The geometry of null systems, Jordan algebras and von Staudt's Theorem

Wolfgang Bertram

Abstract. We characterize an important class of *generalized projective geometries* (X, X') by the following essentially equivalent properties:

(1) (X, X') admits a *central null-system*,

(2) (X, X') admits *inner polarities*,

(3) (X, X') is associated to a *unital Jordan algebra*.

These geometries, called *of the first kind*, play in the category of generalized projective geometries a rôle comparable to the one of the projective line in the category of ordinary projective geometries. In this general set-up, we prove an analogue of von Staudt's theorem which generalizes similar results by L. K. Hua [Hua45].

A.M.S classification: 17 C 37, 51 A 05, 51 A 50, 51 N 25, 53 C 35

Key words: null-system, projective geometry, polar geometry, symmetric space, Jordan algebra

Vandœuvre-lès-Nancy, november 15, 2001

Wolfgang Bertram
Institut Elie Cartan
Université Nancy I, Faculté des Sciences
B.P. 239
54506 Vandœuvre-lès-Nancy, Cedex
France

email: bertram@iecn.u-nancy.fr

0. Introduction

Duality is one of the basic features of linear algebra and of projective geometry: as a matter of principle, a projective space $X = \mathbb{K}\mathrm{P}^n = \mathrm{P}(W)$ over a field $\mathbb{K}$ should always be considered together with its dual space $X' = \mathrm{P}(W^*)$, the space of hyperplanes in X. In general, there are many ways to identify X and X', via *polarities* (given by a non-degenerate quadratic form) or via *null-systems* (given by a symplectic form), but none of these really is canonical. However, there is one exception to this rule: the *projective line* $\mathbb{K}\mathrm{P}^1$ – indeed, hyperplanes and points in $\mathbb{K}\mathrm{P}^1$ are the same objects. Thus to a point $x \in \mathbb{K}\mathrm{P}^1$ there is a canonically associated affine space $V_x = X \setminus \{x\}$ which one might call the "absolute space of x". Of course, the point x itself does not belong to its absolute space V_x; this corresponds to the fact that the identity map $\mathrm{id} : X \to X'$ is a null-system: it comes from the identification of $\mathbb{K}^2$ with its dual space via the, up to a scalar factor canonical, symplectic form on $\mathbb{K}^2$.

In the present work we will show that the correct generalization of the projective line in the category of *generalized projective geometries* (cf. Chapter 1 for the basic notions) is given by spaces corresponding to *unital Jordan algebras* (Theorem 4.1). More precisely, we will prove:

Theorem 0.1. *Under the assumptions formulated in* Theorem 4.1*, the following statements are equivalent for a connected generalized projective geometry over a commutative ring* $\mathbb{K}$ *with* $\frac{1}{2} \in \mathbb{K}$*:*

(1) (X, X') *has a (unique) central null-system,*

(2) (X, X') *has inner polarities,*

(3) *The Jordan pair associated to a base point* (o, o') *in* (X, X') *has invertible elements (i.e. it comes from a unital Jordan algebra).*

Let us illustrate the properties (1) – (3) by the important example of the *Grassmannian geometry* (X, X'): $X = \mathrm{Gras}_p(\mathbb{K}^{p+q})$ is the Grassmannian variety of p-spaces in $\mathbb{K}^{p+q}$, and $X' = \mathrm{Gras}_q(\mathbb{K}^{p+q})$ is its dual space. There exist always *correlations*, i.e. isomorphisms mapping X onto X', given by non-degenerate forms. But the case $p = q$ is very special for in this case we really have equality $X = X'$, and this identification has all formal aspects of a correlation, although it is not given by a form (unless $p = 1$, in which case it is given by a symplectic form on $\mathbb{K}^2$). Geometrically, the identification $n = \mathrm{id}\colon X \to X'$ behaves like a *null-system*: every point $x \in X$ is *isotropic* in the sense that x and $n(x)$ are incident. Moreover, the identification $n : X \to X'$ is *canonical* or *central* in the sense that it commutes with the action of the relevant symmetry group $\mathrm{P\,Gl}(p + q, \mathbb{K})$. Thus, if $p = q$, the Grassmannian geometry (X, X') admits a central null-system, and it is easily seen that for $p \neq q$ this is not the case. This illustrates property (1).

For the example of Grassmannian geometry, the algebraic aspect (3) is given by the Jordan pair of rectangular matrices,

$$(V, V') = (M(p, q; \mathbb{K}), M(q, p; \mathbb{K})), \tag{0.1}$$

with the trilinear product

$$T : V \times V' \times V \to V, \quad (X, Y, Z) \mapsto XYZ + ZYX \tag{0.2}$$

together with its dual. It is well-known that this Jordan pair has invertible elements in the Jordan theoretic sense (cf. [Lo75]) if and only if $p = q$; in this case, it comes from the Jordan algebra $V = V' = M(p,p;\mathbb{K})$ of square matrices with the usual symmetrized matrix product. Note that this case arises precisely if the corresponding Grassmannian geometry admits a central null-system, as we have claimed.

The link between the properties (1) and (3) is property (2) which has both geometric and algebraic aspects. In the example of Grassmannian geometry (X, X'), it corresponds to the *midpoint map*: fix two points $x, y \in X$ and assume that x and y are complementary subspaces of the q-dimensional subspace a; then, since a defines a structure of affine space on the space of complements of a, there is a well-defined midpoint $M_{x,y}(a) \in X$ of x and y with respect to this affine structure. The "midpoint map"

$$M_{x,y} : X' \supset E \to X, \quad a \mapsto M_{x,y}(a) \tag{0.3}$$

is defined on a Zariski-dense part E of X. It is quite easy to find an explicit formula for the map $M_{x,y}$ (cf. Formula (2.14) in Section 2.7). If $M_{x,y}$ extends to a bijection from X' onto X, then it has all formal aspects of a polarity which therefore is called an *inner polarity.* In general, however, $M_{x,y}$ will not extend to a bijection; in fact, the explicit formula shows that inner polarities exist if and only if $p = q$. For instance, in case of ordinary projective geometry $(p = 1)$, the midpoint of x and y for $x \neq y$ always lies on the unique projective line spanned by x and y, and thus the image of $M_{x,y}$ is one-dimensional and hence $M_{x,y}$ cannot extend to a bijection of X' onto X – unless $q = 1$; this is the case of the projective line in which $M_{x,y}$ is equivalent to the polarity given by the symmetric form of signature $(1,1)$. In case $p = q > 1$, the inner polarities are not given by forms; in terms of algebraic geometry, they are *non-standard automorphisms* of the Grassmannians.

Among other examples we would like to mention *Lagrangian geometries* where X is the space of Lagrangian subspaces of some bi- or sesquilinear form: for the Hermitian form of signature $(2,2)$ on $\mathbb{C}^4$, the corresponding Jordan algebra is the Jordan algebra $\mathrm{Herm}(2,\mathbb{C})$ of complex Hermitian 2×2-matrices which is nothing but the *Minkowski space* of special relativity; as explained in [Be00], our Lagrangian geometry X may in this case be seen as the "causal compactification" of our physical space-time; it admits a central null-system. Similarly, in infinite dimension, the geometry associated to the Jordan algebra $\mathrm{Herm}(\mathcal{H})$ of an infinite-dimensional Hilbert space $\mathcal{H}$ shares these features – because of its close relation with quantum theory this case initiated the investigation of Jordan algebras by P. Jordan, J. von Neumann and E. Wigner [JNW34].

Our proof of the main theorem (Theorem 4.1) avoids the use of axiomatic Jordan theory and is based directly on the properties of a generalized projective geometry. It reveals several surprising features which seem very "strange" when one compares with usual projective geometry since we are mostly used to think in terms of dimensions bigger than one. In dimension one, however, we have the "strange" identity of projective classes $[g^*] = [g^{-1}]$ for *all* elements $[g]$ of the projective group $\mathrm{P\,Gl}(2,\mathbb{K})$, where g^* is the adjoint of g with respect to some symplectic form ω on $\mathbb{K}^2$. (This reflects just the fact that $\mathrm{P\,Gl}(2,\mathbb{K})$ preserves ω up to a factor.) In our general context, this translates to the important identity

$$M_{x,y}^{(r)} = M_{y,x}^{(1-r)} = R_{x,y}^{(r^{-1})} = R_{y,x}^{((1-r)^{-1})} = L_{x,y}^{((1-r^{-1})^{-1})} = L_{y,x}^{(1-r^{-1})} \tag{0.4}$$

between certain *inner operators of the geometry,* called operators of *middle-, right- and left multiplication,* with respect to the scalar given by the superscript.

The six scalar values appearing in (0.4) are well-known in projective geometry as the six values of the cross-ratio under permutations of the four arguments. As in classical projective geometry, the special case $r = \frac{1}{2}$ is related to *points in harmonic range* (Chapter 5): $M_{x,y}^{(\frac{1}{2})}$ is nothing but the midpoint map $M_{x,y}$ defined above, and we say that four points (x,y,a,b) are in *harmonic range* if they are pairwise non-incident and if

$$M_{x,y}(a) = b. \tag{0.5}$$

This generalizes the classical notion of harmonic range on a projective line (cf. [B94, 6.4.2]). The notion of harmonic range on Jordan algebras has also been defined by H. Braun via a *generalized cross-ratio* ([Br68]; cf. Remark 5.6); but the notion of cross-ratio, belonging to the topic of *invariant theory on generalized projective geometries*, is much more involved than harmonicity and will be left for later work. In our set-up, we can easily prove a generalization of the well-known theorem by von Staudt (cf. [B94, 6.4.10]) saying that a bijection of the projective line $\mathbb{KP}^1$ preserves harmonic range if and only if it is induced by a semilinear map (Theorem 5.5). In the special case of "matrix geometries" (finite-dimensional Grassmannian and Lagrangian geometries) such a generalization has already been found by L.K. Hua ([Hua45]). However, as W-L. Chow remarks in his related work [Ch49], in these cases already much weaker assumptions on the bijections permit to conclude. Thus it seems that the degree of generality of our version of von Staudt's theorem permits to really appreciate its nature as the most general theorem characterizing the "semi-linear projective group" – indeed, it seems rather unlikely that in our framework weaker assumptions on the bijections could suffice to conclude.

Finally, we discuss the geometric interpretation of *unital Jordan algebras*: a unital Jordan algebra is essentially the same as a Jordan pair together with a distinguished invertible element. Thus the correspondence from Theorem 0.1 gives us a correspondence between

(a) *central null-geometries, together with a distinguished quadruple of harmonic points,*
(b) *unital Jordan algebras.*

An interesting aspect of (real) unital Jordan algebras, featured by the work of Koecher and Vinberg (cf. [FK94] or [Be00]) is that they are related to *symmetric spaces*. In Chapter 6 we generalize this relation to the case of arbitrary dimension and general base fields or rings: as has been shown in [Be01b], the set of non-isotropic points of a polarity p carries naturally the structure of a "symmetric space over $\mathbb{K}$". Now, under the hypotheses of Theorem 4.1, there is one distinguished class of polarites, namely the inner ones. Thus, without any further choices, there is a class of symmetric spaces canonically associated to our geometry. For instance, in the Grassmannian case with $p = q$, this class is represented by the general linear group $\mathrm{Gl}(p,\mathbb{K})$, and in the case of a Euclidean Jordan algebra, it is given by the associated symmetric cone (cf. [FK94]). We describe some important geometric features of these spaces: firstly, they have a *linear realization*, similar to the linear realization $\mathrm{Gl}(p,\mathbb{K}) \subset M(p,p;\mathbb{K})$ and generalizing the *prehomogeneous symmetric spaces* from [Be00, Ch.2]. Secondly, the *Jordan inverse* $j(x) = x^{-1} = Q(x)^{-1}x$ of the associated Jordan algebra plays an important rôle: according to [Lo75, 1.10], the Jordan inverse should correctly be interpreted as a map *exchanging* the dual partners X and X' since $Q(x)$ exchanges the partners. In fact, this is in perfect agreement with our theory since it turns out that, seen this way, j is nothing but the canonical null-system itself. It becomes "visible" only if we use some other way (namely the inner polarity) to identify X and X', and then it can be interpreted as a geometric map from X to X (Section 6.3). Summing up, our approach featuring the canonical null-system can be regarded as a geometric and more general version of the approach by T.A. Springer ([Sp73]) who bases the whole theory of (finite dimensional) Jordan algebras on the Jordan inverse.

Contents of the paper:

Notation. Throughout this paper $\mathbb{K}$ denotes a commutative ring with unit 1 and $\frac{1}{2} \in \mathbb{K}$.

1. Generalized projective geometries

1.1. (Generalized projective geometries.) A *pair geometry* is given by two sets X, X' and a subset $M \subset X \times X'$ such that X is covered by the sets $V_a = \{x \in X \mid (x,a) \in M\}$ for $a \in X'$ and X' is covered by the sets V'_x defined dually for $x \in X$. An *affine pair geometry over* $\mathbb{K}$ is a pair geometry such that all V_a, $a \in X$ carry a structure of an affine space over $\mathbb{K}$ (which shall contain at least two points), and dually for all V'_x, $x \in X$. Then we let, for $(x,a),(y,a) \in M$ and $r \in \mathbb{K}$, $r_{x,a}(y) := ry$ be the product $r \cdot y$ in the $\mathbb{K}$-module V_a with zero vector x, and we define *multiplication maps* by

$$\mu_r : D \supset X \times X' \times X \to X, \quad (x,a,y) \mapsto \mu_r(x,a,y) := r_{x,a}(y), \tag{1.1}$$

with $D = \{(x,a,y) \in X \times X' \times X \mid (x,a) \in M, (y,a) \in M\}$, and dually $\mu'_r : D' \to X'$. *Homomorphisms* are pairs of maps (g,g') which are compatible with the multiplication maps in sense that $g\mu_r(x,a,y) = \mu_r(gx, g'a, gy)$ and dually, and *adjoint pairs* are given by pairs $g : X \to Y$, $h : Y' \to X'$ such that

$$g\mu_r(x,ha,y) = \mu_r(gx,a,gy) \tag{1.2}$$

and dually; we then write $h = g^t$. (Here we allow g and h to be defined not everywhere but at least on some affine part, see [Be01b] for details.) We define operators of *left, right* and *middle multiplication* by

$$\begin{aligned} \mu_r(x,a,y) &= L^{(r)}_{x,a}(y) = R^{(r)}_{a,y}(x) = M^{(r)}_{x,y}(a) \\ \mu'_r(a,x,b) &= L^{(r)}_{a,x}(b) = R^{(r)}_{x,b}(a) = M^{(r)}_{a,b}(x); \end{aligned} \tag{1.3}$$

the superspript (r) will be dropped if the value of r is clear from the context or not relevant. Note that by an elementary property of affine spaces over $\mathbb{K}$, the operators of left and right multiplication are related via the identity $\mu_r(x,a,y) = \mu_{1-r}(y,a,x)$, i.e.

$$L^{(r)}_{x,a} = R^{(1-r)}_{a,x}. \tag{Af3}$$

We say that an affine pair geometry satisfies the *fundamental identities of projective geometry* if the following holds for all $r \in \mathbb{K}$ and $x,y \in X$, $a,b \in X'$ where things are defined:

$$(L_{x,a})^t = L_{a,x}, \quad (R_{a,x})^t = R_{x,a}, \tag{PG1}$$

$$(M_{x,y})^t = M_{y,x}, \quad (M_{a,b})^t = M_{b,a}. \tag{PG2}$$

For invertible r, (PG1) can be rephrased by saying that $(r_{x,a}, r^{-1}_{a,x})$ is an automorphism of (X,X'). The group generated by those automorphisms is called the *inner automorphism group.* It contains the *translations* (with respect to some vectorialization $(o,o') \in M$)

$$(\tau_v, \widetilde{\tau}_v) := (2_{o,o'} 2^{-1}_{v,o'}, 2^{-1}_{o',o} 2_{o',v}) \tag{1.4}$$

for $v \in V := V_{o'}$. The following identity (T) together with its dual,

$$\widetilde{\tau}_{v+w} = \widetilde{\tau}_v \widetilde{\tau}_w \tag{T}$$

is called the *translation property*; it implies that

$$\tau_V := \{(\tau_v, \widetilde{\tau}_v) |\, v \in V\}, \quad \widetilde{\tau}_{V'} := \{(\widetilde{\tau}_a, \tau_a) |\, a \in V'\} \tag{1.5}$$

are (abelian) groups, the former acting on V by translations and the latter by fixing the origin o, and dually on V'. Finally, a *generalized projective geometry* is an affine pair geometry such that, in all scalar extensions of $\mathbb{K}$, the fundamental identities (PG1) and (PG2) and the translation property (T) hold.

1.2. (The associated Jordan pair.) We fix a base point $(o, o') \in M$ and let $(V, V') = (V_{o'}, V'_o)$. This is a pair of $\mathbb{K}$-modules. In [Be01b, Th. 8.6] it is shown that then the expression

$$Q(x)a := M^{(\frac{1}{2})}_{x,-x} a \tag{1.6}$$

is quadratic polynomial in x and linear in a. By polarizing Q, we define a trilinear map

$$T : V \times V' \times V, \quad (x, a, y) \mapsto (Q(x + y) - Q(x) - Q(y))a \tag{1.7}$$

which, together with its dual T', satisfies the axioms of a *linear Jordan pair* (cf. [Be01b, Th. 9.5]). (In [Be01b], $Q(x)$ was defined by taking the opposite sign in (1.6). In turns out that the sign chosen here is more convenient.)

1.3. (Polarities and null-systems.) An *antiautomorphism* of a generalized projective geometry (X, X') is an isomorphism $(p, p') : (X, X') \to (X', X)$ onto the dual geometry (i.e. $p : X \to X'$ and $p' : X' \to X$ are bijections such that the identities $p\mu_r(o, o', x) = \mu'_r(p(o), p'(o'), p(x))$, and dually, hold). A *correlation* of (X, X') is an antiautomorphism (p, p') such that $p' = p^{-1}$; equivalently, $p = p^t$. A correlation is called a *polarity* if there exists $x \in X$ such that x and $p(x)$ are non-isotropic (which means that $(x, p(x)) \in M$) and a *null-system* if all $x \in X$ are isotropic (i.e. $(x, p(x)) \notin M$). A *generalized polar geometry* is generalized projective geometry (X, X') together with a polarity p. The set of non-isotropic points,

$$M^{(p)} = \{x \in X |\, (x, p(x)) \in M\}, \tag{1.8}$$

carries the structure of a *symmetric space over* $\mathbb{K}$ with respect to the "multiplication map" (in the sense of [Lo69]) $\mu(x, y) = \mu_{-1}(x, p(x), y)$ (see [Be01b, Th. 4.2]).

1.4. (Absolute identifications and central correlations.) An *absolute identification* is a pair of maps $n : X \to X'$, $n' : X' \to X$ such that n and n' are bijections which are inverse to each other and such that n commutes with the automorphism group in the sense that

$$\forall (g, g') \in \mathrm{Aut}(X, X') : \quad g' \circ n = n \circ g. \tag{A}$$

A *central correlation* (resp. *central null-system*) is a correlation (resp. a null-system) which is an absolute identification, i.e. which commutes with all automorphisms. A *central null-geometry* is a generalized projective geometry (X, X') together with a central null system (n, n'). We can rewrite (A) by letting all maps act on $X \times X'$: given $n : X \to X'$, $n' : X' \to X$ and an automorphism (g, g'), we let $\widetilde{g} = g \times g' : X \times X' \to X \times X'$ and

$$\widetilde{n} : X \times X' \to X \times X', \quad (x, a) \mapsto (n'(a), n(x)), \tag{1.9}$$

then (A) is equivalent to the condition $\widetilde{n} \circ \widetilde{g} = \widetilde{g} \circ \widetilde{n}$.

1.5. (Inner correlations, inner polar geometries.) We let $r = \frac{1}{2}$ and consider, for any pair $(x,y) \in X \times X$ the "midpoint map"

$$M_{x,y} := M^{(\frac{1}{2})}_{x,y} : V'_x \cup V'_y \to X. \tag{1.10}$$

In fact, the geometric interpretation of $M_{x,y}(a)$ as the midpoint of x and y in the affinization a holds only for $x, y \in V_a$, i.e. for $a \in V'_x \cap V'_y$. If the condition

$$V'_x \cap V'_y \neq \emptyset \tag{1.11}$$

is not satisfied, then $M_{x,y}$ is only defined via the continuation of $2^{-1}_{x,a}$ to an automorphism of X and has no direct geometric interpretation.

We say that a pair $(x,y) \in X \times X$ is *invertible* if $M_{x,y} : V'_x \cup V'_y \to X$ extends to a bijection from X' onto X such that the relation (PG2) $M^t_{x,y} = M_{x,y}$ still holds for the extended map. Then (PG2) implies that $((M_{x,y})^{-1}, M_{y,x})$ is an isomorphism from (X,X') onto (X',X), that is, it is an *antiautomorphism* of (X,X'). Moreover, since $r = 1/2$, we have $M'_{x,y} = (M_{y,x})^{-1} = (M_{x,y})^{-1}$, i.e. $(M_{x,y},(M_{x,y})^{-1})$ is a *correlation*; such a correlation is called *inner*. If the condition (1.11) holds, then $M_{x,y}$ is a polarity: in fact, if $a \in V'_x \cap V'_y$, then $x,y \in V_a$ and hence $\mu_r(x,a,y) \in V_a$, i.e. $M_{x,y}(a) \in V_a$, and therefore $M_{x,y}$ is a polarity which we call an *inner polarity.* An *inner polar geometry* is a generalized projective geometry together with an inner polarity $M_{x,y}$ (i.e. we assume (1.11) to be satisfied. Condition (1.11) is satisfied for *all* $x,y \in M$ if the geometry is *stable*, see below.)

1.6. (Connectedness.) We say that (x,a) and $(y,b) \in M$ are *connected* and write $(x,a) \sim (y,b)$, if there is a sequence $(p_1,q_1),\ldots,(p_n,q_n) \in M$ such that $(p_1,q_1) = (x,a)$, $(p_n,q_n) = (y,b)$ and

$$(p_{j+1},q_{j+1}) \in (V_{q_j}, V'_{p_j}), \quad j = 1,\ldots,n-1.$$

This defines an equivalence relation on M whose equivalence classes are called *connected components.* A geometry (X,X') is called *connected* if M is connected. If (X,X') is connected, then the inner automorphism group acts transitively on M ([Be01b, Th. 5.7]).

1.7. (Stability.) The geometry (X,X') is called *stable* if (1.11) holds for all $x,y \in X$, and dually (cf. [Be01b, Section 5.5]). A stable geometry is connected (but the converse is not true): in fact, if (x,a), $(y,b) \in M$, let $z \in V_a \cap V_b$, $c \in V'_x \cap V'_z$; whence $(z,c) \in M$ and $(z,c) \in (V_a, V'_x)$. Next choose $d \in V'_z \cap V'_y$; then $(z,d) \in M$ and $(z,d) \in (V_c, V'_z)$. Finally, $(y,b) \in (V_d, V'_z)$, and thus the sequence $(x,a),(z,c),(z,d),(y,b)$ connects (x,a) and (y,b).

1.8. (Orders of infinity.) We fix an affinization $o' \in X$ and let $V = V_{o'} \subset X$. The *first order infinity set of* o' is the set

$$\mathrm{I}_\infty = X \setminus V \tag{1.12}$$

(in case of ordinary projective geometry this is the "hyperplane at infinity"). The *second order infinity set of* o' is the set of affinizatons whose affine parts lie entirely at infinity:

$$\mathrm{II}_\infty = \{a \in X' \mid V_a \subset \mathrm{I}_\infty\} = \{a \in X' \mid V_a \cap V = \emptyset\}. \tag{1.13}$$

For a stable geometry, all second order infinity sets are empty. Clearly, one can continue in this way by defining a *third order infinty set*

$$\mathrm{III}_\infty = \{x \in X \mid V'_x \subset \mathrm{II}_\infty\} \tag{1.14}$$

(those points which only belong to affine parts that are entirely located at infinity; in particular $\mathrm{III}_\infty \subset \mathrm{I}_\infty$), and so on. Note that I_∞ is τ_V-invariant, and therefore also II_∞ etc. are τ_V-invariant sets.

1.9. (Big orbits of translation groups.) We will need conditions expressing the fact that first and second order infinity sets are "not too big". Stability is the most restrictive of such conditions. In order to formulate more general conditions, fix an affinization $o' \in X$ and let $\tau_V = \tau_{V_{o'}}$ be the corresponding translation group. We say that the τ_V-orbit of a point $x \in X$ is *big* if the orbit map

$$\pi_x : V \to X, \quad v \mapsto \tau_v(x) \tag{1.15}$$

is *essentially an isomorphism*; by this we mean that it extends to a bijection from X onto X, again denoted by π_x, and such that there exists a bijection $\pi_x' : X' \to X'$ completing it to an isomorphism (π_x, π_x'). In particular, π_x has to be injective, i.e. τ_V acts freely on the orbit $\tau_V.x$. For instance, if x belongs to V, then $\tau_V.x = V$ is a big orbit, and moreover (w.r.t. some fixed origin $o \in V$), $\pi_x(v) = v + x = x + v = \tau_x(v)$, whence $\pi_x = \tau_x$, and (π_x, π_x') with $\pi_x' = \widetilde{\tau}_x$ is an isomorphism. The axioms of a generalized projective geometry do not exclude that, even in the connected case, there exist points $x \notin V$ having a big τ_V-orbit; that is, a translation group may have a big orbit in the first order infinity set I_∞. We say that our geometry has *no big orbits at first order infinity* if all translation groups have precisely one big orbit in X, namely the affine part to which they belong.

A translation group may also have *big orbits in the dual space* X': by this we mean that there exists $a \in X'$ such that the orbit map

$$f_a : V \to X', \quad v \mapsto \widetilde{\tau}_v.a \tag{1.16}$$

extends to an isomorphism $(f_a, f_a') : (X, X') \to (X', X)$ in the sense explained above. As above, we cannot exclude, even in the connected case, that the translation group τ_V has a big orbit in the second order infinity set II_∞. We say that our geometry has *no big orbits in second order infinity* if no translation group has a big orbit in its second order infinity set.

If the geometry is *stable*, then there are no big orbits in first and second order infinity: in fact, all second order infinity sets are empty, and if $W := \pi_x(V)$ is a big orbit of τ_V, then $W = \pi_x(V_{o'}) = V_{\pi_x'(o')}$ intersects the orbit $V = V_{o'}$, and therefore both orbits coincide. In particular, all these properties hold for connected finite dimensional geometries over fields since such geometries are stable (cf. [Be01b, 5.5]). We do not know wether in turn stability is implied by the non-existence of big orbits. However, in order to show that at least the first order infinity sets can become quite large in infinite dimensional geometries, we present in the Appendix a class of examples showing that even a stable geometry may admit many τ_V-orbits at first order infinity on which the translation group acts freely.

2. Absolute identifications and null-geometries

2.1. (Absolute identifications and the permutation group.) In this chapter we assume that $n : X \to X'$, $n' : X' \to X$ is an absolute identification of the generalized projective geometry (X, X') (cf. Section 1.4). If r is invertible in $\mathbb{K}$ and $(x, a) \in M$, then by (PG1), $(g, g') = (r_{x,a}, r_{a,x}^{-1})$ is an automorphism of (X, X'), and we can apply the relation (A) from 1.4 and get

$$r_{a,x}^{-1} \circ n = n \circ r_{x,a} \tag{2.1}$$

which can be rewritten

$$n(\mu_r(x, a, z)) = \mu_{r^{-1}}(a, x, n(z)). \tag{2.2}$$

If, as we shall do later, we identify X and X' via n, then Equation (2.2) describes the behavior of the ternary map μ_r under the permutation $(1,2)$ of the three arguments. On the other hand, the behavior under the permutation $(1,3)$ is known by the identity (Af3) from Section 1.1. Therefore the behavior of μ_r under the full permutation group Σ_3 is now known:

$$\begin{aligned} n(\mu_r(x,a,z)) = \mu_{r^{-1}}(a,x,n(z)) = \mu_{1-r^{-1}}(n(z),x,a) \\ = n(\mu_{1-r}(z,a,x)) = \mu_{(1-r)^{-1}}(a,z,n(x)) = \mu_{1-(1-r)^{-1}}(n(x),z,a). \end{aligned} \tag{2.3}$$

Eliminating the variable z, we get in operator form:

$$\begin{aligned} n \circ L^{(r)}_{x,a} = L^{(r^{-1})}_{a,x} \circ n = R^{(1-r^{-1})}_{x,a} \circ n \\ = n \circ R^{(1-r)}_{a,x} = M^{((1-r)^{-1})}_{a,n(x)} = M^{(1-(1-r)^{-1})}_{n(x),a}. \end{aligned} \tag{2.4}$$

The six values that arise from r in this way are known from projective geometry as the six possible values of the cross-ratio under permutations. In the complex case there are essentially two cases in which several of these values coincide, namely $r = -1$ ("harmonic range") and $r = \frac{1+i\sqrt{3}}{2}$ (where $r^{-1} = 1-r$). For later use, let us write down (2.4) in the case $r = -1$:

$$\begin{aligned} n \circ L^{(-1)}_{x,a} = L^{(-1)}_{a,x} \circ n = R^{(2)}_{x,a} \circ n \\ = n \circ R^{(2)}_{a,x} = M^{(\frac{1}{2})}_{a,n(x)} = M^{(\frac{1}{2})}_{n(x),a}. \end{aligned} \tag{2.5}$$

2.2. (Existence of inner correlations.) Let us assume that $1-r$ and $s := 1-(1-r)^{-1} = r(r-1)^{-1}$ are invertible in $\mathbb{K}$; then also r is invertible in $\mathbb{K}$, and thus $M^{(r)}_{x,y}$ is, for all pairs $(x,y) \in X \times X$, defined on a non-empty domain. If in addition $(x,a) := (x, n'(y))$ belongs to M, then since s is invertible, the operator $n \circ L^{(s)}_{x,a}$ is a bijection from X' onto X, and Equation (2.4) shows now that $M^{(r)}_{x,y}$ extends to a bijection from X' onto X, also denoted by $M^{(r)}_{x,y}$ and given by

$$M^{(r)}_{x,y} = n \circ L^{(s)}_{x,n'(y)}. \tag{2.6}$$

If we specialize (2.6) to the value $r = \frac{1}{2}$, we have $s = -1$, and we see that the midpoint map $M_{x,y} = M^{(\frac{1}{2})}_{x,y}$ extends to a bijection. As has been explained in Section 1.5, $(M_{x,y}, M^{-1}_{x,y})$ is then an inner correlation; it is an inner polarity if (1.11) holds. Moreover, from (2.6) we get, using that $s = -1$ is its own inverse,

$$(M_{x,y})^{-1} = n' \circ M_{x,y} \circ n'. \tag{2.7}$$

2.3. (Absolute identifications are null-systems.) Comparing the six terms of (2.5) among each other, we get the relations (here and in the sequel we use the simpler notation $(-1)_{x,a} = L^{(-1)}_{x,a}$ and omit the supersript $\frac{1}{2}$ in the notation of the midpoint map)

$$n = M_{a,n(x)} \circ (-1)_{x,a} = (-1)_{a,x} \circ M_{a,n(x)}. \tag{2.8}$$

Taking inverses, we get from (2.8)

$$(n, n') = (n, n^{-1}) = (M_{a,n(x)}, M^{-1}_{a,n(x)}) \circ ((-1)_{x,a}, (-1)_{a,x}) \tag{2.9}$$

which means that we can write (n, n') as a composition of an inner correlation and an inner automorphism. Therefore (n, n') is an antiautomorphism which is a correlation since $n' = n^{-1}$. It is central since it is an absolute identification.

We claim that n is not a polarity (hence n is a null-system) : in fact, let $o \in X$ and assume that $(o,o') := (o,n(o)) \in M$, i.e. o is non-isotropic for n. Since (n,n') is central, (g,g') stabilizes $(o,n(o))$ iff $g.o = o$ and $g'n(o) = n(o)$, i.e. iff $g.o = o$; by definition, this means that (g,g') belongs to the stablizer group of o, denoted by P in [Be01b]. But since $(\widetilde{\tau}_a, \tau_a) \in P$ for all $a \in V'_o$, this means that $o' = \tau_a.o' = a$, and hence V'_o is zero-dimensional, which is excluded by definition in Section 1.1. Therefore all points $o \in X$ are isotropic, and thus (n,n') is a central null-system. Summing up, we have shown that absolute identifications are the same as central null-systems.

2.4. (Values on the diagonal.) Once we have shown that the identification (n,n') is a correlation, there is no danger to identify X and X' via n since μ_r and μ'_r give rise to the same map $X \times X \times X \supset D \to X$ which we again denote by μ_r. The set M is identified with a subset of $X \times X$ which does not contain the diagonal since n is a null-system and which is stable under the exchange map $(x,y) \mapsto (y,x)$. For invertible r, the map μ_r is now defined on an *extended domain*

$$\widetilde{D} := \{(x,y,z) |\, (x,y) \,\mathrm{or}\, (x,z) \,\mathrm{or}\, (y,z) \in M\} \subset X \times X \times X \tag{2.10}$$

which is invariant under the permutation group Σ_3, and the relations (2.4), (2.5) can now be written as in the Introduction, Equation (0.4). In particular, if r is invertible and $(x,y) \in M$, we get the following values on the diagonal $x = z$:

$$\mu_{1-r}(x,x,y) = \mu_r(y,x,x) = \mu_{r^{-1}}(x,y,x) = x. \tag{2.11}$$

The preceding relations permit to extend the domain of definition of certain natural operations: for instance, the vector addition $+_{o,o'}$ with respect to a base point $(o,o') \in M$, which will just be denoted by $+$ and defined by $x + y = 2_{o,o'} 2^{-1}_{x,o'}(y)$, is originally defined on $(V \times X) \cup (X \times V)$, but can now be extended to all $(x,y) \in M$ by $x + y = 2_{o,o'}(-1)_{x,y}(o')$. (It can also be defined on the whole diagonal $x = y$ by $x + x = 2_{o,o'} x$). In particular, the translation τ_x can be defined for any $x \in X$ as a not necessarily everywhere defined map by

$$\tau_x(y) = 2_{o,o'} \mu_{-1}(x,y,o') = 2_{o,o'} \circ M^{(-1)}_{x,o'}(y). \tag{2.12}$$

As we will see later, it is reasonable to introduce the notation $\infty := o'$; then we have the addition rule $\infty + y = \infty$ for all $y \in X$.

2.5. (Inner polarities.) Let us assume that the geometry (X,X') admits no big orbits of translation groups in second order infinity (Section 1.9). We claim that then inner polarities exist: first of all, if (n,n') is a central null-system, then, letting for an affinization $o' \in X'$ as usual $V = V_{o'}$, the set $n(V) \subset X'$ is a big orbit of τ_V since $n(\tau_v.o) = \tau_v.n(o)$. Since by assumption this big orbit is not included in the second order infinity set, i.e. $n(V) \subset (X' \setminus \mathrm{II}_\infty)$, for all $y \in V$ the condition $V_{n(y)} \cap V \neq \emptyset$ is satisfied. Therefore all pairs (y,o') with $y \in V$ satisfy condition (1.11), and thus $M_{n(y),o'}$ is an inner polarity. In particular, inner polarities exist.

In general, inner correlations are not unique, but the relation

$$M_{g.x,g.y} = g \circ M_{x,y} \circ g \tag{2.13}$$

for all automorphisms g implies that the conjugation class depends only on the orbits of the group $G = \mathrm{Aut}(X)$ acting on the set of invertible pairs in $X \times X$. By 2.2, M is included in this set, and if the geometry is connected, then $M = G.(o,o')$ is homogeneous under G (cf. 1.5). Under the assumptions made above, M is moreover contained in the set of invertible pairs satisfying (1.11). In the next chapter it will be shown that for connected geometries we have equality, and thus all inner polarities will be conjugate under the automorphism group.

2.6. (Uniqueness of central correlations.) Let us assume that the geometry (X,X') is connected and admits no big orbits at first order infinity. Then central correlations (n,n') are necessarily unique: in fact, via composition with (n,n'), central correlations correspond to elements of the center $Z=Z(\mathrm{Aut}(X,X'))$ of the automorphism group. Let $(z,z')\in Z$, fix $o'\in X'$ and let $V=V_{o'}$. Then $z(V)=\tau_V z(o)$ is a big orbit of τ_V, and therefore by assumption $V\cap z(V)\neq\emptyset$. Fix $o\in(V\cap z(V))$ and let $x:=z(o)$. Then $f:=\tau_x^{-1}\circ z$ commutes with the action of τ_V, and thus, for all $v\in V$,

$$f(v)=f\tau_v(o)=\tau_v z_2(o)=\tau_v.o=v,$$

whence $f=\mathrm{id}_X$ by connectedness. But then $\tau_x=z$ has to be central, forcing $x=o$ and thus $z=\mathrm{id}$ and similarly $z'=\mathrm{id}$.

2.7. Example: Grassmannian geometry. Let E,F be $\mathbb{K}$-modules and $W=E\oplus F$. Let $X=\mathrm{Gras}_E(W)$ and $X'=\mathrm{Gras}_F(W)$ be the sets of submodules of W which are isomorphic to E, resp. to F, and which admit complements. The pair (X,X') is a generalized projective geometry with product maps

$$\mu_r([x],[a],[y])=[(1-r)x(ax)^{-1}+ry(ay)^{-1}], \tag{2.14}$$

where elements of X are identified with injections $x:E\to W$ up to equivalence under $\mathrm{Gl}(E)$ and elements of X' with surjections $a:W\to F$ up to equivalence under $\mathrm{Gl}(E)$ (cf. [Be01a, Ch. 2]). The automorphism group is essentially $\mathrm{P\,Gl}(W)$, acting by

$$(g,g').([x],[a])=([g\circ x],[a\circ g^{-1}]). \tag{2.15}$$

Ordinary projective geometry over $\mathbb{K}$ arises as the special case $E\cong\mathbb{K}$, $X=\mathrm{P}(W)$, $X'=\mathrm{P}(W^*)$, and in this case, formula (2.14) can be rewritten

$$\mu_r([x],[\lambda],[y])=[(1-r)\lambda(y)x+r\lambda(x)y]. \tag{2.16}$$

If, in the general case, we have $E=F$, i.e. W is the direct sum of E with itself, then the identity map $n:X\to X'$ defines an absolute identification: in fact, the relation $n.[g\circ x]=[n(x)\circ g]$ holds for all $g\in\mathrm{P}(\mathrm{Gl}(W))$ since both sides of the equation describe just the push-forward of the subspace $[x]=[n(x)]$ by g.

In the case of the projective line, n is given by an arbitrary symplectic form $\omega:\mathbb{K}^2\times\mathbb{K}^2\to\mathbb{K}$: $n([x])=[\omega(x,\cdot)]$, and we can write μ_r as a ternary map on $X=\mathbb{K}\mathrm{P}^1$ as

$$\mu_r([x],[y],[z])=[(1-r)\omega(y,z)x+r\omega(y,x)z]. \tag{2.17}$$

(This formula still makes sense for higher dimensional projective spaces, but then it describes null-systems which are not central.)

In the general case, if $x:E\to W$ is an injection, then $n([x])$ is the class of some surjection $\widetilde{x}:W\to E$ having $\mathrm{im}(x)$ as kernel, and identifying X and X' via n, (2.14) reads

$$\mu_r([x],[y],[z])=[(1-r)x(\widetilde{y}x)^{-1}+rz(\widetilde{y}z)^{-1}]. \tag{2.18}$$

However, if $\dim E>1$, then there is no “linear version” of the map $y\mapsto\widetilde{y}$. One would be tempted to write $\widetilde{x}=x^t\circ J_x$, where the transpose x^t is taken with respect to some auxiliary form and $J_x:W\to W$ some linear bijection; but for $\dim E>1$ it is not possible to choose a “version” of J_x which is independent of x.

2.8. Example: Lagrangian geometry. Assume b is a symplectic form or the symmetric form on $\mathbb{K}^{2m}$ given by the matrix

$$A=\begin{pmatrix}0 & \mathbf{1}_m\\ \mathbf{1}_m & 0\end{pmatrix}.$$

In both cases there exist pairs of complementary Lagrangian subspaces, and it is easily verified that (X,X'), with $X = X'$= the space of Lagrangians of the form b on $\mathbb{K}^{2m}$, is a generalized projective geometry (cf. [Be01a, Ch. 4]); it is a subgeometry of $(\mathrm{Gras}_E(E\oplus E), \mathrm{Gras}_E(E\oplus E))$, $E = \mathbb{K}^m$, with μ_r given by Formula (2.14), and by the same arguments as in the preceding example it is seen that the identification $n : X \to X'$ is a central null-system.

If b is symplectic, the Lagrangian geometry is also called "symplectic"; in this case (X,X') is always connected. If b is symmetric, given by the matrix A, then the Lagrangian geometry is called "orthogonal". In this case the geometry (X,X) has *two connected components*: first of all, by Witt's theorem, the group $\mathrm{O}(b) = \mathrm{O}(A,\mathbb{K}) = \{g \in \mathrm{Gl}(2m,\mathbb{K}) |\, g^t A g = A\}$ acts transitively on X; we write $X = \mathrm{O}(b)/P$ where P is the stabilizer of the Lagrangian subspace $E\oplus 0 = \mathbb{K}^n \oplus 0$. A simple calculation shows that elements of P are of the form $\left(\begin{smallmatrix} a & * \\ 0 & (a^t)^{-1}\end{smallmatrix}\right)$ with $a \in \mathrm{Gl}(m,\mathbb{K})$. They have all determinant one, and thus $P \subset \mathrm{SO}(b)$. On the other hand, $\mathrm{O}(b)$ does contain elements σ with determinant minus one. Therefore we have a disjoint union $X = X_1 \cup X_2$ with $X_1 = \mathrm{SO}(b)/P$, $X_2 = \sigma(X_1)$. We claim that the connected components of the geometry (X,X) are as follows: if m is even, (X_1,X_1) and (X_2,X_2); if m is odd, (X_1,X_2) and (X_2,X_1). In fact, we can always write $\mathbb{K}^{2m} = E_1 \oplus E_2$ as a direct sum of two Lagrangian subspaces. Since $\mathrm{O}(b)$ acts transitively on the Lagrangian geometry X, there is $g \in \mathrm{O}(b)$ such that $g(E_1) = E_2$. In matrix form, g must then have the form $g = \left(\begin{smallmatrix} 0 & a \\ b & *\end{smallmatrix}\right)$ with $b = (a^t)^{-1}$. It follows that the determinant of g is equal to $(-1)^m \det(a)\det(a^{-1}) = (-1)^m$. Therefore, if m is odd there is no element $g \in \mathrm{SO}(b)$ with $g(E_1) = E_2$, and hence X_1 does not contain complementary pairs of Lagrangians. Thus (X_1,X_1) is not a generalized projective geometry, but (X_1,X_2) and (X_2,X_1) are. However, these geometries are *not* preserved under the central null-system, and therefore they are not central null geometries. If m is even, X_1 does contain complementary pairs of Lagrangians, and hence (X_1,X_1) and similarly (X_2,X_2) are generalized projective geometries which, moreover, are both stable under the central null-system and hence are central null-geometries.

2.9. Example: Conformal geometry. Here $X = \{[x] \in \mathbb{KP}^{n+1} |\, b(x,x) = 0\}$ is the projective quadric of a symmetric non-degenerate bilinear form b on $\mathbb{K}^{n+2}$ and X' is the space of tangent hyperplanes of X. The structure of generalized projective geometry over $\mathbb{K}$ is described in [Be01a, Ch. 5]. The automorphism group of this geometry is the orthogonal group of b. The map $X \to X'$, $[x] \mapsto [\ker(b(x,\cdot))]$ commutes with the action of this group and thus is an absolute identification. The geometry (X,X') is always connected.

3. Inner polar geometries

3.1. We fix the following assumptions for this chapter: (X,X') is a connected inner polar geometry (Sections 1.5 and 1.6): there exists $(x,y) \in X \times X$ such that $M_{x,y}$ extends to a polarity, and $V'_x \cap V'_y \neq \emptyset$. Moreover, we assume that there are no big orbits of translation groups at first order infinity (Section 1.9).

The aim of this chapter is to show that under these assumptions the formula $n(a) := (M_{a,b})^{-1}(a)$ defines an absolute identification of X and X'. (This formula is motivated by Formula (2.11) which implies that, if (n,n') is an absolute identification, then it must satisfy the relation $M_{x,y}(n(x)) = x$, whence $n(x) = (M_{x,y})^{-1}(x)$ if the operator $M_{x,y}$ is invertible. Moreover, we will see in Section 6.3 that this formula is closely related the Jordan theoretic formula $x^{-1} = Q(x)^{-1}x$ for the Jordan inverse in a Jordan algebra.)

3.2. (Faithfulness.) A connected inner polar geometry is *faithful* in the sense that different points $a,b \in X'$ define different affinizations of X, and dually (cf. [Be01b, 2.8]). In fact, assume $p := M_{x,y} : X' \to X$ is an inner polarity. Then by assumption $V'_x \cap V'_y \neq \emptyset$, and by transitivity of the automorphism group on X', we may assume that $a \in V'_x \cap V'_y$. Since $M_{x,y}$ is injective, if $a \neq b \in X'$, we have $M_{x,y}(a) \neq M_{x,y}(b)$. Then either $V_a \neq V_b$; in this case the affinizations given by a and b are different since their domains are different; or $V_a = V_b$; in this case, the affine structures given by a and b on V_a have to be different since they give different midpoints of x and y.

3.3. (Orbit maps of dual translation groups.) Fix $o \in X$, let $V' = V'_o \subset X'$ and recall the definition of the dual translation group $\widetilde{\tau}_{V'}$ acting on $X \times X'$. For $x \in X$ we define the orbit map similarly to the orbit maps defined by Eqn. (1.16):

$$f_x : X' \supset V' \to X, \quad a \mapsto \widetilde{\tau}_a.x. \tag{3.1}$$

The orbit maps commute with the action of the translation group: for all $y \in V'$,

$$f_x \circ \tau_y = \widetilde{\tau}_y \circ f_x \; : \qquad \begin{array}{ccc} X' & \xrightarrow{f_x} & X \\ \tau_y \downarrow & & \downarrow \widetilde{\tau}_y \\ X' & \xrightarrow{f_x} & X \end{array} . \tag{3.2}$$

In fact, $f_x\tau_y(v) = f_x(v+y) = \widetilde{\tau}_{v+y}(x) = \widetilde{\tau}_y\widetilde{\tau}_a(x) = \widetilde{\tau}_y(f_x(v))$. Moreover, if x does not lie in the second order infinity set II_∞ of o, then the orbit map f_x can be completed to an adjoint pair (f_x, f_x^t): in fact, since $V'_x \cap V'_o \neq \emptyset$, we can choose a point $o' \in V'_x \cap V'_o$ and let $V = V_{o'}$. Then $x, o \in V$, the latter taken as zero vector, and we can apply the *symmetry principle* ([Be01b, 6.3]):

$$\widetilde{\tau}_a(x) = \tau_x(-Q(x))\widetilde{\tau}_x(a) \tag{3.3}$$

(where $Q(x)$ is defined by Equation (1.6)), which means that

$$f_x = \tau_x \circ (-Q(x)) \circ \widetilde{\tau}_x, \tag{3.4}$$

and therefore f_x can be completed to the adjoint pair (f_x, f_x^t) given by

$$f_x^t = \tau_{-x} \circ (-Q(x)) \circ \widetilde{\tau}_{-x}. \tag{3.5}$$

Since τ_x and $\widetilde{\tau}_x$ are bijections, it follows from (3.4) and (3.5) that the map $Q(x) = M_{x,-x}$ extends to a bijection if and only if so does f_x, i.e. if and only if the orbit $\widetilde{\tau}_V.x$ is *big* (cf. 1.9). Summing up, for x not belonging to the second order infinity set of o, the following are equivalent:

(a) the orbit $\widetilde{\tau}_V.x \subset X$ is big,

(b) the map $Q(x) = M_{-x,x}$ extends to a bijection; since Q is the quadratic map of the associated Jordan pair (V, V') (cf. Section 1.2), this means that x is *invertible in the Jordan pair* (V, V') (cf. [Lo75, 1.10]),

(c) the maps $M_{x,-x}$, $M_{o,2x}$, $M_{o,x}$ are inner polarities.

(For (c), just note that $(o, x), (o, 2x), (x, -x) \in X \times X$ are conjugate under the action of the automorphism group und use that $M_{g.x,g.y} = gM_{x,y}g'$.) Moreover, any inner polarity $M_{x,y}$ arises in the way described by (c) since we may choose $o' \in V_x \cap V_y$ and $o = M_{x,y}(o')$; then $x = -y$ in the $\mathbb{K}$-module (V, o). In particular, by the assumptions made in Section 3.1, pairs (o, x) with the desired property exist. Assume now that (a) – (c) are satisfied. Then the orbit map f_x extends to an anti-automorphism (f_x, f'_x), given by

$$f'_x = ((f_x)^t)^{-1} = (\tau_x(-Q(x))\widetilde{\tau}_x)' = \widetilde{\tau}_x(-Q(x))'\tau_x : X \to X'. \tag{3.7}$$

The relation (3.2) implies, since our geometry is faithful, that

$$f_x' \circ \tilde{\tau}_y = \tau_y \circ f_x' : \qquad \begin{array}{ccc} X & \xrightarrow{f_x'} & X' \\ \tilde{\tau}_y \downarrow & & \downarrow \tau_y \\ X & \xrightarrow{f_x'} & X' \end{array} . \tag{3.8}$$

Relations (3.2) and (3.8) imply that $\tilde{\tau}_{V'}$ can also be realized as some translation group, but of course with respect to some new affinization: in fact, for $b \in V'$, $\tilde{\tau}_b^{o',o} := \tilde{\tau}_b$ (the superscript indicates affinization and base point with respect to which the dual translation is defined) can be written

$$\tilde{\tau}_b = f_x \circ \tau_b^{o',o} \circ f_x^{-1} = \tau_{f_x(b)}^{f_x(o'),f_x'(o)}; \tag{3.9}$$

this is a translation for the affinization given by $f_x'(o) \in X'$. Put another way: the collection of all translation groups acting on X is the same as the collection of all dual translation groups acting on X. Therefore, assuming that all translation groups have at most one big orbit in X is the same as assuming that all dual translation groups have at most one big orbit in X.

3.4. (The absolute identification.) If $o \in X$ is fixed and x is as above, then we apply both sides of (3.8) to the point o; since o is invariant under $\tilde{\tau}_{V'}$, we get

$$\tau_{V'}(f_x'.o) = f_x'.o. \tag{3.10}$$

To give a more explicit expression for $f_x'.o$, recall Equation (3.7) and note that $Q(x)' = Q(x)^{-1} = (M_{x,-x})^{-1}$; thus

$$f_x'(o) = \tilde{\tau}_x(-M_{x,-x})^{-1}(x) = -(M_{o,-2x})^{-1}(o) = (M_{o,2x})^{-1}(o) \in X'. \tag{3.11}$$

After replacing x by $\frac{x}{2}$, (3.10) and (3.11) show that the point $(M_{o,x})^{-1}.o$ is invariant under $\tau_{V'}$: for all $y \in V'$,

$$\begin{aligned}(M_{o,x})^{-1}.o &= \tau_y(M_{o,x})^{-1}.o \\ &= (M_{\tilde{\tau}_y.o,\tilde{\tau}_y.x})^{-1}(\tilde{\tau}_y.o) \\ &= (M_{o,\tilde{\tau}_y.x})^{-1}(o).\end{aligned}$$

Therefore the point $(M_{o,x})^{-1}.o \in X'$ does not depend on the representative x of the big orbit $\tilde{\tau}_{V'}.x$. Now we are going to use for the first time the assumption that there is at most one such orbit which, therefore, depends only on the point $o \in X$. Let us denote it by $\mathcal{O}_o := \tilde{\tau}_V.x$. Since the automorphism group acts transitively on X and, for $(g,g') \in \mathrm{Aut}(X,X')$, $g(\mathcal{O}_o)$ is a big orbit of a dual translation group for the point $g.o$, it follows that for every point $z \in X$ there exists such an orbit $\mathcal{O}_z$, and since it is unique, we have the relation $g(\mathcal{O}_o) = \mathcal{O}_{g.o}$ for all $(g,g') \in \mathrm{Aut}(X,X')$. Summing up, we have a well-defined map

$$n : X \to X', \quad n(z) := (M_{z,x})^{-1}(z), \quad x \in \mathcal{O}_z. \tag{3.12}$$

Let us prove that it is an absolute identification. First of all, for all $(g,g') \in \mathrm{Aut}(X,X')$,

$$n(g.o) = (M_{g.o,y})^{-1}(g.o) = g'(M_{o,g^{-1}y})^{-1}(o) = g'n(o) \tag{3.13}$$

where $y \in \mathcal{O}_{g.o} = g(\mathcal{O}_o)$ and hence $g^{-1}y \in \mathcal{O}_o$. Thus the relation (A): $n \circ g = g' \circ n$ holds for all $(g,g') \in \mathrm{Aut}(X,X')$, and the arguments leading from (2.1) up to (2.8) show that then necessarily $n = M_{a,n(x)} \circ (-1)_{x,a} = (-1)_{a,x} \circ M_{a,n(x)}$ holds. This implies both that n is actually a bijection with inverse n' given by $n' = (-1)_{a,x} \circ M_{a,n(x)}^{-1}$ and that (n,n') is a correlation which has to be a null-system according to 2.3.

4. The main theorem

Theorem 4.1. *Assume (X, X') is a connected generalized projective geometry over $\mathbb{K}$ having no big orbits at first and second order infinity. Then the following are equivalent:*

(1) *(X, X') has a central null-system.*

(2) *(X, X') has inner polarities.*

(3) *A dual translation group $\widetilde{\tau}_{V'}$ has a big orbit in X.*

(4) *The Jordan pair associated to a base point (o, o') in (X, X') has invertible elements (i.e. it comes from a unital Jordan algebra).*

If these properties hold, then the central null-system is unique, the inner polarities are all conjugate under the automorphism group and any dual translation group has exactly one big orbit in X. The assumptions formulated above are satisfied, in particular, if the geometry (X, X') is stable. Conversely, to every stable Jordan pair with invertible elements we can associate (in a functorial way) a connected generalized projective geometry with base point such that (1) – (4) hold.

Proof. $(1) \Rightarrow (2)$: cf. Section 2.5

$(2) \Rightarrow (1)$: cf. Section 3.4.

$(2) \Leftrightarrow (4)$: this follows immediately from the fact that $M_{x,-x} = Q(x)$ – cf. the equivalence of (b) and (c) in Section 3.3.

$(2) \Leftrightarrow (3)$: this follows from Equation (3.4) – cf. the equivalence of (a) and (b) in Section 3.3; for $(3) \Rightarrow (2)$ we use that, by assumption, the big orbits of $\widetilde{\tau}_{V'}$ do not lie at second order infinity, and therefore the assumption on x made in 3.3 and 3.4 is fulfilled.

The central null-system is unique by 2.6, and as seen is Section 3.4, the big orbit $\mathcal{O}_o$ of the dual translation group $\widetilde{\tau}_{V'_o}$ is unique. The inner polarities are all conjugate under the automorphism group: fixing an origin $o \in X$ as in Chapter 3, any inner polarity $M_{y,z}$ is conjugate to an inner polarity of the form $M_{o,x}$; in Section 3.4 we have seen that then necessarily $x \in \mathcal{O}_o$, and therefore $M_{y,z}$ is conjugate to $M_{o,e}$ with some base point $e \in \mathcal{O}_o$.

Finally, given a Jordan pair, we can construct (in a functorial way) an associated connected geometry (X, X') ([Be01b, Th. 10.1]). The geometry (X, X') is stable (in the sense of 1.7) since the associated Jordan pair is stable (cf. [Lo95, Prop. 3.2]), and it admits inner polarities $M_{x,-x} = Q(x)$ since by assumption the Jordan pair contains invertible elements x. Thus property (2) and hence also the other properties hold. ■

We do not know whether Theorem 4.1 generalizes to a bijection between Jordan pairs having invertible elements and connected geometries satisfying (1) – (4). Properties (2) and (4) are equivalent for any connected geometry, but for the other equivalences we need more specific assumptions. On a Jordan theoretic level, this corresponds to the problem of finding the most general conditions under which Koecher's theory of the Jordan inverse ([BK65], [Koe69]) can essentially be generalized – see Sections 4.3, 6.3 and Problem 7.1.

4.2. (Jordan algebras.) Recall that a unital Jordan algebra is essentially the same as a Jordan pair together with a distinguished invertible element, say e, and with product $xy = \frac{1}{2}T(x, Q(e)^{-1}e, y)$ (cf. [Lo75]). Therefore, under the correspondence of Theorem 4.1, unital Jordan algebras correspond to a geometry (X, X') together with *two* base points (o, o'), (e, e') satisfying the relation of a *harmonic quadruple* – see Section 5.3.

4.3. (Alternative of proof in the finite dimensional case over a field.) If one is willing to use some theory of finite dimensional Jordan algebras, then the implication $(4) \Rightarrow (1)$ from Theorem 4.1

can be proved differently in the finite dimensional case over a field. Let us use here notation and some results from [Be00] which are given there for $\mathbb{K} = \mathbb{R}$; but using results of Koecher [Koe69] these can be extended to the case of an arbitray base field with $\frac{1}{2} \in \mathbb{K}$. Assume V is a finite-dimensional Jordan algebra with unit element e and Jordan inverse $j(x) = x^{-1} = Q(x)^{-1}x$. The *conformal group* $G = \mathrm{Co}(V)$ is the group of birational maps generated by the translations, the structure group $\mathrm{Str}(V)$ of V and the Jordan inverse j. It carries an involution given by $j_*(g) = jgj$. The *conformal completion of* V is the imbedding

$$V \to X = \mathrm{Co}(V)/P, \quad v \mapsto \tau_v P, \quad P = \mathrm{Str}(V) j_*(\tau_V).$$

We let $X' := X$ and put $o := eP$, $o' := j(o)$, $V' = j(V)$; sometimes one writes also $\infty := o'$. For $a = g.o' \in X' = X$ we let $V_a = g(V)$ and $M = \{(x,a) \in X \times X' | x \in V_a\}$. Then (X, X', M) is a generalized projective geometry; in fact, it is easily verified that this is the geometry constructed from the asscociated Jordan pair (V, V) of V in [Be01b, Th. 10.1]. Then the identity map $n = \mathrm{id} : X \to X'$ is a central null-system. The central null-system is identified with the Jordan inverse j if use the realization of X' as $X' = \mathrm{Co}(V)/P'$, $P' = jPj$ (as in [Be00]) – see Chapter 6 for further remarks on the corresponding geometry.

4.4. (Classification.) The well-known classification of finite-dimensional simple complex Jordan algebras (cf. [Lo75] or [FK94]) yields the following list of finite-dimensional complex simple central null-geometries. In the last column we list the isomorphism class of the symmetric space Ω associated to the unique class of inner polarities (cf. Chapter 6); these are the complexifications of the symmetric cones from [FK94] (the symmetric cones and their "satellites" would appear in a similar classification of the real geometries which is too long to be presented here; cf. [Be00, Ch. XII]).

	Jordan algebra V	geometry (X, X'); $X = X'$	symmetric space Ω
1.	$M(n,n;\mathbb{C})$	$X = \mathrm{Gras}_n(\mathbb{C}^{2n})$	$\mathrm{Gl}(n,\mathbb{C})$
2.	$\mathrm{Sym}(n,\mathbb{C})$	$X = \mathrm{Lag}(\mathbb{C}^{2n}, \left(\begin{smallmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{smallmatrix}\right))$	$\mathrm{Gl}(n,\mathbb{C})/\mathrm{O}(n,\mathbb{C})$,
3.	$\mathrm{Asym}(n,\mathbb{C})$	$X = \mathrm{Lag}(\mathbb{C}^{2n}, \left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{smallmatrix}\right))$	$\mathrm{Gl}(2m,\mathbb{C})/\mathrm{Sp}(m,\mathbb{C})$ $(n = 2m)$
4.	$\mathbb{C}^n$	$X = \{[z] \in \mathbb{CP}^{n+1} \vert \sum_i z_i^2 = 0\}$	$(\mathrm{O}(n,\mathbb{C}) \times \mathbb{C}^*)/\mathrm{O}(n-1,\mathbb{C})$
5.	$\mathrm{Herm}(3,\mathbb{O}_\mathbb{C})$	$X = E_7/P$	$(E_6 \times \mathbb{C}^*)/F_4$

In cases 1 and 2 the Jordan product on V is the usual symmetrized matrix product $X \cdot Y = \frac{XY+YX}{2}$ which is obtained from the Jordan triple product $T(X,Y,Z) = XYZ + ZYX$ by fixing the middle element to be the invertible element $\frac{1}{2}\mathbf{1}_n$. The space $\mathrm{Asym}(n,\mathbb{C})$ (case 3) is stable under T; if n is even, it contains invertible elements, for instance the element $J = \left(\begin{smallmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{smallmatrix}\right)$, and thus belongs to a unital Jordan algebra (matrices which are symmetric with respect to J). If n is odd, then $\mathrm{Asym}(n,\mathbb{C})$ does not contain invertible elements. This is geometrically explained in Example 2.8: the geometry (X, X) from line 3 has two connected components; if n is even, then each component is stable under the central null-system, but if n is odd, then this is not the case. In case 4 the Jordan product is given by

$$x \cdot y = b(x,e)y + b(y,e)x - b(x,y)e,$$

where $b(z,w) = \sum_i z_i w_i$ and e such that $b(e,e) = 1$. The corresponding geometry is the one of the complex quadric (cf. Example 2.9). Finally, for definitions in the exceptional case cf. [FK94]; we give the associated geometry in the form $\mathrm{Co}(V)/P$ from Section 4.3 (P is a semidirect product of the structure group $E_6 \times \mathbb{C}^*$ and the translation group $\mathbb{C}^{27}$); it would be nice to have a geometric model in which the central null-system becomes better "visible". Finally, note that the classical series 1. - 4. have natural counterparts in infinite dimension; for another class of infinite-dimensional examples see the Appendix.

5. Harmonic range, Jordan algebras and von Staudt's theorem

5.1. (Harmonic range.) In this chapter (X, X') may be any generalized projective geometry (in section 5.5 we specialize to the interesting case of null-geometries). A quadruple (o, o', a, b) of points in $X \times X' \times X \times X$ is said to be *in harmonic range* if $o, a, b \in V_{o'}$ and one of the two following equivalent conditions is satisfied:

$$\mu_{\frac{1}{2}}(a, o', b) = o, \qquad \mu_{-1}(o, o', a) = b, \tag{5.1}$$

(It is clear that both conditions are equivalent to $a + b = o$ in $(V_{o'}, o)$.) The set $H \subset X \times X' \times X \times X$ of harmonic quadruples is precisely the graph of the multiplication map

$$\mu_{\frac{1}{2}} : X \times X' \times X \supset D \to X,$$

and in a similar way it can also be seen as the graph of μ_{-1}. Moreover, corresponding to the definition of $\mu_{\frac{1}{2}}$ and μ_{-1} on an extended domain, the notion of harmonic range may be extended to quadruples of points such that at least one of them is remote from o'.

5.2. (Harmonic quadruples and orbits of the structure group.) The image of a harmonic quadruple under an automorphism (g, g') of (X, X') clearly is again a harmonic quadruple. However, the action of the automorphism group G on harmonic quadruples is in general not transitive: first of all, since any triple (a, o', b) of pairwise remote points can uniquely be completed to a harmonic quadruple by putting $o = M_{a,b}(o')$, the set of harmonic quaduples is in bijection with the set $D \subset X \times X' \times X$. If (X, X') is connected, then the action of G on $M \subset X \times X'$ is transitive (Section 1.6), and any point in D is conjugate to (o, o', x) for a fixed base point $(o, o') \in M$ and $x \in V_{o'}$. The stabilizer of (o, o') in G is the structure group $\mathrm{Str}(V)$, and hence the G-orbits in D are in bijection with the $\mathrm{Str}(V)$-orbits in V. A classification of harmonic quadruples is therefore equivalent to the classification of the orbits of the structure group. (In the finite dimensional case over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, there is just a finite number of such orbits, chararacterized by invariants such as rank and signature; in the general case this is no longer true.)

5.3. (The associated Jordan algebra.) Assume (o, o', a, b) are in harmonic range, consider (o, o') as base point in (X, X') and let $e := a = -b$. For $x \in V'$ we let

$$x^2 := Q(x)e = M_{x,-x}(e). \tag{5.2}$$

According to [Be01b, Th. 8.6], $Q : V' \to \mathrm{Hom}(V, V')$ is quadratic polynomial, and hence the squaring comes from a bilinear product

$$x \cdot y = \frac{1}{2}(Q(x + y)e - Q(x)e - Q(y)e) \tag{5.3}$$

which is obtained from the associated Jordan pair (T, T') via

$$x \cdot y = \frac{1}{2} T(x, e, y). \tag{5.4}$$

Recall from Section 1.2 that the linearization (T, T') of (Q, Q') is a linear Jordan pair. Therefore it follows from "Meyberg's theorem" ([Lo75, 1.9]) that the product $x \cdot y$ defines a Jordan algebra (without unit in general). Note that this Jordan algebra is naturally defined on the dual space. We would need a "dual harmonic quadruple" in order to define a Jordan algebra on V.

5.4. (Center and centroid.) The *center* of a unital Jordan algebra (V,e) is defined by (cf. [BK65, p.24])

$$Z(V)=\{v\in V\,|\,(v,V,V)=(V,v,V)=(V,V,v)=0\}, \tag{5.5}$$

where

$$(a,b,c)=(ab)c-a(bc) \tag{5.6}$$

is the *associator* of three elements. Thus the center is a commutative and associative subalgebra containing e, and V can be seen as a Jordan algebra over $Z(V)$. For general Jordan pairs the center is replaced by the *centroid* which we define (slightly differently from [Lo75, 1.5]) to be the center of the *multiplication algebra* of (V,V') which is the subalgebra of $\mathrm{End}_{\mathbb{K}}(V\oplus V')$ generated by all inner operators $T(x,y)$, $T'(x,y)$, $Q(x)$, $Q'(y)$ together with the projections onto V and V' (cf. [Lo75, 2.4]). By definition, the centroid is a commutative and associative algebra, and its elements are of the form (z,z'), $z:V\to V$, $z':V'\to V'$ such that T and T' are "trilinear with respect to z,z'". Thus our Jordan pair can be defined over the centroid Z, and therefore it is possible to consider the corresponding connected generalized projective geometry (X,X') to be defined over $\overline{\mathbb{K}}:=Z$.

In this situation we can define *semi-linear automorphisms* of the geometry (X,X'): Let τ be an automorphism of $\overline{\mathbb{K}}$; then (φ,φ') is called a τ*-conjugate* or *semi linear homomorphism* if it is remoteness preserving and satisfies the identity

$$\varphi(\mu_r(x,y,z))=\mu_{\tau(r)}(\varphi(x),\varphi(y),\varphi(z)) \tag{5.7}$$

together with its dual identity.

Theorem 5.5. (von Staudt's theorem) *Assume (X,X') is a connected geometry over $\mathbb{K}$, let $k\subset\mathbb{K}$ be the subring generated by the element $\frac{1}{2}\in\mathbb{K}$ and let $\overline{\mathbb{K}}=Z=Z(V,V')$ be the centroid of one (and hence of all) of the Jordan pairs corresponding to (X,X'). Then for a pair of bijections $\varphi:X\to X$, $\varphi':X'\to X'$ the following are equivalent:*

(1) *φ and φ' map harmonic quadruples onto harmonic quadruples: (a,o',b,c) harmonic implies $(\varphi(a),\varphi'(o'),\varphi(b),\varphi(c))$ harmonic, and dually,*
(2) *(φ,φ') is an automorphism of X, considered as geometry over the subring k,*
(3) *(φ,φ') is a conjugate-linear automorphism of X, considered as a geometry over $\overline{\mathbb{K}}$.*

Proof. If (φ,φ') satisfies (2) or (3), then

$$\varphi(\mu_{\frac{1}{2}}(a,o',b))=\mu_{\frac{1}{2}}(\varphi(a),\varphi'(o'),\varphi(b)),$$

and therefore the harmonic pair (a,b,o',o) is mapped onto the harmonic pair $(\varphi(a),\varphi(b),\varphi(o'),\varphi(o))$. Conversely, if φ preserves harmonic range, then by (5.1) the maps $\mu_{\frac{1}{2}}$ and μ_{-1} are φ-equivarant. But all vector additions can be recovered from these since

$$x+_{o,o'}y=\mu_{-1}(\mu_{\frac{1}{2}}(o,o',x),o',\mu_{-1}(o,o',y)),$$

and therefore all restrictions of $\varphi:(V_{o'},o)\to(\varphi(V_{o'}),\varphi(o))$ are additive. The compatibility with multiplication by scalars from k is an immediate consquence. In order to exhibit their compatibility with multiplication by scalars from $\overline{\mathbb{K}}$, we may and will assume (by composing with an automorphism) that (φ,φ') fixes the base point (o,o'). Then, since the associated Jordan pair is defined by $Q(x)y=M_{x,-x}(y)$, φ preserves the Jordan pair associated to (o,o'):

$$\varphi(T(x,a,y))=T(\varphi(x),\varphi'(a),\varphi(y)),$$

and it follows that, for all $(c,c') \in Z(V)$, also $(\varphi_* c, \varphi'_* c') := (\varphi c \varphi^{-1}, \varphi' c' (\varphi')^{-1})$ belongs to $Z(V)$. Thus

$$\tau := (\varphi_*, \varphi'_*) : Z(V) \to Z(V), \quad (c,c') \to (\varphi_*(c), \varphi'_*(c'))$$

is a ring automorphism, and it follows that (φ, φ') is τ-linear: for all $r \in \overline{\mathbb{K}}$,

$$\varphi(rx) = \tau(r)\varphi(x)$$

and dually. But this is precisely (5.7). ■

In general, nothing can be said about the behaviour of a harmonicity-preserving map with respect to scalars from $\mathbb{K}$ – as an example one may take a geometry which is defined over $\mathbb{C}$, but we take scalars from $\mathbb{K} = \mathbb{R}$ only. Since there exist automorphisms of $\mathbb{C}$ which do not stabilize $\mathbb{R}$, there exist in general harmonicity preserving maps of the geometry which are not defined over $\mathbb{R}$.

5.5. (Case of a null-geometry.) If (X, X') is a connected null-geometry, then, by (2.3), Equation (5.1) is equivalent to

$$\mu_{-1}(b, a, o') = o, \quad \mu_{\frac{1}{2}}(o', b, o) = a. \tag{5.8}$$

It follows that (a, b, o, o') are in harmonic range iff so are (b, a, o, o'), and we may speak of a *harmonic pair* $(o, o'), (a, b)$, having most of the properties known from the classical case of the projective line. Using the polarity $p = (M_{e,-e})^{-1} = Q(e)^{-1}$ $(e = a = -b)$, we identify V and V', and our Jordan pair (T, T') is turned into a Jordan triple system: $T^{(p)}(x, y, z) = T(x, p(y), z)$. Fixing $y = e$, we obtain the squaring

$$x^2 = T^{(p)}(x, e, x) = T(x, p(e), x) = Q(x)Q(e)^{-1}e = Q(x)e \tag{5.9}$$

(recall that e and $Q(e)^{-1}e$ are identified) which defines a Jordan algebra on V with unit e. Conversely, given a unital Jordan algebra (V, e), the element (e, e) is an invertible element of the associated Jordan pair (V, V) and $((e, -e), (o, o'))$ is a harmonic pair. Thus unital Jordan algebras are essentially in bijection with connected null-geometries together with a fixed harmonic pair. In this case, von Staudt's theorem may be announced by forgetting the distinction between X and X', and its proof really follows the lines of the original proof in one dimension, replacing the associated Jordan pair by the associated Jordan algebra and the centroid by the center which, in the one-dimensional case, is just the base field itself.

5.6. (Remark on the cross-ratio.) In [Br68], H. Braun defines an operator-valued cross-ratio $E(w, x, y, z)$ for four generic points in a Jordan algebra V and calls the four points *harmonic* if $E(w, x, y, z) = -\mathrm{id}_V$. This coincides with our definition: in fact, according to our definition, for all x in the associated Jordan algebra V, the quadruple

$$(x^2, e, -x, x) = (Q(x)e, e, -x, x) = (M_{x,-x}(e), e, -x, x)$$

is harmonic; but it it is also harmonic in the sense of H. Braun since in her case, by power associativity, the cross-ratio of $(x^2, e, -x, x)$ can be calculated as for the projective line (loc. cit. p. 27) and thus gives $-\mathrm{id}_V$, whence harmonicity. General harmonic quadruples can always be transformed into harmonic quadruples of this special form by using the invariance of harmonicity under the group $G = \mathrm{Aut}(X, X')$ (denoted by Ξ in [Br68]).

The matrix cases are treated by L.K. Hua in [Hua45]: for four generic matrices $A, B, C, D \in \mathrm{Sym}(n; \mathbb{C})$ he defines a *matrix cross-ratio* by

$$(A, B; C, D) := (A - C)(A - D)^{-1}(B - D)(B - C)^{-1},$$

and calls a quadruple (A,B,C,D) *harmonic* if $(A,B;C,D)=\mathbf{-1}$. Since $(X^2,\mathbf{1},-X,X)=\mathbf{-1}$, all definitions of harmonicity coincide. (The cross-ratios of Hua and Braun differ by the fact that Hua's cross-ratio is an element of V, whereas Braun's cross-ratio is an element of $\mathrm{End}(V)$.) In the same way, Hua treats the other matrix cases (1. and 3. of the table in Section 4.4) and obtains generalizations of von Staudt's theorem which are special cases of ours. However, in these cases the assumption that a bijection $\varphi : X \to X$ be remoteness-preserving (in Hua's terms: preserving "arithmetic distance") is very strong: here, in dimension bigger than one, a version of the fundamental theorem of projective geometry ([Ch49]) already implies that φ is semi-linear. In our general case there is certainly no general analogue of the fundamental theorem of projective geometry (cf. [Be00, Ch. XIII] for a detailed discussion of this theorem), and in view of its proof it seems that our version of von Staudt's theorem is the most general theorem characterizing the semilinear group.

6. Associated symmetric spaces

6.1. (The symmetric space of an inner polarity.) Assume (X,X') satisfies the assumptions of Theorem 4.1. As before, we identify X and X' via the central null-system n. Let us fix $(a,b)\in M$ and the inner polarity $p=M_{a,b}$. (Recall that all inner polarities are conjugate to this one.) As for any polarity, there is an associated symmetric space

$$\Omega := \Omega_{]a,b[} := \{x\in X \,|\, (x, M_{a,b}(x))\in M\}.$$

The isomorphism class of this symmetric space is canonically associated to the geometry (X,X') without adding any additional structure. The points a and b do not belong to $\Omega_{]a,b[}$; we call them the *vertices of* $\Omega_{]a,b[}$. The notation $\Omega_{]a,b[}$ indicates that our space only depends on a and b and not on other choices, and one may think of it as a sort of "generalized interval with vertices a and b" – in fact, if the associated Jordan algebra is *formally real* or *Euclidean* (cf. [FK94]), then a family given by convex connected components of the $\Omega_{]a,b[}$ defines a family of *causal intervals* for the associated *causal structure* on X (cf. [Be00, Th. XI.3.3]).

6.2. (The finite part of $\Omega_{]a,b[}$.) If $x\in(V_a\cap V_b)$, then also $M_{a,b}(x)\in(V_a\cap V_b)$ and hence $(x,M_{a,b}(x))\in M$. Therefore

$$(V_a\cap V_b)\subset\Omega_{]a,b[}.$$

Moreover,

$$\Omega_{]a,b[}\cap V_a=\Omega_{]a,b[}\cap V_b=V_a\cap V_b;$$

in fact, if $(x,a)\in M$ then $(x,M_{a,b}x)\in M$ is equivalent to $(x,2_{x,a}b)\in M$ and hence to $(2_{x,a}^{-1}x,b)=(x,b)\in M$. We call

$$\Omega^f_{]a,b[} := V_a\cap V_b$$

the *finite part of* $\Omega_{]a,b[}$. In the finite dimensional case over a field, $\Omega_{]a,b[}$ is equal to its finite part (if x,y both lie at infinity of V_a, then x and y are always incident, i.e. $(x,y)\notin M$; cf. [Be00] for the real case). Let us show that in general the finite part is a subsymmetric space of $\Omega_{]a,b[}$: the symmetry σ_x with respect to $x\in\Omega_{]a,b[}$ is $\sigma_x=(-1)_{x,p(x)}=M_{x,M_{a,b}(x)}$, whence

$$\begin{aligned}\sigma_x(a)&=M_{x,M_{a,b}(x)}(a)\\&=\mu_{\frac12}(x,a,\mu_{\frac12}(a,x,b))\\&=2_{x,a}^{-1}2_{a,x}^{-1}(b)=2_{x,a}^{-1}2_{x,a}(b)=b.\end{aligned}$$

It follows that $\sigma_x(V_a) = V_b$ and hence $\sigma_x(V_a \cap V_b) = V_a \cap V_b$; thus $V_a \cap V_b$ is a subspace. Since all symmetries σ_x exchange the vertices, any products of two symmetries preserve them. In particular, the *transvection group*

$$G := G(\Omega_{]a,b[}) =< \sigma_x \sigma_y | \, x, y \in \Omega_{]a,b[} >$$

stabilizes both vertices a and b and therefore acts linearly on (V_a, b) and on (V_b, a). (One should "read" this statement in the "unbounded picture", see below.)

6.3. (Base points and Jordan algebras.) A *base point* in $\Omega_{]a,b[}$ is just a point $x \in \Omega_{]a,b[}$. Thus, letting $y := M_{a,b}(x)$, the quadruple of points (x, y, a, b) is harmonic (cf. 5.1). We choose $(o, o') = (x, y)$ as base point, then $b = -a =: e$ and $\Omega = \Omega_{]-e,e[}$. The associated Jordan algebra is defined by the squaring $x^2 = Q(x)e = M_{x,-x}(e)$, and its *Jordan inverse* is

$$j(x) = Q(x)^{-1}x = (M_{x,-x})^{-1}x = n(x),$$

whence $j = n$ is the canonical null-system. Recall from Equation (2.8) that $n = (-1)_{a,b} M_{a,b}$; therefore, if we use the polarity $p = M_{a,b}$ as identification of X and X' (as usual for polar geometries; cf. [Be01b]), then j describes the polarity $p = M_{e,-e} = (-1)_{e,-e}$. The symmetry σ_o with respect to our fixed base point o is given by $(-1)_{o,o'}$ (negative of the identity on $V = (V_{o'}, o)$).

We have just defined the canonical realization of $\Omega = \Omega_{]-e,e[}$ which, in fact, corresponds to the *bounded realization* of a symmetric cone (cf. [Be00, Th. XI.3.3]). However, the most useful realization is the *unbounded linear realization*: recall from Section 5.5 that (x, y, a, b) are harmonic if and only if (a, b, x, y) are harmonic. Thus we may take $(o, o') = (a, b) \in M$ as base point and write $\infty := o'$ (thus we image the vertices to lie at o and ∞ and write $\Omega = \Omega_{]o,\infty[}$). In this picture, the polarity is just multiplication by -1 in $V = (V_{o'}, o)$, and the base point is then written $(x, y) = (e, -e)$ with $e \in \Omega$. As before, the Jordan inverse of the associated Jordan algebra is the canonical null-system and we use the polarity $M_{e,-e}$ in order to identify X and X'. Therefore the Jordan inverse now describes the polarity $M_{e,-e}$ which is nothing but the symmetry σ_e with respect to the point $e \in \Omega$. In this picture, the polarity $p = M_{o,o'}$ is simply multiplication by -1 in $V = (V_{o'}, o)$.

Cleary, any automorphism φ of (X, X') with $\varphi(x) = a$, $\varphi(y) = b$ interchanges these two pictures. There is a distinguished automorphism of this kind, called the *Cayley transform*, cf. Remark 7.3.

Finally, recall from 5.2 that our symmetric spaces need not be homogeneous under their automorphism group. Therefore, although all polarities $M_{a,b}$, $(a, b) \in M$, are equivalent, the pairs "associated symmetric space + base point" need not be all equivalent: the space $\Omega_{]a,b[}$ defines an associated Jordan algebra only up to *isotopy* (cf. Remark 9.9 in [Be01b]).

6.4. (Quadratic prehomogeneous symmetric spaces.) We consider $\Omega = \Omega_{o,\infty}$ in its "linear realization" (see above). The finite part is given by

$$\Omega_f = V \cap V_\infty = \{x \in V | \, (x, -x) \in M\} = \{x \in V | \, Q(x) \, \text{invertible}\},$$

by definition, this is the set $V^\times$ of invertible elements in the Jordan algebra V. The multiplication map of this subsymmetric space is given by the formula

$$\begin{aligned} \mu(x, y) &= \mu_{-1}(x, p(x), y) \\ &= \mu_{-1}(x, -x, y) = \mu_{\frac{1}{2}}(x, n(y), -x) \\ &= M_{x,-x}(M_{y,-y})^{-1} y \\ &= Q(x)Q(y)^{-1} y = Q(x)^{-1} j(y) \end{aligned}$$

where j is the Jordan inverse. The latter formula shows that $\mu(x,y)$ depends quadratically on x. Thus (V,e,Q) a *quadratic prehomogeneous symmetric space (with base point)* in the following sense:

(1) $Q : V \to \mathrm{End}(V)$ is a quadratic polynomial such that $Q(e) = \mathrm{id}_V$,

(2) $V^\times = \{x \in V \,|\, Q(x) \,\text{invertible}\}$ is non-empty, and $\mu(x,y) = Q(x)Q(y)^{-1}y$ defines a map from $V^\times \times V^\times$ to $V^\times$ such that the defining properties (M1) – (M4) of a symmetric space hold:

(M1) $\mu(x,x) = x$

(M2) $\mu(x,\mu(x,y)) = y$

(M3) $\mu(x,\mu(y,z)) = \mu(\mu(x,y),\mu(x,z))$

(M4) the fixed point x of $\sigma_x = \mu(x,\cdot)$ is isolated.

However, the precise formulation of the isolation property (M4) is less clear in the unbounded picture than in the bounded realization. For this reason we do not develop here the axiomatic theory of such spaces – see [Be00, Ch.2] for the finite dimensional case over $\mathbb{K} = \mathbb{R}$ where one can establish a bijection between quadratic prehomogeneous symmetric spaces and unital Jordan algebras.

7. Problems and further topics

7.1. (Non-unital Jordan algebras.) The interpretation of non-unital Jordan algebras is difficult: they appear in Section 5.3, but we have not given a geometric interpretation. One may adjoin a unit element to a non-unital Jordan algebra (cf. [BK65]), but the geometric meaning of this construction in the present context is not clear. It seems possible that certain null-geometries which do not satisfy the hypotheses of Theorem 4.1 correspond in some sense to non-unital Jordan algebras. This is related to the "topological" problems encountered in the proof of 4.1: our assumptions in Th. 4.1 are sufficient, but not necessary; what are the necessary and sufficient "topological" assumptions on the geometry in order to get an equivalence with Jordan pairs having invertible elements ?

7.2. (Cross-ratio.) As mentioned in the introduction and in Remark 5.6, the definition of a *cross-ratio* belongs to the topic of invariant theory on generalized projective geometries. A central rôle is played here by the *Bergman operator* $B(x,y)$ (cf. [Be01b, Ch.6]).

7.3. (Geometric Peirce-theory.) Harmonic quadruples are closely related to *idempotents* of the associated Jordan pair. The latter give rise to homomorphisms of the projective line into the geometry and lift to homomorphisms of $\mathrm{Sl}(2,\mathbb{K})$ into the automorphism group which serves then to define geometrically the associated Peirce-decomposition. Also, the Cayley transform (cf. Section 6.3) is best introduced in this context.

Appendix: A class of infinite-dimensional examples

Assume (Y,Y') is a generalized projective geometry over $\mathbb{K}$ and S a set. Denote by $\mathcal{F}(S,N)$ the set of functions from S to a set N. Then

$$(X,X') := (\mathcal{F}(S,Y), \mathcal{F}(S,Y'))$$

is a generalized projective geometry over $\mathbb{K}$: in fact, this is just the direct product, indexed by elements of S, of (Y,Y') with itself (cf. [Be01b, 2.7]). In terms of functions, the data can be expressed as follows:

$$M = \{(f,g) \in (X,X') | \forall s \in S : (f(s),g(s)) \in M_{Y,Y'}\},$$

$$(\mu_r(f,g,h))(s) = \mu_r(f(s),g(s),h(s)),$$

and dually. Since everything is defined pointwise, the defining relations (PG1), (PG2) etc. for (X,X') follow immediately from those of (Y,Y'), and the scalar extension for (X,X') is simply given by functions into the scalar extension of (Y,Y'). If a base point (o,o') of the geometry (Y,Y') is chosen, then the constant functions

$$(f_o, f_o'), \quad f_o(s) = o, \quad f_{o'}(s) = o',$$

define a base point in (X,X'). If $(V,V') = (V_{o'}, V_o')$ is the corresponding pair of $\mathbb{K}$-modules in (Y,Y'), then

$$\mathbf{V} = \{f \in Y | \forall s \in S : f(s) \in V\} = \mathcal{F}(S,V)$$

together with $\mathbf{V}' = \mathcal{F}(S,V')$ are the corresponding $\mathbb{K}$-modules in (X,X'). The corresponding Jordan pair structure on $(\mathbf{V},\mathbf{V}')$ is given pointwise by the Jordan pair structure of (V,V'). It is clear that, if V carries a structure of Jordan algebra with unit e, then so does $\mathbf{V}$ with unit the constant function $\mathbf{e}(x) = e$. In the general case, the first order infinity set is

$$\mathbf{I}_\infty = \{f \in Y | \exists s \in S : f(s) \in \mathrm{I}_\infty\}.$$

Moreover, it is immediately clear that stability of (Y,Y') implies stability of (X,X'), but connectedness of (Y,Y') does not imply connectedness of (X,X').

The automorphism group of (X,X') will in general be strictly bigger than $\mathcal{F}(S, \mathrm{Aut}(Y,Y'))$, and thus one cannot conclude that an absolute identification $n : Y \to Y'$ induces via $(n(f))(s) = n(f(s))$ an absolute identification of (X,X'). However, since $\mathbf{V}$ is a unital Jordan algebra if so is V, we will get an absolute identification on the connected component generated by $(\mathbf{V},\mathbf{V}')$.

In order to get interesting examples, assume now that S is a finite-dimensional smooth manifold over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and (Y,Y') is a connected finite dimensional geometry over the same field. Then instead of $\mathcal{F}$ we take $\mathcal{C}^\infty$, i.e. sets of smooth functions (one may also take smooth functions with compact support), and define (X,X') as above. Then everything is well-defined, and (X,X') is a, in general infinite-dimensional, geometry over $\mathbb{K}$ whose associated Jordan pair is a Jordan pair of smooth functions from S to V, resp. to V'. Then, in general the translation group $\tau_{\mathbf{V}}$ does have orbits at infinity on which it acts freely: in fact, assume there exists a function $f \in X$ having a finite, non-zero number of “poles”, i.e. the set

$$P_f := \{s \in S | f(s) \notin V\}$$

is finite and non-empty. (For instance, if $S = S^1$ and $V = \mathbb{R}$, i.e. $X = S^1$, then $f(s) = s$ defines such a function.) Then clearly $f \in \mathbf{I}_\infty$, and the stabilizer of f in $\tau_{\mathbf{V}}$ is trivial: $\tau_g(f) = f$ implies $g(s) = o$ for all s with a finite number of exceptions; but since g is continuous, it follows that $g = f_o$ and hence $\tau_{\mathbf{V}}$ acts freely on the orbit $\tau_{\mathbf{V}}.f$.

References

[Ar66] Artin, E., *Geometric Algebra*, (5th ed.), Interscience, New York, 1966.

[B87] Berger, M., "Geometry" (2 volumes), Springer-Verlag, Berlin 1987.

[Be00] Bertram, W., *The geometry of Jordan- and Lie structures*, Springer Lecture Notes in Mathematics **1754**, Berlin 2000.

[Be01a] Bertram, W., "From linear algebra via affine algebra to projective algebra", preprint, Nancy 2001.

[Be01b] Bertram, W., "Generalized projective geometries: general theory and equivalence with Jordan structures", preprint, Nancy 2001 (to appear in Advances in Geometry).

[Br68] Braun, H., "Doppelverhältnisse in Jordan-Algebren", Abh. Math. Sem. Hamburg **32** (1968), 25–51.

[BK65] Braun, H. and M. Koecher, *Jordan-Algebren*, Springer-Verlag, Berlin 1965.

[Ch49] Chow, W.L., "On the geometry of algebraic homogeneous spaces", Ann. Math. **50**, no. 1 (1949), 32 – 75.

[FK94] Faraut, J. and A. Koranyi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford 1994.

[Hua45] Hua, L-K., "Geometries of Matrices. I. Generalizations of von Staudt's theorem", Trans. A.M.S. **57** (1945), 441–481.

[JNW34] Jordan, P., von Neumann, J. and Wigner, E., "On an algebraic generalization of the quantum mechanical formalism", Ann. Math. **35** (1934), 29–64.

[Koe69] Koecher, M., "Gruppen und Lie-Algebren von rationalen Funktionen", Math. Z. **109** (1969), 349–392.

[Lo69] Loos, O., *Symmetric Spaces I*, Benjamin, New York 1969.

[Lo75] Loos, O., *Jordan Pairs*, Springer LN 460, New York 1975.

[Lo95] Loos, O., "Elementary Groups and Stability for Jordan Pairs", K-Theory **9** (1995), 77 - 116.

[Sp73] Springer, T.A., *Jordan Algebras and Algebraic Groups*, Springer Verlag, New York 1973.