introduction) for the *geometric* Hardy space. Next we will discuss the relation of the geometric Hardy space with the classical Hardy spaces (Fact 2). To this end we define a semigroup

$$\mathbb{C}^< = \{z \mapsto tz |\, t \in \mathbb{C}^*, |t| < 1\} \subset S(D).$$

It is clear that $(D, \Sigma, \mathbf{HF}, \mathbb{C}^<)$ are Hardy space data on D. The corresponding Hardy-space of half-forms is denoted by $\mathcal{H}^2(D, \mathbb{C}^<)$.

Theorem 2.3.1. *We have an equality of normed vector spaces*

$$\mathcal{H}^2(D, \widetilde{S}_2(D)) = \mathcal{H}^2(D, \mathbb{C}^<).$$

In particular, $\mathcal{H}^2(D, \mathbb{C}^<)$ *is a complete Hilbert space admitting a reproducing kernel.*

Proof. The inclusion "$\subset$" follows from the fact that the supremum in the definition of $\mathcal{H}^2(D, \mathbb{C}^<)$ is taken over a subsemigroup of $\widetilde{S}_2(D)$ (strictly speaking, we should use the pre-image of $\mathbb{C}^<$ under the covering $\widetilde{S}_2(D) \to S(D)$, but this doesn't change the Hardy-norm).

Moreover, this inclusion is isometric: first we check that $\mathcal{H}^2(D, \mathbb{C}^<)$ satisfies Assumption 1.4.1 and thus admits an isometric boundary value map (Thm. 1.4.2). Since the boundary value map is independent of $X \in \mathrm{int}(C)$, we choose $-X$ to be the Euler vector field (i.e. $X(p) = -p$) and have for all $\omega \in \mathcal{H}^2(D, \mathbb{C}^<)$

$$||\omega||_{\mathcal{H}^2(D,\mathbb{C}^<)} = \lim_{t\to 0} ||(\exp tX)^*\omega)||_{L^2(\Sigma)} = ||\omega||_{\mathcal{H}^2(D,\widetilde{S}_2(D))}.$$

For the proof of the inclusion "$\supset$" we use the proof of Theorem 2.2.3: There we have seen that the subspace $\mathcal{H}_o$ of sections of **HF** which are holomorphic on some neighborhood of D is contained in $\mathcal{H}^2(D, \widetilde{S}_2(D))$. On the other hand, according to Corollary 1.4.3, $\mathcal{H}_o$ is dense in $\mathcal{H}^2(D, \mathbb{C}^<)$. Thus $\mathcal{H}^2(D, \widetilde{S}_2(D))$ is dense in $\mathcal{H}^2(D, \mathbb{C}^<)$, and since this subspace is complete, we actually have equality. ∎

Note that the preceding theorem together with Thm. 2.2.3 shows that $H^2(D)$ is a reproducing kernel space and yields a proof for the formula of the reproducing kernel which is independent of other known proofs. Thus Facts 1 and 3 (Introduction) are explained for $H^2(D)$.

For the Hardy space $H^2(T_\Omega)$ of the tube T_Ω the situation is slightly more difficult: it is easily verified that Hardy space data are given by $(T_\Omega, V, \mathbf{HF}, t_{V+i\Omega})$, where

$$t_{V+i\Omega} := \{z \mapsto z + v + iu |\, v \in V, u \in \Omega\}$$

is a semigroup of strict compressions of the domain $T_\Omega \subset V_\mathbb{C}$ and **HF** is the trivial half-form bundle induced from $V_\mathbb{C}$. Moreover, it is clear that $H^2(T_\Omega)$ is the trivialized picture of the geometric Hardy space associated to these data.

The reason for the problems is the fact that the semigroup $t_{V+i\Omega}$ is *not* a semigroup of strict compressions of the bounded domain $D = C(T_\Omega)$ equivalent to T_Ω. In fact, the points of the Shilov-boundary lying "at infinity", i.e. in $\Sigma \setminus C(V)$, are not mapped into the interior of D under $Ct_{V+i\Omega}C^{-1}$. In other words, $Ct_{V+i\Omega}C^{-1}$ is a semigroup belonging to the boundary of $S(D)$.

Proposition 2.3.2. *The contractive semigroup representation*

$$\widetilde{S}_2(D)^o \to B(\mathcal{H}^2(D))$$

has a unique continuous extension to a contractive semigroup representation

$$\widetilde{S}_2(D) \to B(\mathcal{H}^2(D)).$$

The extension agrees with the representation of $\widetilde{S}_2(D)$ on the holomorphic sections of $\mathbf{HF}|_D$ given by the formula (2.2.1).

Proof. We know from Theorem 1.4.5 that the representation of $\widetilde{S}_2(D)^o$ on $\mathcal{H}^2(D)$ is a holomorphic Hermitian representation by contractions. Since $\widetilde{S}_2(D)^o$ contains an approximate identity we can use [Ne99, Thm. IV.1.27] to show that a continuous extension to $\widetilde{S}_2(D)$ exists. This extension is unique and automatically contractive.

To show also the last claim, we recall that convergence w.r.t. the Hardy norm implies convergence w.r.t. the compact open topology so that the density of $\widetilde{S}_2(D)^o$ in $\widetilde{S}_2(D)$ and the continuity of the representation (2.2.1) imply the claim. ∎

For $\omega \in \mathcal{H}^2(D)$ we write $||\omega|_\Sigma||_{L^2(\Sigma)}$ for the norm of the boundary value $b\omega$. Then the fact that the extended semigroup representation from the preceding proposition is contractive implies that

$$\sup_{s\in\widetilde{S}_2(D)} ||s^*\omega|_\Sigma||_{L^2(\Sigma)} \leq ||\omega||_{\mathcal{H}^2(D)};$$

on the other hand, by definition of the Hardy space we have the converse inequality, and thus equality holds. In other words, the Hardy space $\mathcal{H}^2(D)$ can also be defined by the taking the supremum over the whole contraction semigroup and not only over its interior.

Theorem 2.3.3. *We have an equality of Hilbert spaces*

$$\mathcal{H}^2(D, \widetilde{S}_2(D)) = \mathcal{H}^2(D, C(V), \mathbf{HF}, Ct_{V+i\Omega}C^{-1}).$$

Proof. "$\subset$": Note that $C(V)$ is open dense in Σ. Therefore

$$\sup_{s\in t_{V+i\Omega}} ||s^*\omega||_{L^2(C(V))} = \sup_{s\in t_{V+i\Omega}} ||s^*\omega||_{L^2(\Sigma)} \leq \sup_{s\in \widetilde{S}_2(D)} ||s^*\omega||_{L^2(\Sigma)}.$$

As was remarked before stating the theorem, the last term defines the Hardy space norm of $\mathcal{H}^2(D, \widetilde{S}_2(D))$, and the desired inclusion follows.

"$\supset$": This follows by the same arguments as in the proof of Thm. 2.3.1. ∎

Remark 2.3.4.

(i) Let n be the real dimension and r be the rank of the Euclidean Jordan algebra V. If V is simple and $\frac{n}{2r}$ is an integer, one can define the half-form bundle $\mathbf{HF}$ on all of $(V_{\mathbb{C}})^c$ via an induced represenation. If it is a half-integer, this is no longer possible.

(ii) There is no immediate generalization of the preceding results to bounded symmetric domains which are not of tube type. The reason for this is that in the non-tube type case the Shilov boundary is not a totally real submanifold of M (it contains subspaces of the form $\mathbb{C}^k$ with $k > 0$). There is still a generalization of a Hardy space of functions on a Siegel domain of second kind (cf. [KS72]), but we have no evidence that its reproducing kernel is still a square root of the Bergman kernel. Also no reasonable unitary G-action on this space is known. For these reasons it seems not advisable to try to interprete the Hardy space of non-tube type domains in terms of half-forms.

(iii) The Hardy spaces $H^2_m(T_\Omega)$ defined in [FK94, p.270] do carry a unitary G-action. In these cases the Shilov boundary is replaced by other G-orbits N in ∂T_Ω. However, if $N \neq \Sigma$, then N contains holomorphic arc components, i.e. it is no longer a totally real submanifold. Thus one could try to interprete these spaces as "Hardy spaces with values in Bergman spaces". ∎

3. Bergman- and Hardy spaces on generalized tube domains

3.1. Bergman spaces. Let Ξ be an open domain in a complex homogeneous space $M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is a complex Lie group and $H_{\mathbb{C}}$ a complex closed subgroup. Then the geometric Bergman space $\mathcal{B}^2(\Xi)$ is canonically defined. On the other hand, in harmonic analysis one considers Bergman spaces of functions which are defined under the assumption that there exists a $G_{\mathbb{C}}$-invariant measure μ on $M_{\mathbb{C}}$. The Bergman space associated to such a measure is

$$B^2(\Xi, \mu) := \{f \in \mathcal{O}(\Xi) | \int_\Xi |f(z)|^2 d\mu(z) < \infty\}$$

(cf. e.g. [Kr98], [HK98], [Pe96]). The point one has to observe here is that the measure μ is in general not defined by a volume form, and therefore $B^2(\Xi, \mu)$ is in general not just the trivialized picture of $\mathcal{B}^2(\Xi)$. However, there is a an important class of examples for which one comes close to that situation (cf. [Wa72, Appendix]): Suppose that $G_{\mathbb{C}}$ is reductive so that its modular function $|\det \circ \mathrm{Ad}|^{-1}$ is the constant function 1. Then the existence of the invariant measure implies that the image of the complex representation

$$\delta : H_{\mathbb{C}} \to \mathbb{C}^*, \quad h \mapsto \det(\mathrm{d}_o h)$$

(determinant of the differential at the base point $o = eH$) is contained in the unit circle. Therefore it is constant on each connected component of $H_{\mathbb{C}}$ and in particular trivial on the identity component. If $H_{\mathbb{C}}$ has only finitely many components, $\delta(H_{\mathbb{C}})$ is a finite subgroup of the unit circle and hence the kernel H_1 of δ is a normal subgroup of finite index m in $H_{\mathbb{C}}$. Thus the map

$$p : \widetilde{M_{\mathbb{C}}} := G_{\mathbb{C}}/H_1 \to M_{\mathbb{C}}, \quad gH \mapsto gH_1$$

is an m-fold covering.

Suppose that $M_{\mathbb{C}}$ is actually a complexification of a *real* homogeneous space M, i.e. that there are closed real forms G and H of $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$ such that $M = G/H$. Then the restriction of δ to H is real valued. This means the only possible values for $\delta(h)$ are ± 1. If now each component of $H_{\mathbb{C}}$ intersects H we conclude that $\widetilde{M_{\mathbb{C}}}$ is at most a double cover of $M_{\mathbb{C}}$.

The space $\widetilde{M_{\mathbb{C}}}$ clearly admits a non-trivial holomorphic $G_{\mathbb{C}}$-invariant n-form ν which defines a $G_{\mathbb{C}}$-invariant measure $\widetilde{\mu}$ on $\widetilde{M_{\mathbb{C}}}$. Let $\widetilde{\Xi} := p^{-1}(\Xi)$. Now $B^2(\widetilde{\Xi}, \widetilde{\mu})$ is the usual trivialization of $\mathcal{B}^2(\widetilde{\Xi})$ w.r.t. ν. Up to a constant factor, the pullbacks

$$p^* : \mathcal{B}^2(\Xi) \to \mathcal{B}^2(\widetilde{\Xi}), \quad p^* : B^2(\Xi, \mu) \to B^2(\widetilde{\Xi}, \widetilde{\mu})$$

are isometric imbeddings with image being the respective spaces of "even" elements, i.e. those which are invariant under the group of decktransformations.

Since for $m > 1$ the form ν is "odd", i.e. *not* preserved under decktransformations, the trivialization map $f \mapsto f\widetilde{\mu}$ does not respect the spaces of "even" elements.

3.2. Hardy-spaces. The same remarks as above apply, but dealing with half-form bundles leads to additional and more subtle complications. First, as usual for Hardy spaces, we need a more specific geometric set-up. Here the framework of *compactly causal symmetric spaces* is natural (cf. [HOØ91], [HØ96], [Ol91]). To such a space $M = G/H$ one associates a domain $\Xi = G\exp(iW).o$ in its complexification $M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$. Here W is a certain open convex $\mathrm{Ad}(G)$-invariant cone in the Lie algebra $\mathfrak{g}$ of G. Then $\Gamma_W := G\exp(iW) \subset G_{\mathbb{C}}$ is a complex semigroup. Now one defines a *Hardy space of holomorphic functions* by

$$H^2(\Xi) := \{f \in \mathcal{O}(\Xi) |\ \sup_{s\in\Gamma_W} \int_M |f(sx)|^2 d\rho(x) < \infty\},$$

where ρ is a G-invariant measure on M (cf. [HOØ91]). When trying to relate this space to a geometric Hardy space, we are faced with two problems:

(a) *Existence*: We do not know whether holomorphic half-form bundles exist on Ξ. This can be remedied using the double cover $\widetilde{M_{\mathbb{C}}}$ introduced in the preceding section: The canonical bundle $\mathbf{K}_{\widetilde{M_{\mathbb{C}}}}$ is trivial and admits a non-trivial invariant section, and thus there is a holomorphic half-form bundle $\mathbf{HF}_0$ having the same properties. It is then easy to see that $(\widetilde{\Xi}, \widetilde{M}, \mathbf{HF}_0, \Gamma_W)$ defines (complete) Hardy space data on Ξ and that the trivialization map

$$H^2(\widetilde{\Xi}) \to \mathcal{H}^2(\widetilde{\Xi}, \widetilde{M}, \mathbf{HF}_0, \Gamma_W)$$

is an isometric bijection. As for Bergman spaces, the pull-back $p^* : H^2(\Xi) \to H^2(\widetilde{\Xi})$ is (up to a factor) an isometric imbedding.

(b) *Uniqueness*: We do not know how many essentially different Hardy space data on Ξ exist. This is a subtle question: already in the case where the trivial bundle over Ξ is a holomorphic half-form bundle it may very well be that two inequivalent actions of the same semigroup Γ exist. Once again this can be remedied by introducing a double covering of Ξ (as is done by Koufany and Ørsted [KØ97] for the group case $M = \mathrm{Sp}(n, \mathbb{R})$ for n even, and more generally by Betten and Ólafsson [BO98] for the cases which, in the language of [BH98b], correspond to the case "$\frac{1}{4}$ non-admissible, but $\frac{1}{2}$ admissible"). The nature of this covering is very different from the covering needed for Problem (a). In fact, in the worst case it may be necessary to combine both coverings, leading to a covering of order 4 (as already introduced in [BO98]; this corresponds to the case "$\frac{1}{4}$ and $\frac{1}{2}$ non-admissible" from [BH98b]). The questions related to this problem are fairly involved and will be taken up elsewhere.

3.3. Further problems. Besides the geometric and group-theoretic problem just mentioned, the major problem in the theory of geometric Hardy spaces is to explain the analogues of Facts 1 – 3 (cf. the introduction) for Hardy spaces on Ξ. More precisely, is there an analogue of our Theorem 2.2.3 for Hardy spaces on Ξ? In some important cases the answer is "yes" – this is just the invariant formulation of the main result from [BH98b, Thm. 4]. However, as mentioned in [BH98b, Section 4], the proof of this result is not yet geometric, and the general problem of the relation between Bergman- and Hardy spaces remains open.

Appendix: Reproducing kernels and semigroup actions

A.1. Reproducing kernels on vector bundles. We have seen that geometric Bergman spaces and some geometric Hardy spaces have the property that point evaluations are continuous. This means that they have a *reproducing kernel* which, in the geometric setting we use, is a section of a vector bundle. Therefore we quickly recall how the standard theory of reproducing kernels can be adapted to a vector bundle setting (cf. [BH98a] for details).

Let M be a topological space and $\mathrm{p}: \mathbf{V} \to M$ a complex vector bundle. We assume that the fibers $\mathbf{V}_z$ over $z \in M$ are finite dimensional and denote the complex antilinear dual bundle by $\mathrm{q}: \mathbf{V}^* \to M$. This means that the fiber $\mathbf{V}^*_z$ of $\mathbf{V}^*$ consists of the complex antilinear functionals on $\mathbf{V}_z$. The corresponding evaluation map will be denoted by $\langle \cdot, \cdot \rangle_z: \mathbf{V}^*_z \times \mathbf{V}_z \to \mathbb{C}$. We will use the canonical identification $\mathbf{V}_z \leftrightarrow (\mathbf{V}^*_z)^*, v \mapsto \widehat{v}$, given by $\widehat{v}(\xi) = \overline{\xi(v)}$, and its global analog $\mathbf{V} \cong \mathbf{V}^{**}$ without further mentioning. If we reverse the complex structure on the fibers of $\mathbf{V}$ we write $\overline{\mathbf{V}}$ instead of $\mathbf{V}$. Then $\overline{\mathbf{V}}$ is still a complex vector bundle. If $\mathbf{V}$ is given by a collection of transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{Gl}(\mathbb{C}^k)$, then $\overline{\mathbf{V}}$ is given by the transition functions $\overline{g_{\alpha\beta}}$.

We write $C(M, \mathbf{V})$ for the continuous sections of $\mathbf{V}$. There is a natural *complex conjugation map* $C(M, \mathbf{V}) \to C(M, \overline{\mathbf{V}}), f \mapsto \overline{f}$. It is defined by the ordinary complex conjugation in the local trivializations. With this complex conjugation the identification $\mathbf{V}_z \otimes \overline{\mathbf{V}}_w \cong \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}^*_w, \mathbf{V}_z)$ can be written as

$$\big(f_1(z) \otimes \overline{f}_2(w)\big)(\eta) = \langle \eta, f_2(w) \rangle_w f_1(z) \tag{A.1.1}$$

for $\eta \in \mathbf{V}^*_w$. The point evaluations $f \mapsto f(z)$ will be denoted $\mathrm{ev}_z: C(M, \mathbf{V}) \to \mathbf{V}_z$.

If M is a manifold and $\mathbf{V}$ a smooth vector bundle, we write $C^\infty(M, \mathbf{V})$ for the smooth sections. Moreover, if M is a complex manifold and $\mathbf{V}$ is a holomorphic vector bundle, then we denote the holomorphic sections of $\mathbf{V}$ by $\mathcal{O}(M, \mathbf{V})$. In this case we denote the manifold M, when equipped with the opposite complex structure, by $\overline{M}$. Given $f \in \mathcal{O}(M, \mathbf{V})$ one finds that $\overline{f}$ is an antiholomorphic section of $\overline{\mathbf{V}}$ or, in other words, an element of $\mathcal{O}(\overline{M}, \overline{\mathbf{V}})$.

Definition A.1.1.

(i) A complex vector subspace $\mathfrak{H} \subseteq C(M, \mathbf{V})$ is called a *Hilbert space of sections* if it carries a Hilbert space structure for which the point evaluations $\mathrm{ev}_z: \mathfrak{H} \to \mathbf{V}_z, f \mapsto f(z)$ are continuous.

(ii) A section $K \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$ is called a *positive definite kernel* if for every finite sequence $\xi_1, \ldots, \xi_n \in \mathbf{V}^*$ the expression

$$\sum_{j,k=1}^{n} \langle \xi_k, K\big(\mathrm{q}(\xi_k), \mathrm{q}(\xi_j)\big)\xi_j\rangle_{\mathrm{q}(\xi_k)}$$

is real and non-negative. ■

The basic result now is

Theorem A.1.2. *Let M be a topological space and $\mathrm{p}\colon \mathbf{V} \to M$ a complex vector bundle. Suppose that $K \in C(M \times M, \mathbf{V} \boxtimes \overline{\mathbf{V}})$. Then the following statements are equivalent:*

(1) *K is a positive definite kernel for $\mathbf{V}$.*

(2) *There exists a Hilbert space $\mathfrak{H}_K \subseteq C(M, \mathbf{V})$ such that $\mathrm{ev}_z|_{\mathfrak{H}_K}\colon \mathfrak{H}_K \to \mathbf{V}_z$ is continuous and $K(z,w) = \mathrm{ev}_z \circ \mathrm{ev}_w^* \in \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}_w^*, \mathbf{V}_z)$ for all $z, w \in M$.* ■

The *reproducing property* of the kernel K in this context is

$$(K_\xi \mid f)_{\mathfrak{H}_K} = \langle \xi, f \circ \mathrm{q}(\xi)\rangle_{\mathrm{q}(\xi)}, \tag{A.1.2}$$

where $K_\xi = \mathrm{ev}_z^*(\xi) \in \mathfrak{H}_K \subseteq C(M, \mathbf{V})$ for $\xi \in \mathbf{V}_z^*$. The Hilbert space $\mathfrak{H}_K$ is called the *reproducing kernel Hilbert space* associated to the kernel K, and K is called the *reproducing kernel* of $\mathfrak{H}_K$. Theorem A.1.2 shows that any positive definite kernel can be viewed as the reproducing kernel of Hilbert space of sections. Therefore will call such a kernel simply a reproducing kernel. The argument given in [Ne99, Lemma I.5] shows that for any reproducing kernel Hilbert space $\mathfrak{H} \subseteq C(M, \mathbf{V})$ with reproducing kernel $K(z,w) = \mathrm{ev}_z \circ \mathrm{ev}_w^*$ we have $\mathfrak{H} = \mathfrak{H}_K$.

Suppose that M is a complex manifold and $\mathbf{V} \to M$ a holomorphic vector bundle. If a reproducing kernel $K\colon M \times M \to \mathbf{V} \boxtimes \overline{\mathbf{V}}$ is holomorphic in the first variable, then the space $\mathfrak{H}_K$ consists of holomorphic sections of $\mathbf{V}$ and K is holomorphic when viewed as a map $K\colon M \times \overline{M} \to \mathbf{V} \boxtimes \overline{\mathbf{V}}$. Thus a Hilbert space of holomorphic sections is given by a positive definite kernel in $\mathcal{O}(M \times \overline{M}, \mathbf{V} \boxtimes \overline{\mathbf{V}})$.

A.2. Semigroup actions and invariance properties. We suppose in addition to the notation and assumptions introduced above that S is a semigroup acting from the left on $\mathbf{V}^*$ by *vector bundle morphisms*. This means that S also acts on M from the left by continuous maps and we have

(i) $\mathrm{q}(s.\xi) = s.\mathrm{q}(\xi)$ for all $\xi \in \mathbf{V}^*$ and $s \in S$.

(ii) $s_z\colon \mathbf{V}_z^* \to \mathbf{V}_{s.z}^*, \xi \mapsto s.\xi$ is $\mathbb{C}$-linear.

Then the dual maps $s_z^*\colon \mathbf{V}_{s.z} \to \mathbf{V}_z$, defined by $\langle s_z(\xi), v\rangle_{s.z} = \langle \xi, s_z^*(v)\rangle_z$ yield a right S-action on $C(M, \mathbf{V})$ via

$$(f.s)(z) := (s_z)^* \circ f(s.z) \tag{A.2.1}$$

for $z \in M$, $f \in C(M, \mathbf{V})$, and $s \in S$.

Theorem A.2.1. *Let M be a topological space and $\mathrm{p}\colon \mathbf{V} \to M$ a complex vector bundle. Suppose that $\mathfrak{H}_K \subseteq C(M, \mathbf{V})$ is a reproducing kernel space. Further let $(S, *)$ be an involutive semigroup acting from the left on $\mathbf{V}^*$ by vector bundle morphisms. Then the following are equivalent.*

(1) *$(s_z)^* \circ K(s.z, w) = K(z, s^\sharp.w) \circ (s^\sharp)_w$ for all $z, w \in M$ and $s \in S$.*

(2) *$\mathfrak{H}_K^0$ is invariant under the right action $f \mapsto f.s$ of S on $C(M, \mathbf{V})$, and this action defines a Hermitian representation of S on $\mathfrak{H}_K^0$.*

Proof. [BH98a, Thm.2.1] (which is the adaptation to the vector bundle case of well-known results in the function case, cf. e.g. [Ne99]). ■

If under the hypotheses of Theorem A.2.1 the positive definite kernel K satisfies the equivalent conditions (1) and (2) we call it an *S-invariant kernel* and denote the representation of S on $\mathfrak{H}_K^0$ by π_K. Note that, if, in the situation of the theorem, the semigroup is a group and the involution is the group inversion, then π_K is a unitary representation of S, and property (1) takes the form

$$K(g.z, g.w) = ((g_z)^*)^{-1} \circ K(z,w) \circ (g_w)^{-1}$$

for $g \in G$. This means that K is a G-invariant section of the bundle $\mathbf{V} \boxtimes \overline{\mathbf{V}}$ over $M \times M$ in the usual sense.

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